

A classification of linear differential equations on the Riemann sphere

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Fuchsian ordinary differential equation

$$\mathcal{M} : \frac{du}{dx} = \sum_{1 \leq j < p} \frac{A_j}{x - a_j} u \quad (A_j \in \text{Mat}(n, \mathbb{C}))$$

$A_p = A_\infty := -(A_1 + \cdots + A_{p-1})$: residue matrix at ∞

A_j : eigenvalue $\lambda_{j,\nu}$ with multiplicity $m_{j,\nu}$ ($\nu = 1, \dots, n_j$)

$[A_j] := \{[\lambda_{j,\nu}]_{m_{j,\nu}} \mid \nu = 1, \dots, n_j\}$: spectrum of A_j

$n = m_{j,1} + \cdots + m_{j,n_j}$: spectral type of A_j

$$\text{rank} \prod_{\nu=1}^k (A_j - \lambda_{j,\nu}) = n - m_{j,1} - \cdots - m_{j,k} \quad (k = 1, \dots, n_j)$$

when $m_{j,1} \geq m_{j,2} \geq \cdots \geq m_{j,n_j}$

$$\left\{ \begin{array}{cc} x = a_j \ (j = 1, \dots, p-1) & x = \infty \\ [\lambda_{j,1}]_{m_{j,1}} & [\lambda_{p,1}]_{m_{p,1}} \\ \vdots & \vdots \\ [\lambda_{j,n_j}]_{m_{j,n_j}} & [\lambda_{p,n_p}]_{m_{p,n_p}} \end{array} \right\} : \text{(generalized) Riemann scheme}$$

$$\begin{aligned} \exists \text{sol. } u &\sim C_{j,\nu,i} (x - a_j)^{\lambda_{j,\nu}} & (x \rightarrow a_j, \ i = 1, \dots, m_{j,\nu}, \ \nu = 1, \dots, n_j) \\ &\sim C_{p,\nu,i} \left(\frac{1}{x}\right)^{\lambda_{p,\nu}} & (x \rightarrow \infty, \ i = 1, \dots, m_{p,\nu}, \ \nu = 1, \dots, n_p) \end{aligned}$$

Spectral type

Spectral type \mathbf{m} of \mathcal{M} :

$$\begin{aligned} \mathbf{m} &:= \left(m_{j,\nu} \right)_{\substack{1 \leq \nu \leq n_j \\ 1 \leq j \leq p}} \\ &= m_{1,1} \cdots m_{1,n_1}, \dots, m_{p-1,1} \cdots m_{p-1,n_{p-1}}, m_{p,1} \cdots m_{p,n_p} \\ &= \left[[m_{1,1}, \dots, m_{1,n_1}], \dots, [m_{p-1,1}, \dots, m_{p-1,n_{p-1}}], [m_{p,1}, \dots, m_{p,n_p}] \right] \\ n &= m_{j,1} + \cdots + m_{j,n_j} \quad (j = 1, \dots, p) : \text{tuple of } p \text{ partitions of ord } \mathbf{m} := n \end{aligned}$$

Ex. ${}_nF_{n-1} : n = \overbrace{1 + \cdots + 1}^n = (n-1) + 1 = \overbrace{1 + \cdots + 1}^n$
 $1 \cdots 1, (n-1)1, 1 \cdots 1 = 1^n, (n-1)1, 1^n$
 $[[1, \dots, 1], [n-1, 1], [1, \dots, 1]]$

Gauss: 11, 11, 11 (ord = 2, singular points : $\{0, 1, \infty\}$)

${}_3F_2$: 111, 21, 111 (ord = 3, $\{0, 1, \infty\}$, 2-dim local hol. sol. around 1)

Heun: 11, 11, 11, 11 (ord = 2, 4 singular points \Rightarrow Painlevé VI)

Jordan-Pochhammer : 21, 21, 21, 21 (\Rightarrow Appell's F_1)

Middle convolution

Middle convolution $\text{mc}_\sigma \mathbf{m}$ of a tuple (spectral type) \mathbf{m} :

$$\text{mc}_\sigma \mathbf{m} := \left(m_{j,\nu} - \delta_{\nu,\sigma_j} d_\sigma(\mathbf{m}) \right)_{\substack{1 \leq \nu \leq n_j \\ 1 \leq j \leq p}} \quad (\sigma \in \mathbb{Z}_{\geq 0}^p \leftarrow \text{addition})$$

$$\text{ord mc}_\sigma \mathbf{m} = \text{ord } \mathbf{m} - d_\sigma(\mathbf{m}), \quad \text{mc}_\sigma^2 = \text{id}$$

$$\text{ord } \mathbf{m} := n = m_{j,1} + \cdots + m_{j,n_j}$$

$$d_\sigma(\mathbf{m}) := m_{1,\sigma_1} + \cdots + m_{p,\sigma_p} - (p-2) \cdot \text{ord } \mathbf{m} \quad (\nu > n_j \Rightarrow m_{j,\nu} = 0)$$

$$\Leftarrow \text{addition} : u \mapsto f^\lambda u, \quad \text{middle convolution} : u \mapsto \partial_x^{-\mu} u$$

Ex. $\text{mc}_{(1,1,1)} : \underline{111}, \underline{21}, \underline{111} \xrightarrow{1+2+1-1 \cdot 3} 011, 11, 011 \quad \underline{11}, \underline{11}, \underline{11} \xrightarrow{3-2} 1, 1, 1$
 $\underline{211}, \underline{211}, \underline{1111} \xrightarrow{5-4=1} \underline{111}, \underline{111}, \underline{0111} \xrightarrow{3-3=0} 111, 111, 111$
 $\underline{431}, \underline{3311}, \underline{41111} \xrightarrow{11-8=3} \underline{131}, \underline{311}, \underline{11111} \xrightarrow{7-5=2} \times$

