Neighborhood and algebraic models for predicate modal logics with ω-rules

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Outline

- Equivalence between neighborhood frames with constant domains and complex modal algebras as semantics for predicate modal logic;
- Model existence theorem for predicate modal logics with ω-rules including normal and non-normal cases;
- Completeness of a predicate extension of GL with respect to neighborhood frames with constant domains;
- Kripke incompleteness and neighborhood completeness of a common knowledge predicate logic.

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ω-rules

ω-rules

❄ An ω-rule is an inference rule with countably many premises of the form

$$\frac{p \supset \beta_i \ (i \in \omega)}{p \supset \alpha}, \tag{1.1}$$

❄ The intended meaning of (1.1) with the axiom schemata

$$\alpha \supset \beta_i \ (i \in \omega)$$

is

$$\alpha \equiv \bigwedge_{i \in \omega} \beta_i.$$

❄ Predicate extensions of certain propositional modal logics are not computably enumerable, even when the underlying propositional modal logics are decidable.

❄ By adopting the ω-rules, we sometimes obtain simpler proofs of completeness theorems.

ω-rules

Example

(T. 18). With $\Box p \supset \Box\Box p$, the ω-rule

$$\frac{p \supset \Diamond^n \top \ (n \in \omega)}{p \supset \bot}$$

axiomatizes the provability logic GL.

Example

(Kaneko-Nagashima-Suzuki-T. 02). With $\mathsf{C}p \supset \mathsf{E}^n p$ $(n \in \omega)$, the ω-rules

$$\frac{\gamma \supset \Box_1(\phi_1 \supset \Box_2(\phi_2 \supset \cdots \supset \Box_k(\phi_k \supset \mathsf{E}^n\phi)\cdots)) \quad (n \in \omega)}{\gamma \supset \Box_1(\phi_1 \supset \Box_2(\phi_2 \supset \cdots \supset \Box_k(\phi_k \supset \mathsf{C}\phi)\cdots))},$$

axiomatizes the common knowledge logic, where $k \in \omega$ and each $\Box_i$ $(i = 1, \ldots k)$ is E or C.

Neighborhood frames and modal algebras

Neighborhood semantics

❄ Using neighborhood semantics, we can provide

1. models for non-normal modal logics;
2. constant domain models for predicate modal logics without assuming the Barcan formula.

Distributivity of the modal operator over conjunction

❄ It is well known that neighborhood frames serve as a semantic framework for non-normal modal logics, namely, modal logics that do not validate $\Box\top \equiv \top$ or $\Box(p \wedge q) \equiv \Box p \wedge \Box q$.

❄ If we define $\bigwedge \emptyset$ as $\top$, this can be summarized that neighborhood frames can provide semantics for modal logics without assuming distributivity of the modal operator over finite conjunction.

❄ In fact, neighborhood frames can be used to interpret infinitary modal logics in which the distributivity of the modal operator over conjunction of cardinality κ does not hold, for any infinite cardinal κ (Minari 16, T. 21).

❄ On the other hand, for any infinite set X of formulas, Kripke frame validates

$$\Box \bigwedge X \equiv \bigwedge_{x \in X} \Box x.$$

Neighborhood frames

Definition

A *neighborhood frame* is a pair $\langle C, \mathcal{N} \rangle$, where C is a non-empty set and $\mathcal{N}$ maps each $c \in C$ to a subset $\mathcal{N}(c)$ of $\mathcal{P}(\mathcal{P}(C))$. A neighborhood frame $\langle C, \mathcal{N} \rangle$ is said to be *monotonic* (MT), *topped* (TP), or *closed under finite intersections* (CF) if it satisfies the following conditions:

- **MT** for any $c \in C$, $\mathcal{N}(c)$ is an upward closed subset of $\mathcal{P}(C)$ ordered by inclusion;
- **TP** for any $c \in C$, $\mathcal{N}(c)$ contains C;
- **CF** for any $c \in C$, if $X,\ Y \in \mathcal{N}(c)$ then $X \cap Y \in \mathcal{N}(c)$.

