Uniform counting of quadratic fields with local conditions

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Abstract

We sharpen the uniform counting estimates of Bhargava, Thorne, and the author for quadratic fields with prescribed local conditions. Combining the Burgess and Pólya–Vinogradov bounds, we improve the dependence of the error term on the primes supporting the local conditions and make explicit the factors previously absorbed into an arbitrary ϵ-power of their product. The resulting estimate has an absolute implied constant.

1 Introduction

For each prime p, let A_p denote the set of isomorphism classes of étale quadratic $\mathbb{Q}_p$-algebras. Let n be a positive squarefree integer. A family of nonempty proper subsets σ_p of A_p, one for each prime $p \mid n$, is called a *local specification* for quadratic fields and is denoted by $\sigma = (\sigma_p)_{p|n}$. We call n the *support* of σ. A quadratic field F is said to satisfy σ if $F \otimes \mathbb{Q}_p \in \sigma_p$ for every $p \mid n$. Write $\mathcal{F}(\sigma)$ for the set of isomorphism classes of quadratic fields satisfying σ, and, for each sign $\pm$ and $X > 0$, put

$$N_2^{\pm}(X, \sigma) := \#\{F \in \mathcal{F}(\sigma) \mid 0 < \pm \mathrm{Disc}(F) < X\}.$$

Our main theorem gives an estimate for this counting function that is uniform in X and σ. We call σ *pure* if, for every odd prime $p \mid n$, σ_p does not contain both an unramified algebra and a ramified algebra. In this section, we state the main theorem for pure local specifications, which are particularly important in applications. For general local specifications, see Corollary 6 in Section 4.

To state the theorem, we define several quantities associated with a pure local specification σ. For $i = 1, 2$, let u_i be the product of the odd primes $p \mid n$ for which σ_p consists of i unramified algebras, and let r_i be the product of the odd primes $p \mid n$ for which σ_p consists of i ramified algebras. Thus $n = u_1u_2r_1r_2$ if n is odd, and $n = 2u_1u_2r_1r_2$ if n is even. For each odd prime $p \mid n$, define the local density $c_p(\sigma_p)$ by $c_p(\sigma_p) := ip/(2(p+1))$ if $p \mid u_i$, and by $c_p(\sigma_p) := i/(2(p+1))$ if $p \mid r_i$ $(i = 1, 2)$. If $2 \mid n$, assign the local density $1/(3 \cdot 2^e)$ to each étale quadratic $\mathbb{Q}_2$-algebra whose discriminant has 2-adic valuation e, and define $c_2(\sigma_2)$ as the sum of these densities over the elements of σ_2.

For a positive squarefree integer q, let $\omega(q)$ denote the number of distinct prime divisors of q, and put $\lambda_2(q) := \prod_{p|q}(1 + p^{-1/2})$. Our main result is as follows.

Theorem 1. *For every pure local specification σ and $X > 0$, we have*

$$N_2^{\pm}(X, \sigma) = \frac{3}{\pi^2} X \prod_{p|n} c_p(\sigma_p) + O\left(\frac{X^{1/2}\, u_1^{3/16}}{r_1^{5/16} r_2^{1/2}} \left(\frac{3}{2}\right)^{\omega(u_1r_1)} (\log(2u_1r_1))^{5/4} \lambda_2(u_2r_2) \right). \tag{1}$$

The implied constant is absolute.

When $n = 1$, so that no local conditions are imposed, the count reduces to Gegenbauer's count of squarefree integers [7]. For a fixed local specification σ, one can count squarefree integers in residue classes as in [7], obtaining (1) with an error term $O_\sigma(X^{1/2})$. Our aim is to obtain an error bound with more precise dependence on σ.

The dependence of counting estimates on local conditions has also been studied extensively. For counts with coprimality conditions, corresponding to the case $u_1 = r_1 = r_2 = 1$ here, Cohen [3,

Lemma 5.2] gave a uniform estimate, and Suryanarayana [15, Corollary 1] and Robert and Tenenbaum [12, (10.1)] refined the dependence of the error term on the modulus. Theorem 1 further improves that dependence in this case. For counting squarefree integers in residue classes, Cohen and Robinson [4, Corollary 1] and Schwarz [13, Lemma 8] obtained estimates uniform in the modulus and the residue class. Bhargava, Thorne, and the author [1, Proposition 4.2] treated arbitrary σ, obtaining, for pure σ, the bound

$$O_\epsilon\Big(X^{1/2}\, u_1^{1/4} u_2^{1/4} r_1^{-1/4} r_2^{-1/2} (u_1 u_2 r_1 r_2)^\epsilon\Big) \tag{2}$$

for the error term in (1), where $\epsilon > 0$ is arbitrary and the implied constant depends only on ϵ. This estimate permits the specification of individual ramified local extensions while retaining the saving $r_1^{-1/4} r_2^{-1/2}$ from the ramified primes. Our estimate (1) improves the power factors $u_1^{1/4} u_2^{1/4} r_1^{-1/4}$ to $u_1^{3/16} r_1^{-5/16}$ and makes explicit the factors absorbed into the ϵ-power of the support in (2).