Remark. $m_{j,\nu}$ is allowed to be 0 or n

$$\text{idx } \mathbf{m} := \sum_{\substack{1 \leq \nu \leq n_j \\ 1 \leq j \leq p}} m_{j,\nu}^2 - (p-2)(\text{ord } \mathbf{m})^2 : \text{index of rigidity of } \mathbf{m}$$

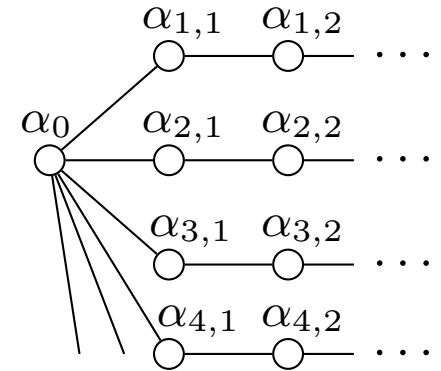
$$\text{idx mc}_\sigma \mathbf{m} = \text{idx } \mathbf{m}$$

Star-shaped Kac-Moody root system

$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi := \{\alpha_0, \alpha_{j,\nu} \mid j \geq 1, \nu \geq 1\})$$

$$(\alpha_0|\alpha_{j,\nu}) = -\delta_{\nu,1}$$

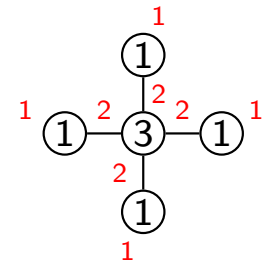
$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \neq j \text{ or } |\mu - \nu| > 1) \\ -1 & (i = j \text{ and } |\mu - \nu| = 1) \end{cases}$$



$$s_\alpha : x \mapsto x - (x|\alpha)\alpha \quad (\alpha \in \Pi), \quad W := \langle s_\alpha \mid \alpha \in \Pi \rangle$$

$$\alpha_{\mathbf{m}} := n\alpha_0 + \sum_{j \geq 1} \sum_{i \geq 1} \left(\sum_{\nu > i} m_{j,\nu} \right) \alpha_{j,i} \quad (n = \text{ord } \mathbf{m})$$

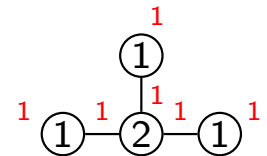
$$= n\alpha_0 + \sum_{j \geq 1} \sum_{\nu \geq 1} n_{j,\nu} \alpha_{j,\nu}$$



$$m_{j,\nu} = n_{j,\nu-1} - n_{j,\nu} \quad (j = 1, 2, \dots, \nu = 1, 2, \dots, n_{j,0} = n)$$

$$11, 11, 11 \leftrightarrow \alpha = 2\alpha_0 + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1}$$

$$21, 21, 21, 21 \leftrightarrow \alpha = 3\alpha_0 + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1}$$



$$111, 21, 111 : \begin{array}{c} 1 \\ \textcircled{1} \\ | \\ \textcircled{2} - \textcircled{3} - \textcircled{2} - \textcircled{1} \\ | \\ \textcircled{2} \\ | \\ \textcircled{1} \end{array}$$

$$33, 222, 111111 : \begin{array}{c} 3 \\ \textcircled{3} \\ | \\ \textcircled{2} - \textcircled{4} - \textcircled{6} - \textcircled{5} - \textcircled{4} - \textcircled{3} - \textcircled{2} - \textcircled{1} \\ | \\ \textcircled{2} \\ | \\ \textcircled{1} \end{array}$$

Deligne-Simpson problem and Fundamental tuples

Th. $\alpha_{mc_{(1,\dots,1)} \mathbf{m}} = s_{\alpha_0} \alpha_{\mathbf{m}}$

$\alpha_{\mathbf{m}'} = s_{\alpha_{j,\nu}} \alpha_{\mathbf{m}} \quad (\mathbf{m}' = \cdots m_{j,\nu+1} \overbrace{m_{j,\nu}} \cdots)$

$\text{idx } \mathbf{m} = (\alpha_{\mathbf{m}} | \alpha_{\mathbf{m}}), \quad d_{(1,\dots,1)} \mathbf{m} = (\alpha_0 | \alpha_{\mathbf{m}})$

Deligne-Simpson Problem (additive). For given $\bar{A}_1, \dots, \bar{A}_p \in \text{Mat}(n, \mathbb{C})$,

\exists irred. A_1, \dots, A_p with $A_j \sim \bar{A}_j$ and $\sum_j A_j \in \mathbb{C}$?

Th [Kostov, Katz, Crawley-Boevey, Oshima].

\exists irred. A_1, \dots, A_p in the above $(\stackrel{\text{def}}{\Leftrightarrow} \mathbf{m} : \text{irreducibly realizable})$

$\Leftrightarrow \alpha_{\mathbf{m}}$ is a **positive root** and moreover $\text{GCD}\{m_{j,\nu}\} = 1$ when $\text{idx } \mathbf{m} = 0$

Th [Katz]. $\Rightarrow \dim\{(A_1, \dots, A_p) \mid A_j \sim \bar{A}_j, \sum A_j \in \mathbb{C}\} / GL(n, \mathbb{C}) = 2 - \text{idx } \mathbf{m}$

Th [Katz]. \mathbf{m} is irreducibly realizable

$\Leftrightarrow mc_{\sigma} \mathbf{m}$ is irreducibly realizable or $\text{ord } \mathbf{m} = 1$