A neighborhood frame $\langle C, \mathcal{N} \rangle$ is called a *Kripke frame* if it satisfies MT and $\bigcap \mathcal{N}(c) \in \mathcal{N}(c)$ for any $c \in C$.

Modal algebras

Definition

(Došen). An algebra $\langle A; \vee, \wedge, -, \Box, 0, 1\rangle$ is a *modal algebra*, if its reduct $\langle A; \vee, \wedge, -, 0, 1\rangle$ is a Boolean algebra and $\Box$ is a unary operator on A. A modal algebra is said to be *complete* if its underlying Boolean algebra is complete. A complete modal algebra is *completely multiplicative*, if

$$\Box \bigwedge_{x \in X} x = \bigwedge_{x \in X} \Box x$$

holds, for any $X \subseteq A$. A modal algebra A is said to be *monotonic* (MT), *topped* (TP), or *closed under finite intersections* (CF), if it satisfies the following conditions:

- **MT** for any x and y in A, $\Box(x \wedge y) \leq \Box x \wedge \Box y$;
- **TP** $\Box 1 = 1$;
- **CF** for any x and y in A, $\Box x \wedge \Box y \leq \Box(x \wedge y)$.

The complex modal algebra of a neighborhood frame

Definition

(Došen). Let $Z = \langle C, \mathcal{N} \rangle$ be a neighborhood frame. Define the *complex modal algebra* of Z, which is denoted by $\mathsf{Alg}(Z)$, by

$$\mathsf{Alg}(Z) = \langle \mathcal{P}(C); \cup, \cap, C \setminus -, \Box_Z, \emptyset, C \rangle,$$

where

$$\Box_Z X = \{c \in C \mid X \in \mathcal{N}(c)\}$$

for any $X \subseteq C$.

Equivalence of neighborhood frames and their complex modal algebras

Neighborhood models

Definition

A *neighborhood model* for predicate modal logics is a 4-tuple $\langle C, \mathcal{N}, \mathcal{D}, \mathcal{I} \rangle$, where

- $\langle C, \mathcal{N} \rangle$ is a neighborhood frame;
- $\mathcal{D}$ is a non-empty set called the *domain*;
- and $\mathcal{I}$ is a mapping called the *interpretation* such that for each $n \in \omega$, each $P \in \mathsf{Pred}(n)$, and each $c \in C$, $\mathcal{I}$ maps (c, P) to an n-ary relation $P^{\mathcal{I}}(c) \subseteq \mathcal{D}^n$ over $\mathcal{D}$.

An *assignment* $\mathcal{A}$ to $\mathcal{D}$ is a mapping from V to $\mathcal{D}$. For any assignment $\mathcal{A}$, any variable x, and any $d \in \mathcal{D}$, define an assignment $[d/x]\mathcal{A}$ as follows:

$$[d/x]\mathcal{A}(z) = \begin{cases} \mathcal{A}(z) & \text{if } z \neq x \\ d & \text{if } z = x \end{cases}.$$

Neighborhood models

For each neighborhood model $M = \langle C, \mathcal{N}, \mathcal{D}, \mathcal{I} \rangle$ and each assignment $\mathcal{A}$, the valuation $v_{\mathcal{I},\mathcal{A}}$ of a formula $\phi \in \Phi$ on M is defined inductively as follows:

1. $v_{\mathcal{I},\mathcal{A}}(\top) = C$, $v_{\mathcal{I},\mathcal{A}}(\bot) = \emptyset$;
2. $v_{\mathcal{I},\mathcal{A}}(P(x_1, \dots, x_n)) = \{c \mid (\mathcal{A}(x_1), \dots, \mathcal{A}(x_n)) \in P^{\mathcal{I}}(c)\}$, for any $n \in \omega$, $P \in \mathsf{Pred}(n)$, and $x_1, \dots, x_n \in \mathsf{V}$;
3. $v_{\mathcal{I},\mathcal{A}}(\phi \wedge \psi) = v_{\mathcal{I},\mathcal{A}}(\phi) \cap v_{\mathcal{I},\mathcal{A}}(\psi)$;
4. $v_{\mathcal{I},\mathcal{A}}(\neg\phi) = C \setminus v_{\mathcal{I},\mathcal{A}}(\phi)$;
5. $v_{\mathcal{I},\mathcal{A}}(\forall x \phi) = \bigcap_{d \in \mathcal{D}} v_{\mathcal{I},[d/x]\mathcal{A}}(\phi)$;
6. $v_{\mathcal{I},\mathcal{A}}(\Box\phi) = \{c \mid v_{\mathcal{I},\mathcal{A}}(\phi) \in \mathcal{N}(c)\}$.

