Hierarchy of logical axioms over intuitionistic logic and arithmetic

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Arithmetical Hierarchy of Logical Principles

- There are several logical principles that are valid in classical logic but not provable in intuitionistic logic.
- Such axioms include
  LEM (Law of Excluded Middle): $\varphi \vee \neg\varphi$;
  DML (De Morgan's Law): $\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$;
  DNE (Double Negation Elimination): $\neg\neg\varphi \to \varphi$;
  WLEM (Weak LEM): $\neg\varphi \vee \neg\neg\varphi$;
  WDML (Weak DML): $\neg(\neg\varphi \wedge \neg\psi) \to \neg\neg\varphi \vee \neg\neg\psi$.

A formula φ is called Σ^0_n if φ is of the form $\exists x_1 \forall x_2 \cdots Q x_n \varphi_{\mathrm{qf}}$, where x_i are number variables. The fragments of the logical principles restricted to Σ^0_n-formulas have the following hierarchy:

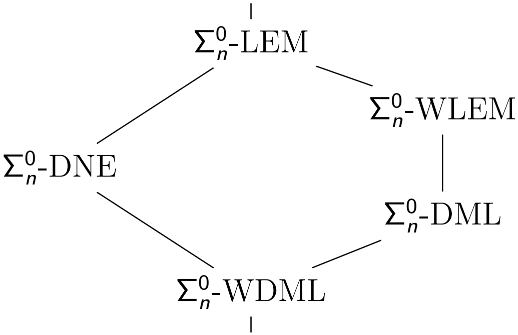
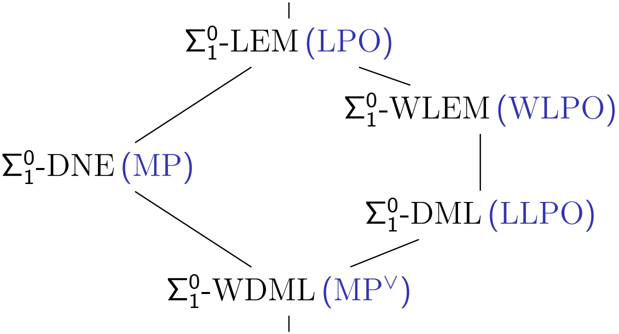


Figure: Arithmetical hierarchy of classical principles over HA + Σ^0_{n-1}-DNE (Akama et al. 2004, Fujiwara and Kurahashi 2022)

In constructive mathematics, the Σ_1^0-variants of the logical principles are known as LPO (limited principle of omniscience), LLPO (lesser limited principle of omniscience), WLPO (weak limited principle of omniscience), MP (Markov's principle), and $\mathrm{MP}^{\vee}$ (disjunctive Markov's principle) respectively.



They have been a driving force for developing constructive reverse mathematics, where we seek to determine axioms that are necessary and sufficient to prove each math. theorem.

- In fact, such logical principles has its roots in Brouwer's intuitionistic mathematics, in which only constructive reasonings are accepted entirely in the proofs.
- In analogy with a "counterexample" which shows a statement is false, Brouwer constructed so-called a " weak counterexample " for some mathematical statements by proving (constructively) that the statement implies $\mathrm{LPO}/\mathrm{LLPO}$, which should not be accepted in his intuitionistic standpoint.

Separations

- By proof interpretations with respect to finite-type arithmetic, one can obtain a lot of separation results between the Σ^0_1-fragments of the logical principles over $\mathsf{HA}^\omega + \mathrm{AC}$.
- However, weak logical principles are sound for all the proof interpretations, and hence, they cannot be separated by those methods.
- A technique on propositional Kripke models, which was invented by Ishihara-Nemoto-Suzuki-Yokoyama-F. 2023, works quite well for separating the logical principles.
- The purpose of this talk is to introduce the idea behind this technique and to present the induced hierarchy of Σ-variants of the logical principles over predicate logic.