$\mathbf{m} : \text{rigid} \stackrel{\text{def}}{\Leftrightarrow} \mathbf{m} : \text{irred. realizable and } \text{idx } \mathbf{m} = 2 \Leftrightarrow \alpha_{\mathbf{m}} : \text{real positive root}$

$$\mathbf{m} : \text{ordered} \stackrel{\text{def}}{\Leftrightarrow} \begin{cases} m_{j,1} \geq m_{j,2} \geq \cdots \geq m_{j,n_j} > 0 & (1 \leq j \leq p) \\ m_{j,\nu} = m_{j+1,\nu} \ (1 \leq \nu < k) \text{ and } m_{j,k} > m_{j+1,k} \text{ or } k = n_j + 1 \end{cases}$$

$$\sum_{j=1}^p \sum_{\nu=1}^{n_j} m_{j,\nu} (m_{j,1} - m_{j,\nu}) = d_{(1,\dots,1)}(\mathbf{m}) \cdot \text{ord } \mathbf{m} - \text{idx } \mathbf{m}$$

- $\mathbf{m} : \text{idx } \mathbf{m} = 2$ and ordered $\Rightarrow d_{(1,\dots,1)} \mathbf{m} > 0 \Rightarrow \text{ord } m_{(1,\dots,1)} \mathbf{m} < \text{ord } \mathbf{m}$
 \Rightarrow rigid \mathcal{M} is reduced to trivial equation by additions and middle convolutions

$\mathbf{m} : \text{rigid} \Leftrightarrow \alpha_{\mathbf{m}} \in W\alpha_0 \Rightarrow \alpha_{\mathbf{m}} : \text{positive real root}$

- $\mathbf{m} : \text{ordered}, \text{idx } \mathbf{m} < 0, d_{(1,\dots,1)} \mathbf{m} \leq 0 \Rightarrow$ irreducibly realizable (C-B)

$$\mathcal{F}_\ell := \{ \mathbf{m} : \text{ordered} \mid d_{(1,\dots,1)} \mathbf{m} \leq 0, \text{idx } \mathbf{m} = -\ell, \text{GCD}\{m_{j,\nu}\} = 1 \text{ if } \ell = 0 \}$$

$\mathbf{m} : \text{fundamental tuple} \in \mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_{2k} \quad (\mathcal{F}_\ell \neq \emptyset \Leftrightarrow \ell \in 2\mathbb{Z}_{\geq 0}) : \text{irred. realizable}$

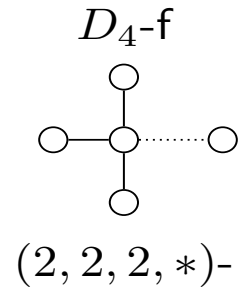
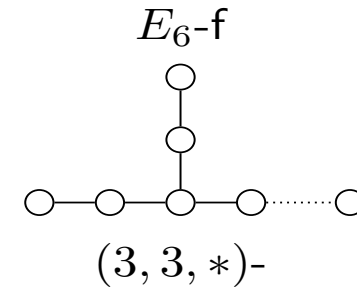
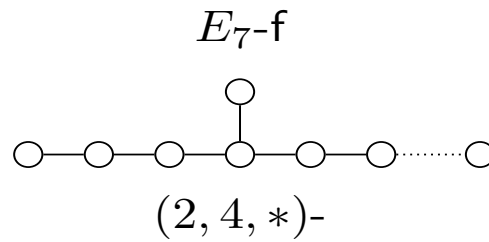
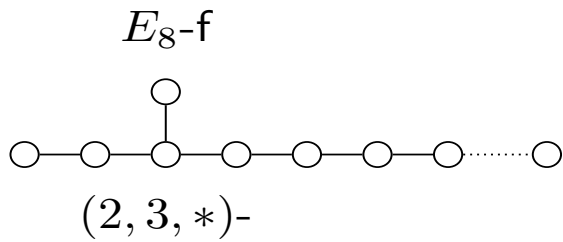
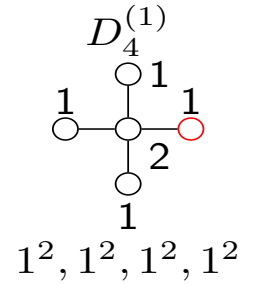
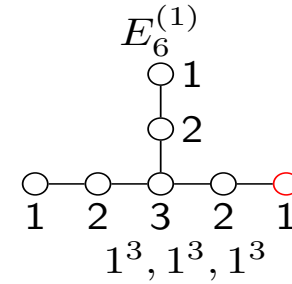
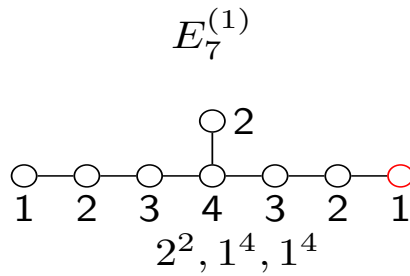
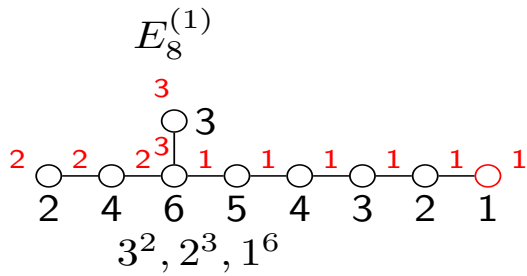
Th [Kostov]. $\mathcal{F}_0 = \{33, 222, 111111 \ 22, 1111, 1111 \ 111, 111, 111 \ 11, 11, 11, 11\}$

Th [O]. $\#\mathcal{F}_\ell < \infty$

$$\mathcal{P} := \{ \mathbf{m} \mid \text{idx } \mathbf{m} \leq 0 \text{ and irreducibly realizable} \} = W\mathcal{F} \quad (\mathcal{P}/W \simeq \mathcal{F})$$

$$= \{ \mathbf{m} \mid \alpha_{\mathbf{m}} : \text{positive imaginary roots and } \text{GCD}\{m_{j,\nu}\} = 1 \text{ if } \text{idx } \mathbf{m} = 0 \}$$