The key to the proof of Theorem 1 is an estimate for the character sums over squarefree integers in (9). Expressing the indicator of the local condition at each prime $p \mid u_1 r_1$ in terms of quadratic characters and expanding the product leads to such sums. For sums of this type, Granville and Soundararajan [8, Lemma 4.1] obtained a bound using the Pólya–Vinogradov inequality, while Munsch [10, Lemma 2.3] obtained one using Burgess's inequality [2]. After Möbius inversion, we estimate the inner character sums in (11) by the Pólya–Vinogradov inequality or the Burgess bound of Jain-Sharma, Khale, and Liu [9, Theorem 1.1], according to the length of the summation interval. We track the dependence on the local conditions through explicit prime-by-prime factors.

Uniform estimates for counting cubic fields with local conditions, due to Thorne and the author [16] and Bhargava, Thorne, and the author [1], have found applications to various problems in arithmetic statistics. Such arguments sometimes require counts of lower-degree fields with local conditions to isolate the contributions of reducible étale algebras. For example, in [1, §4], counts of quadratic fields with local conditions are used to estimate the contribution of reducible cubic étale algebras $\mathbb{Q} \times F$, where $F/\mathbb{Q}$ is quadratic. Shankar and Tsimerman [14, §3.1] likewise use such counts to estimate the contribution of reducible quartic étale algebras in counting quartic fields. Our result sharpens the uniform estimates for the quadratic-field counts used in these arguments. Other applications of counts of quadratic fields with local conditions include Pollack's [11, §5.2] study of the average least inert prime in quadratic fields and bounds on ℓ-torsion in class groups obtained by Ellenberg, Pierce, and Wood [6, §6]. In the latter work, the quadratic field counts serve as input to a sieve. We expect Theorem 1 and its generalization, Corollary 6, to find further applications in arithmetic statistics.

The paper is organized as follows. In Section 2, we combine the Burgess bound and the Pólya–Vinogradov inequality to obtain a uniform bound for a sum of absolute values of character sums. In Section 3, we use this bound and Möbius inversion to derive a uniform counting formula for squarefree integers with prescribed congruence conditions and Legendre-symbol values. In Section 4, we reduce the counting of quadratic-field discriminants to the problem studied in Section 3 and prove Theorem 1. We also deduce Corollary 6 for general local specifications. Unless otherwise stated, the implied constants in Landau's O-notation and Vinogradov's $\ll$-notation are absolute.

Acknowledgments

This work was supported by JSPS KAKENHI Grant Numbers JP23K22386 and JP24K21512, and by the Research Institute for Mathematical Sciences (RIMS), an International Joint Usage/Research Center at Kyoto University. Part of this work was carried out during the author's participation in the Thematic Programme on Arithmetic Statistics at the Lodha Mathematical Sciences Institute. The author is deeply grateful to the organizers and staff of the programme for providing a truly exceptional research environment.

Note on translation and AI assistance

This paper was originally written in Japanese, the author's native language. The author subsequently translated it into English with the assistance of ChatGPT (GPT-6) in October and November 2026. During this process, the author also consulted ChatGPT on mathematical notation and terminology. The author has carefully reviewed and revised the entire translation and takes full responsibility for the content of the paper. Both the Japanese and English versions are available on the author's website.

2 Character sum estimates

Throughout this section, χ denotes a nontrivial primitive Dirichlet character of conductor q. For a nonnegative integer N and a real number b, define the N-term character sum

$$S(\chi, N; b) := \sum_{b<j\le b+N} \chi(j).$$

For a positive integer q, write $\omega(q)$ for the number of distinct prime factors of q, $\tau(q)$ for the number of positive divisors of q, and $\varphi(q)$ for the order of $(\mathbb{Z}/q\mathbb{Z})^\times$. Define

$$\kappa(q) := q^{3/16}(2^{\omega(q)}\tau(q))^{3/4}\left(\frac{q}{\varphi(q)}\right)^{1/2}. \tag{3}$$

Since $2^\omega, \tau, \varphi$ are multiplicative, so is κ. We use the following explicit form of Burgess's bound [2] for character sums, due to Jain-Sharma, Khale, and Liu [9, Theorem 1.1].