Hierarchy as Intermediate Propositional Logics

- Propositional connectives: $\bot, \wedge, \vee, \to$ $(\neg\varphi := \varphi \to \bot)$
- In addition to the before-mentioned logical principles, let us consider Δ_1^0-variant of LEM

$$\mathrm{RLEM}: (\varphi \leftrightarrow \neg\psi) \to \varphi \vee \neg\varphi$$

and the following variations of linearity axiom:

$$\mathrm{LIN}_1: (\varphi \to \psi) \vee (\psi \to \varphi);$$
$$\mathrm{LIN}_2: (\varphi \to \neg\psi) \vee (\neg\psi \to \varphi).$$
$$\mathrm{LIN}_3: (\neg\varphi \to \neg\psi) \vee (\neg\psi \to \neg\varphi);$$
$$\mathrm{LIN}_4: (\neg\varphi \to \neg\neg\psi) \vee (\neg\neg\psi \to \neg\varphi);$$
$$\mathrm{LIN}_5: (\neg\neg\varphi \to \neg\neg\psi) \vee (\neg\neg\psi \to \neg\neg\varphi);$$
$$\mathrm{LIN}_6: (\varphi \to \neg\neg\psi) \vee (\neg\neg\psi \to \varphi);$$
$$\mathrm{LIN}_7: (\varphi \to \psi) \vee (\neg\neg\psi \to \varphi);$$
$$\mathrm{LIN}_8: (\neg\neg\varphi \to \psi) \vee (\neg\neg\psi \to \varphi).$$

Derivations and Substitutions

A set **L** of propositional formulae s.t. IPC $\subseteq$ **L** $\subseteq$ CPC is called **intermediate propositional logic** if the following hold:

1. if $\varphi \to \psi$ and φ are in **L**, then ψ is in **L**;
2. if φ is in **L**, then any substitution instance of φ is in **L**.

Fact. (Hierarchy of Intermediate Propositional Logics)

$$\begin{gathered}\mathrm{LEM} = \mathrm{DNE} = \mathrm{LIN}_8 \\ \cup \\ \mathrm{LIN}_7 \\ \cup \\ \mathrm{LIN}_1 \\ \cup \\ \mathrm{WLEM} = \mathrm{RLEM} = \mathrm{DML} = \mathrm{WDML} \\ = \mathrm{LIN}_2 = \mathrm{LIN}_3 = \mathrm{LIN}_4 = \mathrm{LIN}_5 = \mathrm{LIN}_6.\end{gathered}$$