Def. A fundamental tuple of partitions $\mathbf{m} = (m_{j,\nu})_{\substack{1 \leq \nu \leq n_j \\ 1 \leq j < p}} \in \mathcal{F}$ is E_k -fundamental for $k = 8, 7, 6$ if $p = 3$ and $(n_1, n_2) = (2, 3), (2, 4), (3, 3)$, resp. D_4 -fundamental if $p = 4$ and $(n_1, n_2, n_3) = (2, 2, 2)$.



$$E_8\text{-f} : n = a_1 + a_2 = b_1 + b_2 + b_3 = c_1 + c_2 + \cdots + c_k,$$

$$a_1 \geq a_2 > 0, \quad b_1 \geq b_2 \geq b_3 > 0, \quad c_1 \geq c_2 \geq \cdots \geq c_k > 0$$

$$\text{fundamental} \Leftrightarrow n \geq a_1 + b_1 + c_1 \Leftrightarrow a_2 \geq b_1 + c_1$$

$$\Rightarrow \frac{n}{3} \leq b_1 < a_2 = n - a_1 \leq \frac{n}{2} \leq a_1 < \frac{2n}{3}, \quad c_1 \leq \frac{n}{6}, \quad k \geq 6$$

$$b_3 - c_1 \geq (n - b_1 - b_2) - (n - a_1 - b_1) = a_1 - b_2 > 0$$

Numbers of all, E_8 -f, 3, 4, 5 tuples with a given index

| | | | | | | | | | | |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| idx | 0 | -2 | -4 | -6 | -8 | -10 | -12 | -14 | -16 | -18 |
| all | 4 | 13 | 37 | 69 | 113 | 198 | 291 | 415 | 647 | 884 |
| E_8 | 1 | 2 | 5 | 7 | 10 | 15 | 18 | 21 | 33 | 36 |
| 3 | 3 | 9 | 25 | 46 | 73 | 127 | 182 | 249 | 395 | 522 |
| 4 | 1 | 3 | 9 | 17 | 29 | 50 | 76 | 115 | 172 | 243 |
| 5 | 0 | 1 | 2 | 4 | 8 | 15 | 24 | 36 | 55 | 83 |
| idx | -20 | -22 | -24 | -26 | -28 | -30 | -32 | -34 | -36 | -38 |
| all | 1186 | 1682 | 2183 | 2889 | 3841 | 4831 | 6203 | 8012 | 9789 | 12290 |
| E_8 | 40 | 55 | 62 | 70 | 92 | 96 | 114 | 142 | 148 | 168 |
| 3 | 680 | 963 | 1230 | 1577 | 2105 | 2570 | 3257 | 4187 | 5009 | 6144 |
| 4 | 345 | 478 | 625 | 861 | 1126 | 1430 | 1890 | 2411 | 2982 | 3832 |
| 5 | 111 | 164 | 229 | 310 | 416 | 570 | 716 | 940 | 1214 | 1546 |
| idx | -40 | -42 | -44 | -46 | -48 | -50 | -52 | -54 | -56 | -58 |
| all | 15379 | 18683 | 22910 | 28096 | 33387 | 40617 | 48703 | 57481 | 68622 | 81728 |
| E_8 | 212 | 220 | 246 | 298 | 314 | 351 | 412 | 429 | 486 | 570 |
| 3 | 7731 | 9153 | 11072 | 13493 | 15768 | 18878 | 22594 | 26040 | 30819 | 36495 |
| 4 | 4721 | 5785 | 7227 | 8763 | 10484 | 12953 | 15392 | 18259 | 22056 | 26041 |
| 5 | 1939 | 2491 | 3046 | 3805 | 4675 | 5700 | 6916 | 8521 | 10114 | 12203 |

Confluence and Unfolding

Irregular singularity at ∞ with Poincaré rank 1:

$$\left\{ \begin{array}{ll} x = \infty & x = a_j \quad (j = 2, \dots, p-1) \\ [\lambda_{1,1}x + \lambda_{1,1}^{(0)}]_{m_{1,1}^{(0)}} & [\lambda_{j,1}]_{m_{j,1}} \\ \vdots & \vdots \\ [\lambda_{1,n_1}x + \lambda_{n_1,R_{n_1}}^{(0)}]_{m_{n_0,R_{n_0}}^{(0)}} & [\lambda_{j,n_j}]_{m_{j,n_j}} \end{array} \right\} : \text{Riemann scheme}$$

$$\exists \text{ sol } u \sim C_{\nu,k,\ell} \left(\frac{1}{x}\right)^{\lambda_{\nu,k}^{(0)}} e^{-\lambda_{1,\nu}x} \quad (x \rightarrow \infty, 1 \leq \ell \leq m_{\nu,k}^{(0)}, 1 \leq k \leq R_\nu)$$

Spectrum at ∞ : $\{[\lambda_{1,\nu}x + \lambda_{\nu,k}^{(0)}]_{m_{\nu,k}^{(0)}} \mid 1 \leq k \leq R_\nu, 1 \leq \nu \leq n_1\}$

$$m_{1,\nu} := \sum_{k=1}^{R_\nu} m_{\nu,k}^{(0)} \Rightarrow \{m_{\nu,k}^{(0)}\} \text{ is a refinement of } n = m_{1,1} + \dots + m_{m,n_1}$$

$$n = m_{1,1} + \dots + m_{1,n_1}$$

Spectral type $\mathbf{m} : (m_{1,1}^{(0)} \dots m_{1,R_1}^{(0)}) \dots (m_{n_1,1}^{(0)} \dots m_{n_1,R_{n_1}}^{(0)}), m_{2,1} \dots, \dots, m_{p-1,1} \dots m_{p-1,n_{p-1}}$
 $= m_{1,1}^{(0)} m_{1,2}^{(0)} \dots m_{n_1,R_{n_1}}^{(0)} \mid m_{1,1} \dots m_{1,n_1}, m_{2,1} \dots, \dots, m_{p-1,1} \dots m_{p-1,n_{p-1}}$