Lemma 2 (Burgess bound [9])**.** *For all $N \le q^{5/8}$, we have*

$$|S(\chi, N; b)| \ll \sqrt{N}\,\kappa(q)(\log q)^{1/4}. \tag{4}$$

The assumption in [9, Theorem 1.1] that the conductor is sufficiently large can be removed by enlarging the implied constant. The next lemma is the familiar Pólya–Vinogradov inequality; see, for example, Davenport [5, §23] for a proof.

Lemma 3 (Pólya–Vinogradov)**.** *For all χ, N, b, we have*

$$|S(\chi, N; b)| \ll \sqrt{q}\,\log q.$$

Combining the Burgess and Pólya–Vinogradov bounds, we obtain the following proposition.

Proposition 4. *Let $Y > 0$. Suppose that, for each integer $a \ge 1$, N_a is a nonnegative integer satisfying $N_a \le Y/a^2$. Then*

$$\sum_{a\ge1} |S(\chi, N_a; b_a)| \ll \sqrt{Y}\,\kappa(q)(\log q)^{5/4}, \tag{5}$$

where the b_a are arbitrary real numbers.

Proof. Put $A = \sqrt{Y}\,q^{-5/16}$. For $a \le A$, Lemma 3 gives

$$\sum_{a\le A} |S(\chi, N_a; b_a)| \ll A\sqrt{q}\,\log q = \sqrt{Y}\,q^{3/16}\log q \ll \sqrt{Y}\,\kappa(q)(\log q)^{5/4}.$$

If $a > A$, then $N_a \le Y/A^2 = q^{5/8}$. If $N_a > 0$, then $N_a \ge 1$ by integrality, and hence $a \le \sqrt{Y}$. Thus Lemma 2 gives

$$\sum_{a>A} |S(\chi, N_a; b_a)| \ll \sum_{A<a\le\sqrt{Y}} \sqrt{N_a}\,\kappa(q)(\log q)^{1/4} \le \sqrt{Y}\,\kappa(q)(\log q)^{1/4} \sum_{A<a\le\sqrt{Y}} \frac{1}{a}.$$

The last sum is bounded by

$$\sum_{A<a\le\sqrt{Y}} \frac{1}{a} \le 1 + \log\frac{\sqrt{Y}}{A} = 1 + \frac{5}{16}\log q.$$

The bounds for $a \le A$ and $a > A$ together prove the proposition. □

3 Counting squarefree integers

Let n_1, n_2 be coprime positive odd squarefree integers, and let $c \in (\mathbb{Z}/8\mathbb{Z})^\times$. For each $p \mid n_1$, choose a sign $\varepsilon(p) \in \{+1, -1\}$, and write $\varepsilon = (\varepsilon(p))_{p\mid n_1}$. For $Y > 0$, let $T(Y; c, n_1, n_2, \varepsilon)$ be the number of integers $0 < d < Y$ satisfying the following conditions:

$$d \text{ is squarefree;} \quad d \equiv c \pmod 8; \quad (d, n_1 n_2) = 1; \quad \left(\frac{d}{p}\right) = \varepsilon(p) \text{ for every } p \mid n_1.$$

By definition, this counting function is

$$T(Y; c, n_1, n_2, \varepsilon) = \sum_{\substack{0<d<Y \\ d\equiv c \ (8) \\ (d, n_1 n_2)=1}} \mu(d)^2 \prod_{p\mid n_1} \frac{1 + \varepsilon(p)\left(\frac{d}{p}\right)}{2}. \tag{6}$$

Here μ is the Möbius function. Define a multiplicative function λ_1 on the positive squarefree integers by prescribing, for each prime p,

$$\lambda_1(p) := \frac{\kappa(p) + \lambda_2(p)}{2} = \frac{\sqrt{2}}{(1-p^{-1})^{1/2}}\, p^{3/16} + \frac{1 + p^{-1/2}}{2}. \tag{7}$$

We prove the following uniform estimate.

Proposition 5.

$$T(Y; c, n_1, n_2, \varepsilon) = \frac{Y}{2^{\omega(n_1)}\pi^2} \prod_{p\mid n_1 n_2} \frac{p}{p+1} + O\Big(\sqrt{Y}\,(\log(2n_1))^{5/4} \lambda_1(n_1)\lambda_2(n_2)\Big). \tag{8}$$

Proof. Extend ε to a multiplicative function on the divisors of n_1 by putting $\varepsilon(h) := \prod_{p\mid h} \varepsilon(p)$ for each $h \mid n_1$. The product in (6) expands as

$$\prod_{p\mid n_1} \frac{1 + \varepsilon(p)\left(\frac{d}{p}\right)}{2} = \frac{1}{2^{\omega(n_1)}} \sum_{h\mid n_1} \varepsilon(h) \left(\frac{d}{h}\right).$$