Σ_n^0-hierarchy over HA + Σ_{n-1}^0-DNE

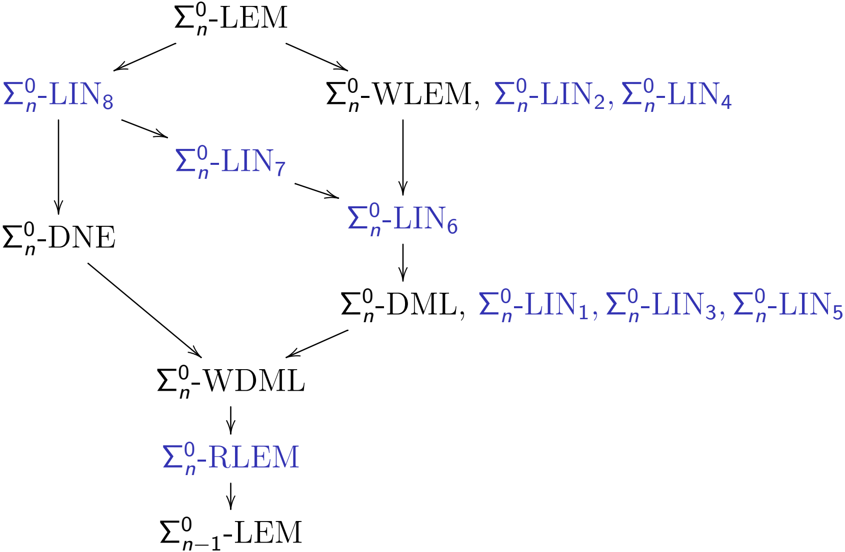
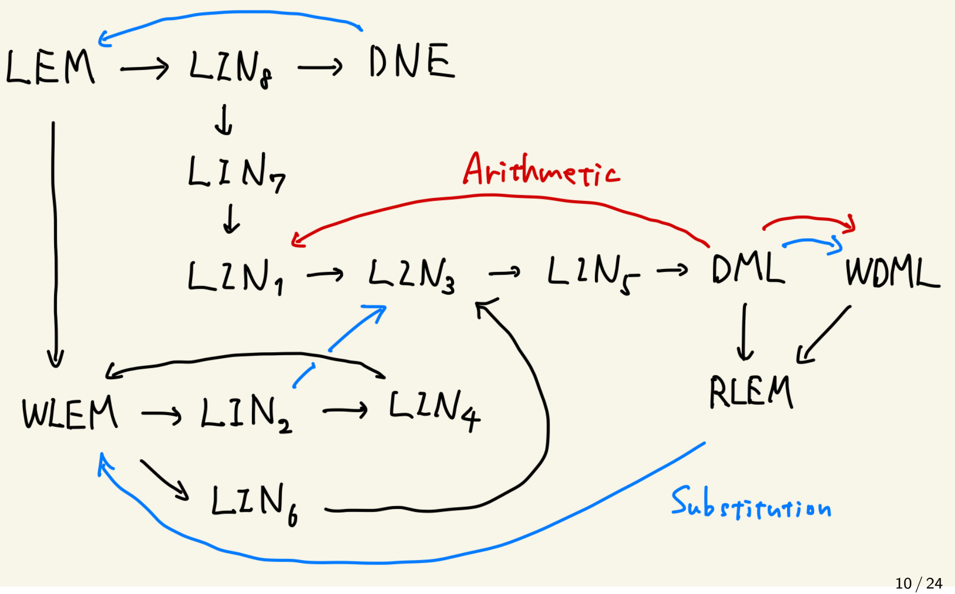


Figure: Refined hierarchy of the logical principles over HA + Σ_{n-1}^0-DNE

What Happens ?



$\mathsf{HA} + \Sigma^0_n\text{-DML} + \Sigma^0_{n-1}\text{-DNE} \vdash \Sigma^0_n\text{-LIN}_1$.

Proof. Fix Σ^0_n-formulae φ_1 and φ_2. W.l.o.g, assume $n > 0$. We show $(\varphi_1 \to \varphi_2) \lor (\varphi_2 \to \varphi_1)$ within $\mathsf{HA} + \Sigma^0_n\text{-DML} + \Sigma^0_{n-1}\text{-DNE}$. Let φ_1 and φ_2 be $\exists x \varphi_1'(x)$ and $\exists x \varphi_2'(x)$ where $\varphi_1'(x)$ and $\varphi_2'(x)$ are Π^0_{n-1}-formulae respectively. Consider the following formulae:

$$\psi_1(x) \equiv \quad \varphi_1'(x) \land \forall y \leq x \neg \varphi_2'(y);$$
$$\psi_2(x) \equiv \quad \varphi_2'(x) \land \forall y \leq x \neg \varphi_1'(y).$$