Unfolding of \mathbf{m} :

$$m_{1,1}^{(0)} m_{1,2}^{(0)} \dots m_{n_1,R_{n_1}}^{(0)}, m_{1,1} \dots m_{1,n_1}, m_{2,1} \dots, \dots, m_{p-1,1} \dots m_{p-1,n_{p-1}}$$

Ex. 33, 222, 111111 $\xrightarrow{\text{confluence}}$

$$111111|222, 33 = (11)(11)(11), 33$$

$$111111|33, 222 = (111)(111), 222$$

42, 21111, 21111 $\xrightarrow{\text{confluence}}$

$$11112|42, 21111 = (1111)(2), 21111$$

$$21111|42, 21111 = (211)(11), 21111$$

$$21111|21111, 42 = (2)(1)(1)(1), 42$$

$$11112|11112|42 = ((1)(1)(1)(1))((2))$$

$$21111|21111|42 = ((2)(1)(1))((1)(1))$$

Th [OH]. The number of fundamental spectral types of linear Pfaffian system with unramified irregular/regular singularities and a fixed index of rigidity is finite.

Th [Hiroe]. \mathbf{m} : a tuple of partitions with a refinement structure

\mathbf{m} : irreducibly realizable \Leftrightarrow the unfolding of \mathbf{m} is irreducibly realizable

Th [O, Ka, Hi]. \exists versal unfolding of a linear Pfaffian system with unramified irregular singularities.

$$\mathcal{M} : \frac{du}{dx} = \frac{A_1}{1 - bx} u + \sum_{j=2}^{p-1} \frac{A_j}{x - a_j} u \quad (\text{versal unfolding, } |b| \ll 1)$$

Harnad dual

Harnad dual Hd_σ (\leftarrow Laplace tranf. $(x, \frac{d}{dx}) \mapsto (-\frac{d}{dx}, x)$ and addition)

$$\begin{aligned} & (m_{1,1}^{(0)} \cdots m_{1,R_1}^{(0)}) \cdots (m_{n_1,1}^{(0)} \cdots m_{n_1,R_{n_1}}^{(0)}), \underline{m_{2,1} m_{2,2} \cdots m_{2,n_2}}, \dots, \underline{m_{q,1} m_{q,2} \cdots m_{q,n_q}} \\ &= m_{1,1}^{(0)} \cdots m_{1,R_1}^{(0)} \cdots m_{n_1,1}^{(0)} \cdots m_{n_1,R_{n_1}}^{(0)} \mid m_{1,1} \cdots m_{1,n_1}, m_{2,1} \cdots m_{2,n_2}, m_{q,1} \cdots m_{q,n_q} \\ &\xrightarrow{\text{Hd}_{(1,\dots,1)}} \\ & (m_{2,2} \cdots m_{2,n_2}) \cdots (m_{q,2} \cdots m_{q,n_q}), \underline{m_{1,0}^{(0)} m_{1,1}^{(0)} \cdots m_{1,R_1}^{(0)}}, \dots, \underline{m_{n_1,0}^{(0)} m_{n_1,1}^{(0)} \cdots m_{n_1,R_{n_1}}^{(0)}} \end{aligned}$$

$\text{idx Hd}_\sigma \mathbf{m} = \text{idx } \mathbf{m}, \quad \text{Hd}_\sigma^2 = \text{id}$

Transformation : $\mathbf{m}^{(0)} \xrightarrow{\text{confluence}} \mathbf{m}^{(1)} \xrightarrow{\text{Harnad dual}} \mathbf{m}^{(2)} \xrightarrow{\text{unfolding}} \mathbf{m}^{(3)}$

$$\begin{aligned} 33, 222, 111111 &\xrightarrow{\text{conf}} (11)(11)(11), \underline{33} \xrightarrow{\text{Hd}} (3), \underline{111}, \underline{111}, \underline{111} \xrightarrow{\text{unf}} 111, 111, 111 \\ &: 33, 222, 111111 \xrightarrow{(1^2)(1^2)(1^2)} 111, 111, 111 \end{aligned}$$

$$\begin{aligned} 33, 222, 111111 &\xrightarrow{\text{conf}} (111)(111), \underline{222} \xrightarrow{\text{Hd}} (22), \underline{1111}, \underline{1111} \xrightarrow{\text{unf}} 22, 1111, 1111 \\ &: 33, 222, 111111 \xrightarrow{(1^3)(1^3)} 22, 1111, 1111 \end{aligned}$$