Here $\left(\frac{d}{h}\right)$ is the Jacobi symbol. For each $h \mid n_1$, put

$$T_h(Y) := \sum_{\substack{0<d<Y \\ d\equiv c \ (8) \\ (d, n_1 n_2)=1}} \mu(d)^2 \left(\frac{d}{h}\right). \tag{9}$$

Then

$$T(Y; c, n_1, n_2, \varepsilon) = \frac{1}{2^{\omega(n_1)}} \sum_{h\mid n_1} \varepsilon(h) T_h(Y). \tag{10}$$

We suppress the dependence of $T_h(Y)$ on c, n_1, n_2 from the notation. We now substitute $\mu(d)^2 = \sum_{a^2|d} \mu(a)$ into $T_h(Y)$ and put $d = a^2k$. The original conditions give $(a, 2n_1n_2) = (k, n_1n_2) = 1$; since $a^2 \equiv 1 \pmod 8$ and $\left(\frac{a^2}{h}\right) = 1$, we obtain

$$T_h(Y) = \sum_{\substack{a \le \sqrt{Y} \\ (a, 2n_1n_2)=1}} \mu(a) \sum_{\substack{0<k<Y/a^2 \\ k \equiv c \ (8) \\ (k, n_1n_2)=1}} \left(\frac{k}{h}\right).$$

As $\left(\frac{k}{h}\right) = 0$ if $(k, h) > 1$, the condition $(k, n_1n_2) = 1$ in the inner sum may be replaced by $(k, n_1n_2/h) = 1$. Inclusion–exclusion gives

$$\mathbf{1}_{(k, n_1n_2/h)=1} = \sum_{e | (n_1n_2/h, k)} \mu(e) = \sum_{\substack{e | n_1n_2/h \\ e | k}} \mu(e).$$

Put $k = e\ell$. Since e is odd, it has an inverse e^{-1} in $(\mathbb{Z}/8\mathbb{Z})^\times$, and $k \equiv c \pmod 8$ becomes $\ell \equiv ce^{-1} \pmod 8$. Interchanging the sums over e and a gives

$$T_h(Y) = \sum_{e | n_1n_2/h} \mu(e) \left(\frac{e}{h}\right) \sum_{\substack{a \le \sqrt{Y/e} \\ (a, 2n_1n_2)=1}} \mu(a) \sum_{\substack{0<\ell<Y/ea^2 \\ \ell \equiv ce^{-1} \ (8)}} \left(\frac{\ell}{h}\right). \tag{11}$$

We first estimate $T_1(Y)$. In this case, the sum over ℓ in (11) is $Y/(8ea^2) + O(1)$. On summing over a, both the total contribution of the $O(1)$ terms and the error from extending the sum over a in the main term to infinity are $O(\sqrt{Y/e})$. Thus, since $\sum_{e|n_1n_2} e^{-1/2} = \lambda_2(n_1n_2)$, we have

$$\begin{aligned} T_1(Y) &= \frac{Y}{8} \sum_{e | n_1n_2} \frac{\mu(e)}{e} \sum_{\substack{a \ge 1 \\ (a, 2n_1n_2)=1}} \frac{\mu(a)}{a^2} + O\Big(\sqrt{Y}\, \lambda_2(n_1n_2)\Big) \\ &= \frac{Y}{8} \prod_{p | n_1n_2} (1 - p^{-1}) \prod_{p \nmid 2n_1n_2} (1 - p^{-2}) + O\Big(\sqrt{Y}\, \lambda_2(n_1n_2)\Big). \end{aligned}$$

Since $\prod_{p \ne 2} (1 - p^{-2}) = 8/\pi^2$, the main term of $2^{-\omega(n_1)} T_1(Y)$ agrees with the main term of (8).

We next estimate $T_h(Y)$ for $h > 1$. For each $e \mid n_1n_2/h$, choose an integer r satisfying $r \equiv ce^{-1} \pmod 8$, and an integer b satisfying $8b \equiv r \pmod h$. Let N_a denote the number of integers ℓ such that $0 < \ell < Y/ea^2$ and $\ell \equiv r \pmod 8$. Writing $\ell = r + 8j$, we have $\ell > 0$ if and only if $j > -r/8$, and $\left(\frac{\ell}{h}\right) = \left(\frac{8}{h}\right) \left(\frac{j+b}{h}\right)$. Consequently,

$$\sum_{\substack{0<\ell<Y/ea^2 \\ \ell \equiv ce^{-1} \ (8)}} \left(\frac{\ell}{h}\right) = \left(\frac{8}{h}\right) S\Big(\Big(\frac{\cdot}{h}\Big), N_a; b - \frac{r}{8}\Big), \qquad N_a \le \frac{Y}{ea^2}.$$