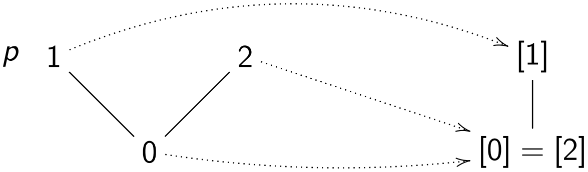
Then we have $\mathsf{HA} \vdash \neg(\exists x \psi_1(x) \land \exists x \psi_2(x))$ trivially. Since $\neg\varphi_2'(y)$ and $\neg\varphi_1'(y)$ are equivalent to some Σ^0_{n-1}-formulae respectively in the presence of $\Sigma^0_{n-1}\text{-DNE}$, we have that$\forall y \leq x \neg \varphi_2'(y)$ and $\forall y \leq x \neg\varphi_1'(y)$ are equivalent to some Σ^0_{n-1}-formulae respectively. Therefore we have that $\exists x \psi_1(x)$ and $\exists x \psi_2(x)$ are equivalent to some Σ^0_n-formulae respectively in our theory. Applying $\Sigma^0_n\text{-DML}$, we have $\neg\exists x \psi_1(x) \lor \neg \exists x \psi_2(x)$. In the former case, if $\varphi_1'(x)$, then we have $\neg \forall y \leq x \neg \varphi_2'(y)$, equivalently, $\neg\neg\exists y \leq x \varphi_2'(y)$. Then we have $\exists y \leq x \varphi_2'(y)$ by using $\Sigma^0_{n-1}\text{-DML}$ and $\Sigma_{n-2}\text{-DNE}$. Thus we have shown $\exists x \varphi_1'(x) \to \exists x \varphi_2'(x)$. In the latter case, we have $\exists x \varphi_2'(x) \to \exists x \varphi_1'(x)$ similarly. □

Meta-theorem 1. (Ishihara-Nemoto-Suzuki-Yokoyama-F. 2023)

Let $\mathcal{K} = (K, \leq, \Vdash)$ be a finite IPC-Kripke model s.t. the induced frame $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ is a rooted tree and the induced extended frame $\mathcal{E}_{\mathcal{K}}$ is locally directed. If $\mathcal{K} \nVdash \varphi$, then for all n,

$$\mathsf{HA} + \Sigma^0_{n-1}\text{-}\mathrm{LEM} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}}) \nvdash \Sigma^0_n\text{-}\varphi.$$

A crucial idea underlying this meta-theorem is to restrict possible evaluations on the Kripke frame by using the **extended frame** generated by a given Kripke model.



$\mathcal{K}_1$ $(I_{\mathcal{K}_1}, \leq_{\mathcal{K}_1})$

$[k] := \{k' \in K_1 \mid k \in U \leftrightarrow k' \in U$ for any evaluation set U of $\mathcal{K}_1\}$.

Definition (Ishihara-Nemoto-Suzuki-Yokoyama-F. 2023)

An extended frame $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ is a triple of frames $(K, \leq)$ and $(I, \leq_I)$, and a monotone mapping f between them, that is, $k \leq k'$ implies $f(k) \leq_I f(k')$ for each $k, k' \in K$.

- Each IPC-Kripke model $\mathcal{I} = (I, \leq_I, \Vdash)$ induces an IPC-Kripke model $\mathcal{K}_{\mathcal{E},\mathcal{I}} = (K, \leq, \Vdash_{\mathcal{E},\mathcal{I}})$ by defining

  $$k \Vdash_{\mathcal{E},\mathcal{I}} p :\Leftrightarrow f(k) \Vdash p$$

  for each $k \in K$ and propositional variable p.
- A propositional formula φ is valid on $\mathcal{E}$ if $\mathcal{K}_{\mathcal{E},\mathcal{I}} \Vdash_{\mathcal{E},\mathcal{I}} \varphi$ for each IPC-Kripke model $\mathcal{I} = (I, \leq_I, \Vdash)$, that is, for each valuation $\Vdash$ on $(I, \leq_I)$; we then write $\mathcal{E} \models \varphi$.
- For an extended frame $\mathcal{E}$, define $T(\mathcal{E}) := \{\varphi \mid \mathcal{E} \models \varphi\}$.