The frames defined by formulas and the logic defined by frames

❄ Let $M = \langle C, \mathcal{N}, \mathcal{D}, \mathcal{I} \rangle$ be a neighborhood model. For any $\phi \in \Phi$ and $c \in C$, we write $c \models_M \phi$, if $c \in v_{\mathcal{I},\mathcal{A}}(\phi)$ for any assignment $\mathcal{A}$. If $c \models_M \phi$ for every $c \in C$, we write $M \models \phi$.

❄ Let $Z = \langle C, \mathcal{N} \rangle$ be a neighborhood frame. We write $Z \models \phi$, if for any domain $\mathcal{D}$ and any interpretation $\mathcal{I}$, the neighborhood model $M = \langle C, \mathcal{N}, \mathcal{D}, \mathcal{I} \rangle$ satisfies $M \models \phi$. Let Γ be a set of formulas. If $Z \models \phi$ for any $\phi \in \Gamma$, we write $Z \models \Gamma$.

❄ Let $\mathcal{C}$ be a class of neighborhood frames. We write $\mathcal{C} \models \phi$ if $Z \models \phi$ for every $Z \in \mathcal{C}$, and write $\mathcal{C} \models \Gamma$ if $\mathcal{C} \models \phi$ for every $\phi \in \Gamma$.

Algebraic models

Definition

An *algebraic model* for predicate modal logics is a triple $\langle A, \mathcal{D}, \mathcal{J} \rangle$, where A is a complete modal algebra, $\mathcal{D}$ is a nonempty set, and $\mathcal{J}$ maps each n-ary predicate symbol to a mapping $P^{\mathcal{J}} : \mathcal{D}^n \to A$.

Let $\mathcal{A}$ be an assignment to $\mathcal{D}$. The function $u_{\mathcal{J},\mathcal{A}}$ from the set Φ of formulas to A is defined inductively as follows:

1. $u_{\mathcal{J},\mathcal{A}}(\top) = 1$, $u_{\mathcal{J},\mathcal{A}}(\bot) = 0$;
2. $u_{\mathcal{J},\mathcal{A}}(P(x_1, \ldots, x_n)) = P^{\mathcal{J}}(\mathcal{A}(x_1), \ldots, \mathcal{A}(x_n))$ for any $n \in \omega$, $P \in \mathsf{Pred}(n)$, and $x_1, \ldots, x_n \in \mathsf{V}$;
3. $u_{\mathcal{J},\mathcal{A}}(\phi \wedge \psi) = u_{\mathcal{J},\mathcal{A}}(\phi) \wedge u_{\mathcal{J},\mathcal{A}}(\psi)$;
4. $u_{\mathcal{J},\mathcal{A}}(\neg\phi) = -u_{\mathcal{J},\mathcal{A}}(\phi)$;
5. $u_{\mathcal{J},\mathcal{A}}(\forall x \phi) = \bigwedge_{d \in \mathcal{D}} u_{\mathcal{J},[d/x]\mathcal{A}}(\phi)$;
6. $u_{\mathcal{J},\mathcal{A}}(\Box\phi) = \Box u_{\mathcal{J},\mathcal{A}}(\phi)$.