Since the Jacobi symbol $\left(\frac{\cdot}{h}\right)$ is a primitive character of conductor h, Proposition 4 gives

$$\sum_{\substack{a \le \sqrt{Y/e} \\ (a, 2n_1n_2)=1}} \Big| S\Big(\Big(\frac{\cdot}{h}\Big), N_a; b - \frac{r}{8}\Big)\Big| \ll \sqrt{\frac{Y}{e}}\, \kappa(h) (\log h)^{5/4}.$$

Using $h < 2n_1$ and $\sum_{e | n_1n_2/h} e^{-1/2} = \lambda_2(n_1n_2/h)$, we obtain

$$\sum_{1 \ne h | n_1} |T_h(Y)| \ll \sqrt{Y}\, (\log(2n_1))^{5/4} \sum_{1 \ne h | n_1} \kappa(h) \lambda_2(n_1n_2/h). \tag{12}$$

Since $\kappa(1) = 1$, the error bound for $T_1(Y)$ and (12) can be combined into a sum over all $h \mid n_1$. By the multiplicativity of λ_2 and κ and the definition (7), we have

$$\sum_{h|n_1} \kappa(h)\lambda_2(n_1n_2/h) = \lambda_2(n_2) \prod_{p|n_1} \big(\lambda_2(p) + \kappa(p)\big) = 2^{\omega(n_1)}\lambda_1(n_1)\lambda_2(n_2). \tag{13}$$

Together with (10), this proves (8). □

4 Proof of the main theorem and general local specifications

We first prove Theorem 1 for a pure local specification σ, and then deduce Corollary 6 for general σ.

Proof of Theorem 1. We first describe the local condition at 2 in terms of the discriminant $D = \mathrm{Disc}(F)$ of a quadratic field F. Put

$$e = \mathrm{ord}_2(D) \in \{0, 2, 3\}, \qquad c \equiv D/2^e \pmod 8, \quad c \in (\mathbb{Z}/8\mathbb{Z})^\times.$$

Then the isomorphism class of $F \otimes \mathbb{Q}_2$ is determined by (e, c). There are exactly eight possible pairs (e, c):

$$(0,1), (0,5), (2,3), (2,7), (3,1), (3,3), (3,5), (3,7).$$

The density of the corresponding quadratic $\mathbb{Q}_2$-algebra is $1/(3 \cdot 2^e)$ in each case.

It suffices to prove the theorem when $2 \mid n$ and σ_2 is a singleton. Indeed, partitioning according to the isomorphism class of $F \otimes \mathbb{Q}_2$ (and replacing n by $2n$ if necessary) expresses the general count as a sum of at most eight counts of this form, without changing u_1, u_2, r_1, r_2. The main terms add by additivity of local densities (which sum to 1 over A_2), and the error bound is preserved up to an absolute factor. Write (e, c) for the pair corresponding to the unique element of σ_2. Then $c_2(\sigma_2) = 1/(3 \cdot 2^e)$.

The product of the odd primes at which ramification is prescribed is r_1r_2. Thus the discriminant D of any quadratic field F counted by $N_2^{\pm}(X, \sigma)$ can be written as $D = \pm 2^e r_1 r_2 d$ for some integer d. We express the conditions on D in terms of d. Putting $Y = X/(2^e r_1 r_2)$, we have

$$d \text{ is squarefree;} \qquad 0 < d < Y; \qquad (d, n) = 1; \qquad d \equiv \pm c(r_1r_2)^{-1} \pmod 8.$$

Among the odd primes $p \mid n$, the product of those with $\#\sigma_p = 1$ is u_1r_1, and the product of those with $\#\sigma_p = 2$ is u_2r_2. For $p \mid u_1r_1$, we also have a condition $\left(\frac{d}{p}\right) = \varepsilon(p)$. Indeed, $F \otimes \mathbb{Q}_p$ is determined by $\left(\frac{D}{p}\right)$ if $p \mid u_1$, and by $\left(\frac{D/p}{p}\right)$ if $p \mid r_1$. Substituting $D = \pm 2^e r_1 r_2 d$ gives a condition of the form $\left(\frac{d}{p}\right) = \pm 1$ in either case. For $p \mid u_2r_2$, no condition remains beyond the already imposed $(d, p) = 1$. Conversely, the integers d satisfying these conditions correspond bijectively to the quadratic fields F being counted, apart from the case $D = 1$. Hence

$$N_2^{\pm}(X, \sigma) = T\big(Y; \pm c(r_1r_2)^{-1}, u_1r_1, u_2r_2, \varepsilon\big) - \delta_1,$$

where $\delta_1 = 1$ if the term corresponding to $D = 1$ is counted on the right, and $\delta_1 = 0$ otherwise.