Remark

- For a frame $(K, \leq)$, the set

$$L(K, \leq) = \{\varphi \mid (K, \leq) \models \varphi\}$$

  of propositional formulae is an intermediate propositional logic.
- In contrast, for an extended frame $\mathcal{E}$, $T(\mathcal{E})$ is not an intermediate propositional logic in general. In particular, $T(\mathcal{E})$ may not be closed under substitution.

- Let $\mathcal{K} = (K, \leq, \Vdash)$ be an IPC-Kripke model, and define a set $\Phi_{\mathcal{K}}$ of upward closed subsets of K by

$$\Phi_{\mathcal{K}} := \{\{k \in K \mid k \Vdash p\} \mid p \in \mathcal{V}\}.$$

- Define binary relations $\preceq_{\mathcal{K}}$ and $\sim_{\mathcal{K}}$ on K by

$$k \preceq_{\mathcal{K}} k' :\Leftrightarrow k \in U \text{ implies } k' \in U \text{ for all } U \in \Phi_{\mathcal{K}},$$
$$k \sim_{\mathcal{K}} k' :\Leftrightarrow k \preceq_{\mathcal{K}} k' \text{ and } k' \preceq_{\mathcal{K}} k.$$

  Then $\preceq_{\mathcal{K}}$ is a preorder and $\sim_{\mathcal{K}}$ is an equivalence relation on K.

- Let $I_{\mathcal{K}} := K/\sim_{\mathcal{K}}$, $[k] \leq_{\mathcal{K}} [k'] :\Leftrightarrow k \preceq_{\mathcal{K}} k'$, and $f_{\mathcal{K}}(k) := [k]$, where $[k]$ is the equivalence class of k with respect to $\sim_{\mathcal{K}}$.
  Then $\mathcal{E}_{\mathcal{K}} := ((K, \leq), f_{\mathcal{K}}, (I_{\mathcal{K}}, \leq_{\mathcal{K}}))$ is an extended frame, and we call it the extended frame generated by the IPC-Kripke model $\mathcal{K}$.

Definition

- For a propositional formula $\varphi[p_1, \ldots, p_m]$, Σ^0_n-φ denotes a schema $\varphi[\chi_1/p_1, \ldots, \chi_m/p_m]$, where $\chi_1, \ldots, \chi_m$ are Σ^0_n-formulae of HA, and Σ-φ denotes the following schema of HA :

$$\forall x(\psi_1(x) \vee \neg\psi_1(x)) \wedge \ldots \wedge \forall x(\psi_m(x) \vee \neg\psi_m(x))$$
$$\to \varphi[\exists x \psi_1(x)/p_1, \ldots, \exists x \psi_m(x)/p_m].$$

- For an extended frame $\mathcal{E}$, Σ-$T(\mathcal{E})$ is the schema (of HA) consisting of Σ-φ where $\varphi \in T(\mathcal{E})$.
- For $k \in K$, let $\uparrow k$ denote $\{k' \in K \mid k \leq k'\}$.
- An extended frame $\mathcal{E}$ is locally directed if $f^{-1}(\uparrow i) \cap \uparrow k$ is directed for all $i \in I$ and $k \in K$, that is, for each $i \in I$ and $k \in K$, if $l, l' \in f^{-1}(\uparrow i) \cap \uparrow k$, then there exists $l'' \in f^{-1}(\uparrow i) \cap \uparrow k$ such that $l'' \leq l$ and $l'' \leq l'$.

Meta-theorem 1. (revisited)

Let $\mathcal{K} = (K, \leq, \Vdash)$ be a finite IPC-Kripke model s.t. the induced frame $(I_\mathcal{K}, \leq_\mathcal{K})$ is a rooted tree and the induced extended frame $\mathcal{E}_\mathcal{K}$ is locally directed. If $\mathcal{K} \nVdash \varphi$, then for all n,

$$\mathsf{HA} + \Sigma^0_{n-1}\text{-LEM} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_\mathcal{K}) \nvdash \Sigma^0_n\text{-}\varphi,$$

where $L(K, \leq)^*$ is the set of schemata of $\varphi[\psi_1/p_1, \ldots, \psi_m/p_m]$ for propositional formulae $\varphi[p_1, \ldots, p_m] \in L(K, \leq)$.