Equivalence between neighborhood frames and their complex algebras

Lemma

Let $Z = \langle C, \mathcal{N} \rangle$ *be a neighborhood frame and* $A = \langle \mathsf{Alg}(Z), \mathcal{D}, \mathcal{J} \rangle$ *be an algebraic model. Let* $M^A = \langle C, \mathcal{N}, \mathcal{D}, \mathcal{I} \rangle$ *be a neighborhood model, where* $\mathcal{I}$ *is defined by*

$$c \in P^{\mathcal{J}}(d_1, \ldots, d_n) \;\Leftrightarrow\; (d_1, \ldots, d_n) \in P^{\mathcal{I}}(c) \tag{3.1}$$

for any n-ary predicate symbol P. *Then,* $v_{\mathcal{I},\mathcal{A}}(\phi) = u_{\mathcal{J},\mathcal{A}}(\phi)$ *for any formula* ϕ *and any assignment* $\mathcal{A}$.

Lemma

Let $Z = \langle C, \mathcal{N} \rangle$ *be a neighborhood frame and* $M = \langle C, \mathcal{N}, \mathcal{D}, \mathcal{I} \rangle$ *be a neighborhood model. Let* $A^M = \langle \mathsf{Alg}(Z), \mathcal{D}, \mathcal{J} \rangle$ *be an algebraic model, where* $\mathcal{J}$ *is defined by* (3.1). *Then,* $u_{\mathcal{J},\mathcal{A}}(\phi) = v_{\mathcal{I},\mathcal{A}}(\phi)$ *for any formula* ϕ *and any assignment* $\mathcal{A}$.

Equivalence between neighborhood frames and their complex algebras

Theorem

Let $Z = \langle C, \mathcal{N} \rangle$ *be a neighborhood frame. For any formula* ϕ*,*

$$Z \models \phi \;\Leftrightarrow\; \mathsf{Alg}(Z) \models \phi.$$

Model existence theorem

Predicate modal logics

Definition

A set L of formulas is a *predicate modal logic*, if it contains all classical predicate tautologies, is closed under the modus ponens, uniform substitution of formulas, and satisfies the following conditions:

1. for any $\phi \in \Phi$ and $x \in \mathsf{V}$, if $\phi \in \mathsf{L}$ then $\forall x \phi \in \mathsf{L}$;
2. for any ϕ and ψ in Φ, if $\phi \equiv \psi \in \mathsf{L}$, then $\Box\phi \equiv \Box\psi \in \mathsf{L}$.

A predicate modal logic L is said to be *monotonic*, *topped*, or *closed under finite intersections*, which are written by MT, TP, or CF in symbols, if $\Box(p \wedge q) \supset \Box p \wedge \Box q \in \mathsf{L}$, $\Box\top \in \mathsf{L}$, or $\Box p \wedge \Box q \supset \Box(p \wedge q) \in \mathsf{L}$, respectively. A predicate modal logic L is *normal* if it satisfies MT, TP, and CF.

Predicate modal logics

❄ A predicate modal logic L is *consistent* if $\bot \notin$ L.

❄ Let $\mathcal{C}$ be a class of neighborhood frames. A predicate modal logic L is *sound* with respect to $\mathcal{C}$ if $\phi \in$ L implies $\mathcal{C} \models \phi$ for every formula ϕ, and *complete* with respect to $\mathcal{C}$ if the converse holds.

❄ A predicate modal logic L is said to be *neighborhood complete* if there exists a class $\mathcal{C}$ of neighborhood frames such that L is sound and complete with respect to $\mathcal{C}$.

Predicate modal logics with ω-rules

A predicate logic L is said to *admit* a pair of the form

$$\alpha \supset \beta_i \ (i \in \omega), \qquad \frac{p \supset \beta_i \ (i \in \omega)}{p \supset \alpha}, \tag{4.1}$$

if it satisfies the following conditions:

1. $\alpha \supset \beta_i \in \mathsf{L}$ for any $i \in \omega$;
2. for each formula ϕ, $\phi \supset \alpha \in \mathsf{L}$ whenever $\phi \supset \beta_i \in \mathsf{L}$ for every $i \in \omega$.