By Proposition 5, the main term of $N_2^{\pm}(X, \sigma)$ is

$$\frac{X}{2^{e+\omega(u_1r_1)} r_1 r_2 \pi^2} \prod_{2 \neq p \mid n} \frac{p}{p+1} = \frac{3X}{\pi^2} \prod_{p|n} c_p(\sigma_p).$$

Consequently,

$$N_2^{\pm}(X, \sigma) = \frac{3}{\pi^2} X \prod_{p|n} c_p(\sigma_p) + O\left(\sqrt{\frac{X}{r_1r_2}}\, (\log(2u_1r_1))^{5/4} \lambda_1(u_1r_1)\lambda_2(u_2r_2)\right). \tag{14}$$

The contribution of δ_1 can be absorbed into this error term. Since $\lambda_1(p)/p^{3/16} \to \sqrt{2} < 3/2$ as $p \to \infty$, we have $\lambda_1(q) \ll q^{3/16}(3/2)^{\omega(q)}$ for every positive squarefree integer q. Applying this to (14) proves (1). □

We now turn to the case of a general local specification $\sigma = (\sigma_p)_{p|n}$. For each odd prime $p \mid n$, let σ_p^{ur} be the subset of unramified algebras in σ_p, and let σ_p^{rm} be the subset of ramified algebras. Put $a_p := \#\sigma_p^{\mathrm{ur}}, b_p := \#\sigma_p^{\mathrm{rm}}$. Call σ_p *pure* if $a_p b_p = 0$, and *mixed* otherwise. Thus σ is pure precisely when σ_p is pure for every odd prime $p \mid n$.

We group the odd primes $p \mid n$ according to the numbers (a_p, b_p) of unramified and ramified algebras in σ_p, and define the following five factors of n:

$$u_1 := \prod_{a_p=1} p, \quad u_2 := \prod_{\substack{a_p=2\\ b_p=0}} p, \quad u_2' := \prod_{\substack{a_p=2\\ b_p=1}} p, \quad r_1 := \prod_{\substack{a_p=0\\ b_p=1}} p, \quad r_2 := \prod_{\substack{a_p=0\\ b_p=2}} p. \tag{15}$$

These five integers are pairwise coprime, and their product equals the odd part of n. If σ is pure, then $u_2' = 1$, and the definitions of u_1, u_2, r_1, r_2 agree with those in Section 1. For each odd prime $p \mid n$, set $c_p(\sigma_p) := (a_p p + b_p)/(2(p+1))$; the definition of the local density at $p = 2$ remains unchanged.

Bhargava, Thorne, and the author [1, Proposition 4.2] proved, for general local specifications σ and every $\epsilon > 0$, that

$$N_2^{\pm}(X, \sigma) = \frac{3}{\pi^2} X \prod_{p|n} c_p(\sigma_p) + O_\epsilon\Big(X^{1/2}\, u_1^{1/4} (u_2 u_2')^{1/4} r_1^{-1/4} r_2^{-1/2} (u_1 u_2 u_2' r_1 r_2)^{\epsilon} \Big), \tag{16}$$

where the implied constant depends only on ϵ. The following corollary generalizes Theorem 1 and improves this estimate.

Corollary 6. *For every local specification σ and $X > 0$, we have*

$$N_2^{\pm}(X, \sigma) = \frac{3}{\pi^2} X \prod_{p|n} c_p(\sigma_p) + O\left(\frac{X^{1/2}\, u_1^{3/16}}{r_1^{5/16} r_2^{1/2}} \left(\frac{3}{2}\right)^{\omega(u_1 r_1 u_2')} (\log(2u_1 r_1))^{5/4} \lambda_2(u_2 r_2) \right). \tag{17}$$

Proof. As in the proof of Theorem 1, it suffices to consider the case where $2 \mid n$ and σ_2 consists of a single element. Let m be the product of the odd primes p for which σ_p is mixed. By definition, $u_2' \mid m \mid u_1 u_2'$. For each divisor t of m, define a local specification $\sigma^{(t)} = (\sigma_p^{(t)})_{p|n}$ with the same support n as σ by setting, for $p \mid m$,

$$\sigma_p^{(t)} := \begin{cases} \sigma_p^{\mathrm{rm}}, & p \mid t, \\ \sigma_p^{\mathrm{ur}}, & p \mid m/t \end{cases}$$

and taking $\sigma_p^{(t)} := \sigma_p$ for $p \nmid m$. Splitting each mixed condition σ_p into its unramified and ramified parts yields the disjoint union $\mathcal{F}(\sigma) = \sqcup_{t|m} \mathcal{F}(\sigma^{(t)})$. Therefore,

$$N_2^{\pm}(X, \sigma) = \sum_{t|m} N_2^{\pm}(X, \sigma^{(t)}). \tag{18}$$

Since each $\sigma^{(t)}$ is pure, we can apply (14) to each term on the right. The main terms sum to the desired main term by additivity of the local densities.