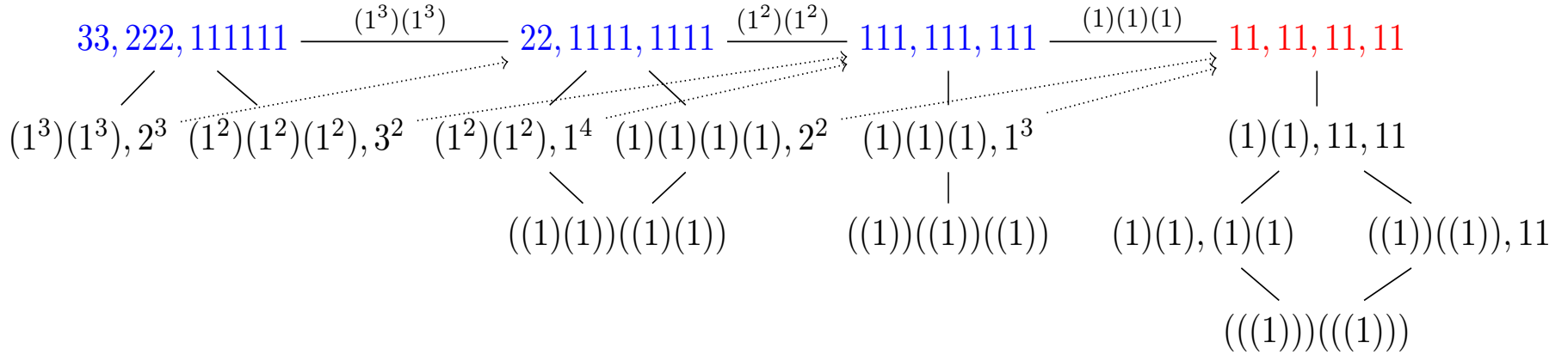
Def [Kawakami]. \mathbf{m}, \mathbf{m}' : irreducibly realizable tuples of partitions

$\mathbf{m} \sim \mathbf{m}' \stackrel{\text{def}}{\iff} \mathbf{m}$ is transformed to \mathbf{m}' by applications of these transformations:

the above transformations, mc_σ and permutations of partitions (=singular points)

idx=0 (Pidx=1), 4 basic (fundamental) tuples

| | | | |
|------------------------------|-----------------------------|----------------------------|----------------------------|
| 6: 33,222,111111 $E_8^{(1)}$ | 4: 22,1111,1111 $E_7^{(1)}$ | 3: 111,111,111 $E_6^{(1)}$ | 2: 11,11,11,11 $D_4^{(1)}$ |
|------------------------------|-----------------------------|----------------------------|----------------------------|



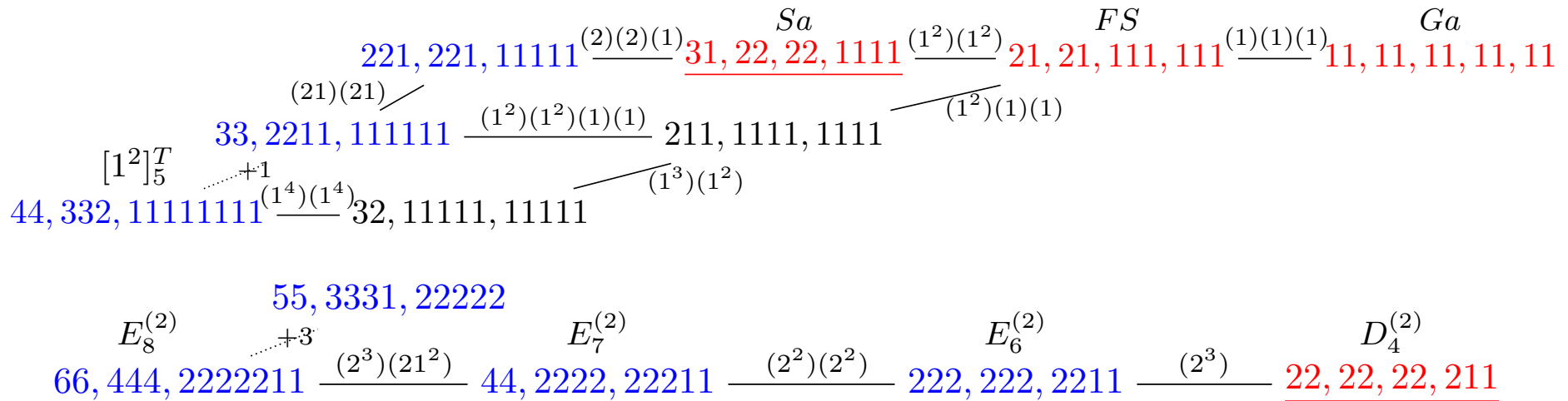
index : $2 - 2m$ ($m \geq 1$)

$$\begin{aligned}
 [11]_{m+3} &: \overbrace{11, 11, \dots, 11}^{m+3} = [11]_{m+3} & [11]_{m+3}^T &: (m+2)^2, (m+1)^2 2, 1^{2m+4} \\
 FS_m &: m1, m1, 1^{m+1}, 1^{m+1} & Sa_m &: m^2, m^2, (2m-1)1, 1^{2m} \\
 D_4^{(m)} &: m^2, m^2, m^2, m(m-1)1 & E_6^{(m)} &: m^3, m^3, m^2(m-1)1 \\
 E_7^{(m)} &: (2m)^2, m^4, m^3(m-1)1 & E_8^{(m)} &: (3m)^2, (2m)^3, m^5(m-1)1
 \end{aligned}$$

idx = 0 ($m = 1$) : $D_4^{(1)} = [11]_4 = Ga_1 = FS_1 = Sa_1$

idx=-2 (Pidx=2), 13 basic (fundamental) tuples (cf. [01, HKNS, Ka])

| | | | |
|--------------------------------|----------------------------|------------------------------|-----------------------------|
| 8: 44,332,11111111 $[11]_5^T$ | 6: 33,2211,111111 | 5: 32,11111,11111 | 5: 221,221,11111 |
| 4: 211,1111,1111 | 4: 31,22,22,1111 <i>Sa</i> | 3: 21,21,111,111 <i>FS</i> | 2: 11,11,11,11,11 $[11]_5$ |
| 12: 66,444,2222211 $E_8^{(2)}$ | 10: 55,3331,22222 | 8: 44,2222,22211 $E_7^{(2)}$ | 6: 222,222,2211 $E_6^{(2)}$ |
| 4: 22,22,22,211 $D_4^{(2)}$ | | | |



$$\begin{aligned}
 &44, 332, 11111111 \xrightarrow[+1]{mc} 11111 \cdot 211 | 54, \underline{333} \xrightarrow{Hd} 33, \underline{111111}, \underline{2211} \\
 &\qquad\qquad\qquad 54, 333, 21111111 \\
 &66, 444, 2222211 \xrightarrow[+3]{mc} 2^4 \cdot 3^2 1 | 87, 5^3 \xrightarrow{Hd} 5^2, \underline{22^4}, \underline{33^2} 1 = 55, 3331, 22222 \\
 &66, 444, 2^5 1^2 \xleftarrow{mc} 76, 544, 2^6 1 \xleftarrow{mc} 77, 554, 32^5 1 \xleftarrow{mc} 87, 555, 3^2 2^4 1
 \end{aligned}$$

Problem [Kawakami]. Study \mathcal{P}/\sim , namely, equivalence classes of linear ODE's!