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$$\mathsf{HA} + \Sigma_{n-1}^0\text{-}\mathrm{LEM} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_\mathcal{K}) \nvdash \Sigma_n^0\text{-}\varphi,$$

where $L(K, \leq)^*$ is the set of schemata of $\varphi[\psi_1/p_1, \ldots, \psi_m/p_m]$ for propositional formulae $\varphi[p_1, \ldots, p_m] \in L(K, \leq)$.

Corollary. (De Jongh's theorem)

If $\varphi[p_1 \ldots, p_m] \notin \mathsf{IPC}$, then $\mathsf{HA} \nvdash \varphi[\chi_1/p_1, \ldots, \chi_m/p_m]$ for some Σ_1^0-formulae $\chi_1, \ldots, \chi_m$ of HA.

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Observation.

The Σ^0_n-substitution instances of our logical principles can be separated uniformly by the technique.

Σ-hierarchy over IQC

- Recall that Σ-φ denotes the following schema:

$$\forall x(\psi_1(x) \vee \neg\psi_1(x)) \wedge \ldots \wedge \forall x(\psi_m(x) \vee \neg\psi_m(x))$$
$$\to \varphi[\exists x\psi_1(x)/p_1, \ldots, \exists x\psi_m(x)/p_m].$$

- In the presence of

$$\mathrm{AC}^{0,0} : \forall x\exists y\psi(x,y) \to \exists f\forall x\psi(x,f(x)),$$

Σ-φ is equivalent to Σ^0_n-φ with function parameters for all natural number n.
- The principle $\mathrm{AC}^{0,0}$ is constructively acceptable (e.g., in the Martin-Löf type theory).
- On the other hand, Σ-φ is strictly stronger than Σ^0_n-φ in the context of HA (without function parameters).
For example, HA + Σ^0_n-DNE does not prove

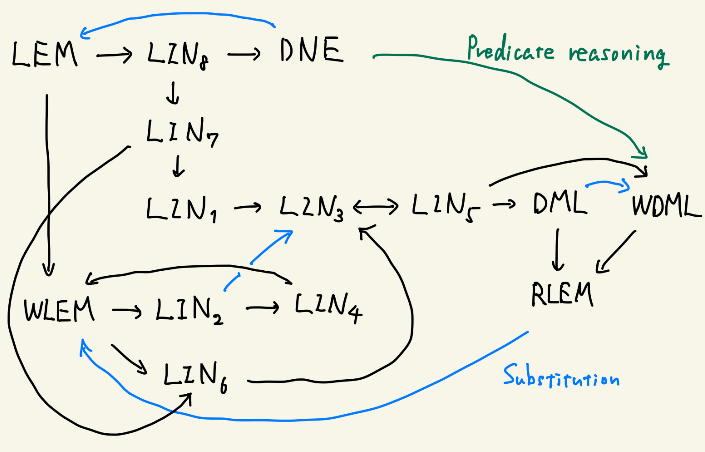
$$\forall x(\psi(x) \vee \neg\psi(x)) \to (\neg\neg\exists x\psi(x) \to \exists x\psi(x)).$$

Remark.

For each logical principles φ_1 and φ_2, we have

$$\varphi_1 \to_{\mathrm{IPC}} \varphi_2 \implies \Sigma\text{-}\varphi_1 \to_{\mathrm{IQC}} \Sigma\text{-}\varphi_2 \implies \Sigma\text{-}\varphi_1 \to_{\mathrm{HA}} \Sigma\text{-}\varphi_2,$$

while the converse of the latter implication is not always the case.