For $\sigma^{(t)}$, let $n_1^{(t)}$ be the product of the odd primes $p \mid n$ at which exactly one local algebra is allowed, and let $n_2^{(t)}$ be the corresponding product where exactly two algebras are allowed. Their product is the odd part of n. For each $t \mid m$, put $t_0 := (t, u_2')$. If $p \mid u_2'$, then $\#\sigma_p^{(t)} = 1$ precisely

when the ramified part is selected, that is, when $p \mid t$. This observation and (15) give $n_1^{(t)} \mid u_1 r_1 t_0$. Hence

$$\log(2n_1^{(t)}) \le \log(2u_1 r_1 t_0) \ll \log(2u_1 r_1) \cdot t_0^{1/5}.$$

The product of the odd primes where $\sigma^{(t)}$ prescribes ramification is $r_1 r_2 t$. Summing the errors in (14) for $N_2^{\pm}(X, \sigma^{(t)})$ over $t \mid m$, we obtain

$$\begin{aligned}\left| N_2^{\pm}(X,\sigma) - \frac{3X}{\pi^2} \prod_{p \mid n} c_p(\sigma_p) \right| &\ll \sqrt{\frac{X}{r_1 r_2}} \sum_{t \mid m} t^{-1/2} (\log(2n_1^{(t)}))^{5/4} \lambda_1(n_1^{(t)}) \lambda_2(n_2^{(t)}) \\ &\ll \sqrt{\frac{X}{r_1 r_2}} (\log(2u_1 r_1))^{5/4} \sum_{t \mid m} \frac{t_0^{1/4}}{t^{1/2}} \lambda_1(n_1^{(t)}) \lambda_2(n_2^{(t)}) \qquad (19)\end{aligned}$$

The desired error bound now follows from the next lemma. □

Lemma 7. *With the notation in the proof of Corollary 6, we have*

$$\sum_{t \mid m} \frac{t_0^{1/4}}{t^{1/2}} \lambda_1(n_1^{(t)}) \lambda_2(n_2^{(t)}) \ll \left(\frac{3}{2}\right)^{\omega(u_1 r_1 u_2')} (u_1 r_1)^{3/16} \lambda_2(u_2 r_2). \qquad (20)$$

Proof. By multiplicativity, the divisor sum on the left-hand side of (20) factors as

$$\prod_{\substack{2 \ne p \mid n \\ p \nmid m}} \lambda_{a_p + b_p}(p) \prod_{p \mid m / u_2'} \big(\lambda_{a_p}(p) + p^{-1/2} \lambda_{b_p}(p)\big) \prod_{p \mid u_2'} \big(\lambda_2(p) + p^{-1/4} \lambda_1(p)\big). \qquad (21)$$

Indeed, for $p \mid m$ with $p \nmid u_2'$, the local factor is the sum of $\lambda_{a_p}(p)$, arising when $p \nmid t$, and $p^{-1/2}\lambda_{b_p}(p)$, arising when $p \mid t$. For $p \mid u_2'$, we have $(a_p, b_p) = (2, 1)$, and the extra weight $t_0^{1/4}$ gives the factor $\lambda_2(p) + p^{-1/4}\lambda_1(p)$. If $p \nmid m$, only $p \nmid t$ occurs; the factor is a single term, which can be written as $\lambda_{a_p + b_p}(p)$ because $a_p b_p = 0$.

Since $\lambda_2(p) < \kappa(p)$, we have $\lambda_2(p) < \lambda_1(p)$. Thus, for $p \mid u_1$, the factor in (21) in the mixed case satisfies

$$\lambda_1(p) + p^{-1/2}\lambda_{b_p}(p) \le \lambda_1(p)(1 + p^{-1/2}) = \lambda_1(p)\lambda_2(p),$$

whereas in the pure case it is $\lambda_1(p)$. In either case, the factor is at most $\lambda_1(p)\lambda_2(p)$. The factor for $p \mid u_2$ in (21) is $\lambda_2(p)$, and that for $p \mid r_i$ is $\lambda_i(p)$. Consequently,

$$\begin{aligned}\sum_{t \mid m} \frac{t_0^{1/4}}{t^{1/2}} \lambda_1(n_1^{(t)}) \lambda_2(n_2^{(t)}) &\le \lambda_1(r_1)\lambda_2(u_2 r_2) \prod_{p \mid u_1} \lambda_1(p)\lambda_2(p) \prod_{p \mid u_2'} \big(\lambda_2(p) + p^{-1/4}\lambda_1(p)\big) \\ &\le \lambda_2(u_2 r_2) \prod_{p \mid u_1 r_1} \lambda_1(p)\lambda_2(p) \prod_{p \mid u_2'} \big(\lambda_2(p) + p^{-1/4}\lambda_1(p)\big).\end{aligned}$$