Results

Def. $\mathcal{F}^T := \{\mathbf{m} \in \mathcal{F} \mid \mathbf{m} \text{ is } T\text{-fundamental}\}$ ($T : E_8, E_7, E_6 \text{ or } D_4$)

$\mathcal{F}_\ell := \{\mathbf{m} \in \mathcal{F} \mid \text{idx } \mathbf{m} = -\ell\}$, $\mathcal{F}_\ell^T := \mathcal{F}^T \cap \mathcal{F}_\ell$.

$\mathcal{F}_{\mathbf{n}} := \{\mathbf{m} \in \mathcal{F} \mid \mathbf{m} \sim \mathbf{n}\}$, $\mathcal{F}_{\mathbf{n}}^T := \mathcal{F}^T \cap \mathcal{F}_{\mathbf{n}}$ ($\mathbf{n} \in \mathcal{F}^{E_8}$).

Theorem (1) $\mathcal{F} = \bigsqcup_{\mathbf{n} \in \mathcal{F}^{E_8}} \mathcal{F}_{\mathbf{n}}$ (i.e. $\mathcal{P}/\sim \simeq \mathcal{F}^{E_8}$)

$\text{ord } \mathbf{m} < \text{ord } \mathbf{n}$ ($\forall \mathbf{m} \in \mathcal{F}_{\mathbf{n}} \setminus \{\mathbf{n}\}, \mathbf{n} \in \mathcal{F}^{E_8}$).

(2) For $\mathbf{m} \in \mathcal{F}^T$ with $T = D_4, E_6, E_7$, we explicitly get $\mathbf{n} \in \mathcal{F}^{E_8}$ with $\mathbf{n} \sim \mathbf{m}$.

In particular, for ordered $\mathbf{n} = \bar{m}_{11}\bar{m}_{12}, \bar{m}_{21}\bar{m}_{22}\bar{m}_{23}, \bar{m}_{31}\bar{m}_{32}\bar{m}_{33} \cdots \bar{m}_{3n_3} \in \mathcal{F}^{E_8}$

- $\mathcal{F}_{\mathbf{n}}^{D_4} \neq \emptyset \Leftrightarrow \#\mathcal{F}_{\mathbf{n}}^{D_4} = 1$ and $\bar{m}_{11} = \bar{m}_{12}, \bar{m}_{21} = \bar{m}_{22}, \bar{m}_{23} = 2\bar{m}_{31} = 2\bar{m}_{32}$
- $\exists \bar{\mathbf{n}} \in \mathcal{F}^{E_8}$ such that $\mathcal{F}_{\bar{\mathbf{n}}}^{E_7} = \mathcal{F}_{\bar{\mathbf{n}}}^{E_6} = \emptyset$ ($\Rightarrow \text{idx } \bar{\mathbf{n}} < -100$)

We sometimes express $\mathbf{m} = m_{11} \cdots m_{1n_1}, m_{21} \cdots m_{2n_2}, \cdots, m_{p1} \cdots m_{pn_p}$ by

$$[[m_{11}, \dots, m_{1n_1}], [m_{21}, \dots, m_{2n_2}], \dots, [m_{p1}, \dots, m_{pn_p}]]$$

Example. $\mathcal{F}_{\bar{\mathbf{n}}}^{E_7} = \mathcal{F}_{\bar{\mathbf{n}}}^{E_6} = \mathcal{F}_{\bar{\mathbf{n}}}^{D_4} = \emptyset$ and $\text{idx } \bar{\mathbf{n}} = -630$ for

$$\bar{\mathbf{n}} = [[50, 49], [34, 33, 32], [14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4]] \in \mathcal{F}^{E_8}$$

In Theorem (2), for example, when $T = E_7$, we have the following:

For an ordered tuple

$$[[m_{11}, m_{12}], [m_{21}, m_{22}, m_{23}, m_{24}], [m_{31}, \dots, m_{3n_3}]] \in \mathcal{F}^{E_7},$$

the tuple $\mathbf{n} \in \mathcal{F}^{E_8}$ satisfying $\mathbf{n} \sim \mathbf{m}$ equals

- $[[m_{11} + m_{24}, m_{12} + m_{24}], [m_{21} + m_{24}, m_{22} + m_{24}, m_{23} + m_{24}],$
 $[m_{24}, m_{24}, m_{31}, \dots, m_{3n_3}]]$ if $m_{11} < m_{22} + m_{23}$,
- $[[m_{12} + m_{23}, m_{12} + m_{24}], [m_{12}, m_{12}, m_{23} + m_{24}],$
 $[m_{12} - m_{22}, m_{12} - m_{21}, m_{31}, \dots, m_{3n_3}]]$ if $m_{11} > m_{21} + m_{23}$,
- $[[m_{22} + m_{23} + m_{24}, m_{22} + m_{23} + m_{24}], [m_{22} + m_{23}, m_{22} + m_{24}, m_{23} + m_{24}],$
 $[m_{11} - m_{21}, m_{12} - m_{21}, m_{31}, \dots, m_{3n_3}]]$ if $m_{11} < m_{21} + m_{24}$,
- $[[m_{22} + m_{23} + m_{24}, m_{12} + m_{24}], [m_{12}, m_{22} + m_{24}, m_{23} + m_{24}],$
 $[m_{24}, m_{12} - m_{21}, m_{31}, \dots, m_{3n_3}]]$
 if $m_{11} \geq m_{22} + m_{23}$, $m_{11} \leq m_{21} + m_{23}$ and $m_{11} \geq m_{21} + m_{24}$.