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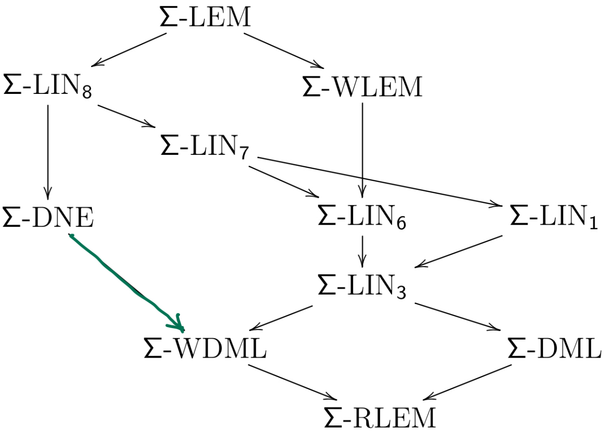

Remark.

No implication hold whenever it does not follow by transitivity from the displayed implications.

All the separations of our Σ-variants can be established by using the following meta-theorem:

Meta-theorem 2. (Ishihara-Nemoto-Suzuki-Yokoyama-F. 2023)

Let $\mathcal{K} = (K, \leq, \Vdash)$ be a finite IPC-Kripke model s.t. the induced extended frame $\mathcal{E}_\mathcal{K}$ is locally directed. If $\mathcal{K} \nVdash \varphi$, then

$$\mathrm{IQC} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_\mathcal{K}) \nvdash \Sigma\text{-}\varphi,$$

where $L(K, \leq)^*$ is the set of schemata of $\varphi[\psi_1/p_1, \ldots, \psi_m/p_m]$ for propositional formulae $\varphi[p_1, \ldots, p_m] \in L(K, \leq)$.

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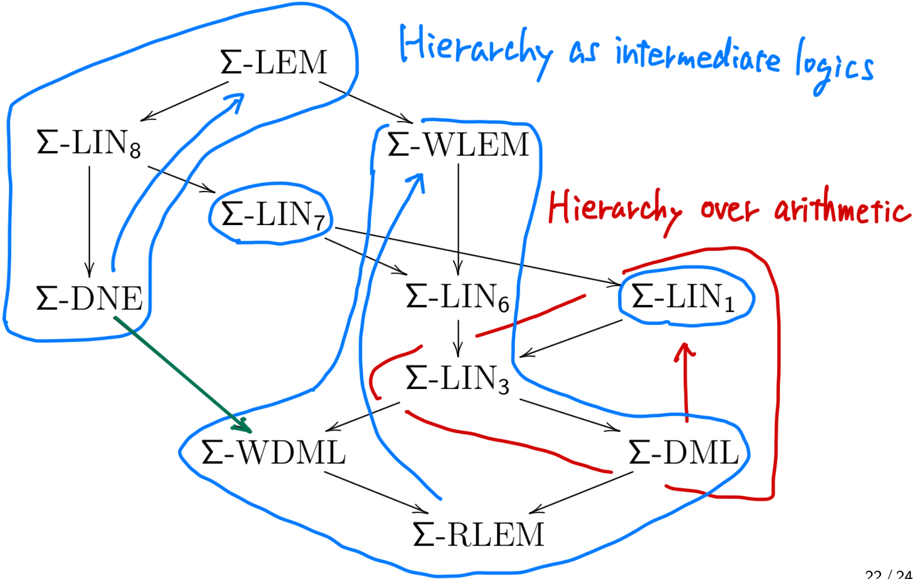
From the Viewpoint of the Σ-hierarchy



Possible Future Works

1. Exploring the relation between propositional intermediate theories and first-order intermediate logics obtained by the Σ-variants of the logical principles (Ongoing with Tenyo Takahashi).
2. Exploring the hierarchy of the logical principles in the framework of type theories (modern framework of constructive mathematics).
3. Inventing a solver for the separation of the logical principles by using propositional Kripke models (of the form of a finite tree).

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