We restrict our attention to pairs of the form (4.1) that satisfy the following condition:

$$|\{\{s(\beta_i) \mid i \in \omega\} \mid s \in \mathsf{Sub}\}| \leq \aleph_0,$$

where Sub is the set of substitutions of closed formulas into predicate symbols.

Model existence theorem

Theorem

Let L *be a consistent predicate modal logic that admits countably many pairs of a set of axiom schemata and an* ω*-rule. Then, there exists a neighborhood model* $M = \langle C, \mathcal{N}, \mathcal{D}, \mathcal{I} \rangle$ *and an assignment* $\mathcal{A}$ *such that*

$$v_{\mathcal{I},\mathcal{A}}(s(\alpha)) = \bigcap_{i \in \omega} v_{\mathcal{I},\mathcal{A}}(s(\beta_i))$$

for any uniform substitution s*, and*

$$\phi \in \mathsf{L} \;\Leftrightarrow\; M \models \phi.$$

for any closed formula $\phi \in \Phi$*. Moreover, if* L *satisfies properties among* MT*,* TP*, and* CF *then the model satisfies the properties corresponding to those of* L*.*

Corollary

If $Z = \langle C, \mathcal{N} \rangle$ *satisfies* $Z \models \mathsf{L}$, L *is complete with respect to a class of neighborhood frames with constant domains.*

Constant domain models for a predicate extension of GL

The proof system $\mathrm{PS}_{\mathsf{QGL}}$

Definition

The proof system $\mathrm{PS}_{\mathsf{QGL}}$ for the predicate GL consists of:

1. all tautologies of classical predicate logic;
2. $\Box(p \supset q) \supset (\Box p \supset \Box q)$ and $\Box p \supset \Box\Box p$;
3. modus ponens, uniform substitution rule, necessitation rule, and generalization rule;
4. $\dfrac{p \supset \Diamond^n \top \ (n \in \omega)}{p \supset \bot}$.

Define the logic QGL as the set of all formulas that are derivable in $\mathrm{PS}_{\mathsf{QGL}}$.

❄ QGL is sound and complete with respect to the class of conversely well-founded Kripke frames.

❄ Kripke complete predicate extension of GL is not computably enumerable (Rybakov 02).

GL-frames

Definition

A neighborhood frame $Z = \langle C, \mathcal{N} \rangle$ is called a *GL-frame* if $\mathsf{Alg}(Z)$ satisfies MT, TP, CF, as well as the following additional conditions:

1. $\Box_Z X \subseteq \Box_Z \Box_Z X$ for every $X \in \mathcal{P}(C)$;
2. $\bigcap_{n \in \omega} {\Diamond_Z}^n C = \emptyset$.

We write $\mathcal{C}_{\mathsf{GL}}$ for the class of all GL-frames.

Theorem

QGL *is sound and complete with respect to* $\mathcal{C}_{\mathsf{GL}}$.

Kripke incompleteness of a common knowledge logic

Common knowledge logics

Definition

A bi-modal logic L with two modal operators E and C is a *common knowledge logic* if it satisfies the following properties:

1. for any formula ϕ and any $n \in \omega$, $\mathsf{C}\phi \supset \mathsf{E}^n\phi \in \mathsf{L}$;
2. for any formulas ϕ and ψ, if $\psi \supset \mathsf{E}^n\phi \in \mathsf{L}$ for any $n \in \omega$, then $\psi \supset \mathsf{C}\phi \in \mathsf{L}$.

The proof system $\mathrm{PS}_{\mathsf{QCKL}}$

The proof system $\mathrm{PS}_{\mathsf{QCKL}}$ for the common knowledge logic consists of:

1. all tautologies of classical predicate logic;
2. $\Box(p \supset q) \supset (\Box p \supset \Box q)$, where $\Box = \mathsf{E}$ or $\Box = \mathsf{C}$;
3. for any $n \in \omega$, $\mathsf{C}p \supset \mathsf{E}^n p$;
4. $\forall x \Box \phi \supset \Box \forall x \phi$, where $\Box = \mathsf{E}$ or $\Box = \mathsf{C}$;
5. modus ponens, uniform substitution rule, necessitation rule for the modal operators E and C, and generalization rule;
6. for each $k \in \omega$ and $\{\Box_i \mid 1 \leq i \leq k\} \subseteq \{\mathsf{E}, \mathsf{C}\}$,

$$\frac{\gamma \supset \Box_1(\phi_1 \supset \Box_2(\phi_2 \supset \cdots \supset \Box_k(\phi_k \supset \mathsf{E}^n \phi) \cdots)) \quad (n \in \omega)}{\gamma \supset \Box_1(\phi_1 \supset \Box_2(\phi_2 \supset \cdots \supset \Box_k(\phi_k \supset \mathsf{C}\phi) \cdots))}.$$

Define the logic QCKL as the set of all formulas that are derivable in $\mathrm{PS}_{\mathsf{QCKL}}$.

The proof system $\mathrm{PS}_{\mathsf{QCKL}^-}$

❄ Kripke complete common knowledge logic is not computably enumerable (Wolter 00).

❄ When $k = 0$, the inference rule

$$\frac{\gamma \supset \Box_1(\phi_1 \supset \Box_2(\phi_2 \supset \cdots \supset \Box_k(\phi_k \supset \mathsf{E}^n\phi)\cdots)) \quad (n \in \omega)}{\gamma \supset \Box_1(\phi_1 \supset \Box_2(\phi_2 \supset \cdots \supset \Box_k(\phi_k \supset \mathsf{C}\phi)\cdots))} \tag{6.1}$$

means

$$\frac{\gamma \supset \mathsf{E}^n\phi \quad (n \in \omega)}{\gamma \supset \mathsf{C}\phi}. \tag{6.2}$$

❄ The proof system $\mathrm{PS}_{\mathsf{QCKL}^-}$ is defined by replacing inference rule (6.1) of $\mathrm{PS}_{\mathsf{QCKL}}$ with (6.2), and by removing the Barcan Formula.

CKL^--algebras and CKL^--frames

Definition

A complete modal algebra A with two modal operators E and C is called a *CKL⁻-algebra*, if it satisfies MT, TP, CF, and

$$\mathsf{C}x = \bigwedge_{n \in \omega} \mathsf{E}^n x$$

for any $x \in A$. We write $\mathcal{A}_{\mathsf{CKL}^-}$ for the class of all CKL^--algebras.

Definition

A neighborhood frame $Z = \langle C, \mathcal{N}_{\mathsf{E}}, \mathcal{N}_{\mathsf{C}} \rangle$ is called a *CKL⁻-frame* if $\mathsf{Alg}(Z) \in \mathcal{A}_{\mathsf{CKL}^-}$. We write $\mathcal{C}_{\mathsf{CKL}^-}$ for the class of all CKL^--frames.

Theorem

$\mathbf{QCKL}^-$ *is sound and complete with respect to both* $\mathcal{A}_{\mathsf{CKL}^-}$ *and* $\mathcal{C}_{\mathsf{CKL}^-}$.

Kripke incompleteness of $\mathbf{QCKL}^-$

Lemma

There exists a modal algebra $A \in \mathcal{A}_{\mathsf{CKL}^-}$ and $x \in A$ such that

$$\mathsf{C}x \not\leq \mathsf{EC}x.$$

If $A \in \mathcal{A}_{\mathsf{CKL}^-}$ is completely multiplicative, then, for any $x \in A$,

$$\mathsf{C}x = \bigwedge_{n \in \omega} \mathsf{E}^n x \leq \mathsf{E} \bigwedge_{n \in \omega} \mathsf{E}^n x = \mathsf{EC}x.$$

Lemma

Let Z be a Kripke frame. If $Z \models \mathbf{QCKL}^-$ then:

1. $\mathsf{C}X = \bigcap_{n \in \omega} \mathsf{E}^n X$ *for any* $X \in \mathsf{Alg}(Z)$;
2. $Z \models \mathsf{C}p \supset \mathsf{EC}p$.

Theorem

$\mathbf{QCKL}^-$ *is Kripke incomplete.*

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Thank you for your attention.