As $p \to \infty$ through the primes,

$$\frac{\lambda_1(p)\lambda_2(p)}{p^{3/16}} \longrightarrow \sqrt{2} < \frac{3}{2}, \qquad \lambda_2(p) + p^{-1/4}\lambda_1(p) \longrightarrow 1 < \frac{3}{2}.$$

These bounds prove (20). □

Remark 8. *The factor* $3/2$ *in the error terms of* (1) *and* (17) *(and also on the right-hand side of* (20)*) may be replaced by any fixed real number* $B > \sqrt{2}$ *(for example,* $B = 1.42$*). The implied constant may then depend on* B*.*

References

[1] M. Bhargava, T. Taniguchi and F. Thorne, *Improved error estimates for the Davenport–Heilbronn theorems*, Math. Ann. **389** (2024), no. 4, 3471–3512. DOI: 10.1007/s00208-023-02684-w.

[2] D. A. Burgess, *On character sums and L-series. II*, Proc. London Math. Soc. (3) **13** (1963), 524–536. DOI: 10.1112/plms/s3-13.1.524.

[3] E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math. Z. **74** (1960), 66–80. DOI: 10.1007/BF01180473.

[4] E. Cohen and R. L. Robinson, *On the distribution of the k-free integers in residue classes*, Acta Arith. **8** (1962/63), 283–293; errata, **10** (1965), 443; correction, **16** (1970), 439. DOI: 10.4064/aa-8-3-283-293.

[5] H. Davenport, *Multiplicative Number Theory*, 2nd ed., Graduate Texts in Mathematics 74, Springer-Verlag, New York, 1980.

[6] J. Ellenberg, L. B. Pierce and M. Matchett Wood, *On ℓ-torsion in class groups of number fields*, Algebra Number Theory **11** (2017), no. 8, 1739–1778. DOI: 10.2140/ant.2017.11.1739.

[7] L. Gegenbauer, *Asymptotische Gesetze der Zahlentheorie*, Denkschr. Kaiserl. Akad. Wiss. Wien, Math.-Naturwiss. Cl. **49** (1885), 37–80.

[8] A. Granville and K. Soundararajan, *The distribution of values of $L(1,\chi_d)$*, Geom. Funct. Anal. **13** (2003), no. 5, 992–1028; errata, **14** (2004), no. 1, 245–246. DOI: 10.1007/s00039-003-0438-3; errata DOI: 10.1007/s00039-004-0456-9.

[9] N. Jain-Sharma, T. Khale and M. Liu, *Explicit Burgess bound for composite moduli*, Int. J. Number Theory **17** (2021), no. 10, 2207–2219. DOI: 10.1142/S1793042121500834.

[10] M. Munsch, *Character sums over squarefree and squarefull numbers*, Arch. Math. **102** (2014), no. 6, 555–563. DOI: 10.1007/s00013-014-0658-9.

[11] P. Pollack, *The average least quadratic nonresidue modulo m and other variations on a theme of Erdős*, J. Number Theory **132** (2012), no. 6, 1185–1202. DOI: 10.1016/j.jnt.2011.12.015.

[12] O. Robert and G. Tenenbaum, *Sur la répartition du noyau d'un entier*, Indag. Math. (N.S.) **24** (2013), no. 4, 802–914. DOI: 10.1016/j.indag.2013.07.007.

[13] W. Schwarz, *Über die Summe $\sum_{n\le x}\varphi(f(n))$ und verwandte Probleme*, Monatsh. Math. **66** (1962), 43–54. DOI: 10.1007/BF01418877.

[14] A. Shankar and J. Tsimerman, *Secondary terms in the counting functions of quartic fields II*, arXiv:2508.08527 (2025).

[15] D. Suryanarayana, *Uniform O-estimates of certain error functions connected with k-free integers*, J. Austral. Math. Soc. **11** (1970), 242–250. DOI: 10.1017/S144678870000642X.

[16] T. Taniguchi and F. Thorne, *Secondary terms in counting functions for cubic fields*, Duke Math. J. **162** (2013), no. 13, 2451–2508. DOI: 10.1215/00127094-2371752.

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