The third and the last tuples are ordered. Note that $m_{12} - m_{21} \geq m_{31}$.

Proof and Examples

Example. We calculated \mathcal{F}_ℓ/\sim for $\ell \leq 100$.

Numbers of all, E_8, E_7, E_6, D_4 -fundamental tuples with a given index

| ℓ | $\#\mathcal{F}_\ell$ | $\#3$ pts | max ord | max $\#\mathcal{F}_\mathbf{n}$ | $\#\mathcal{F}_\ell^{E_8}$ | $\#\mathcal{F}_\ell^{E_7}$ | $\#\mathcal{F}_\ell^{E_6}$ | $\#\mathcal{F}_\ell^{D_4}$ |
|--------|----------------------|-----------|---------|--------------------------------|----------------------------|----------------------------|----------------------------|----------------------------|
| 0 | 4 | 3 | 6 | 4 | 1 | 1 | 1 | 1 |
| 2 | 13 | 9 | 12 | 8 | 2 | 3 | 2 | 2 |
| 4 | 37 | 25 | 18 | 9 | 5 | 6 | 6 | 4 |
| 6 | 69 | 46 | 24 | 12 | 7 | 12 | 10 | 6 |
| 8 | 113 | 73 | 30 | 23 | 10 | 15 | 12 | 8 |
| 10 | 198 | 127 | 36 | 36 | 15 | 25 | 20 | 9 |
| 20 | 1186 | 680 | 66 | 104 | 40 | 80 | 67 | 25 |
| 50 | 40617 | 18878 | 156 | 757 | 351 | 931 | 701 | 134 |
| 100 | 1461132 | 522509 | 306 | 7613 | 3181 | 10389 | 7500 | 743 |

Fact. For $\forall \mathbf{n} \in \mathcal{F}^{E_8}$ with $\text{idx } \mathbf{n} \geq -100$,

$\mathcal{F}_\mathbf{n}^{E_6} \neq \emptyset$ and $\exists \tilde{\mathbf{n}} \in \mathcal{F}_\mathbf{n}^{E_7}$ such that $\text{ord } \tilde{\mathbf{n}} \geq \text{ord } \mathbf{m}$ for $\forall \mathbf{m} \in \mathcal{F}_\mathbf{n} \setminus \{\mathbf{n}\}$.

We conjectured the above in general, but there is a counterexample.

Outline of the proof

Def. $\mathbf{m} = m_{11} \cdots m_{1n_1}, m_{21} \cdots m_{2n_2}, \dots, m_{p1} \cdots m_{pn_p} \in \mathcal{P}$

$\mathbf{m} \underset{\text{mc}}{\sim} \mathbf{m}' \stackrel{\text{def.}}{\Leftrightarrow} \exists \text{mc}_{\sigma(1)}, \dots, \text{mc}_{\sigma(k)}$ such that $\mathbf{m}' = \text{mc}_{\sigma(1)} \cdots \text{mc}_{\sigma(k)} \mathbf{m}$

$\mathcal{R}\mathbf{m} \in \mathcal{F}$ satisfying $\mathcal{R}\mathbf{m} \underset{\text{mc}}{\sim} \mathbf{m}$ ($W\mathcal{F} = \mathcal{P}$)

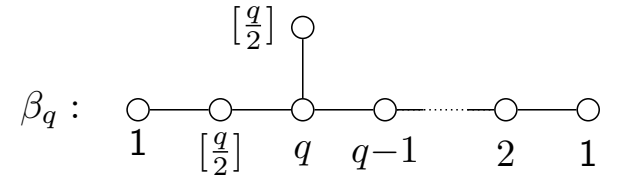
$\mathcal{S}\mathbf{m} := [[(p-1)n, n], \overbrace{[n, \dots, n]}^p, [m_{11}, \dots, m_{p1}, \{m_{j\nu}\}_{\nu=2, \dots, n_j, j=1, \dots, p}]]$
 : unfolding of Harnad dual of $n, \underline{0}m_{11} \cdots, \underline{0}m_{1n_1}, \dots, \underline{0}m_{p1} \cdots m_{pn_p}$
 : unfolding of $(m_{11} \cdots m_{1n_1}) \cdots (m_{p1} \cdots m_{pn_p}), \underline{((p-1)n)n}$

$\mathcal{S}^2\mathbf{m} := [[2pn, pn], [pn, pn, pn], [(p-1)n, \overbrace{[n, \dots, n]}^{p+1}, \{m_{j,\nu}\}]]$

Remark. $\mathcal{RS}^2 : \mathcal{P} \rightarrow \mathcal{F}^{E_8}, \quad \text{mc}_{\sigma} \mathbf{m} = \text{Hd}_2\text{Hd}_{\sigma}(n, \mathbf{m})$

Lemma. (1) $\mathbf{m} : (2, 3, *)\text{-type} \Rightarrow \mathcal{S}\mathbf{m} \underset{\text{mc}}{\sim} \mathbf{m}$

$\mathbf{m} : (*, *, *)\text{-type (i.e. } p = 3) \Rightarrow \mathcal{S}^2\mathbf{m} \underset{\text{mc}}{\sim} \mathcal{S}\mathbf{m}$



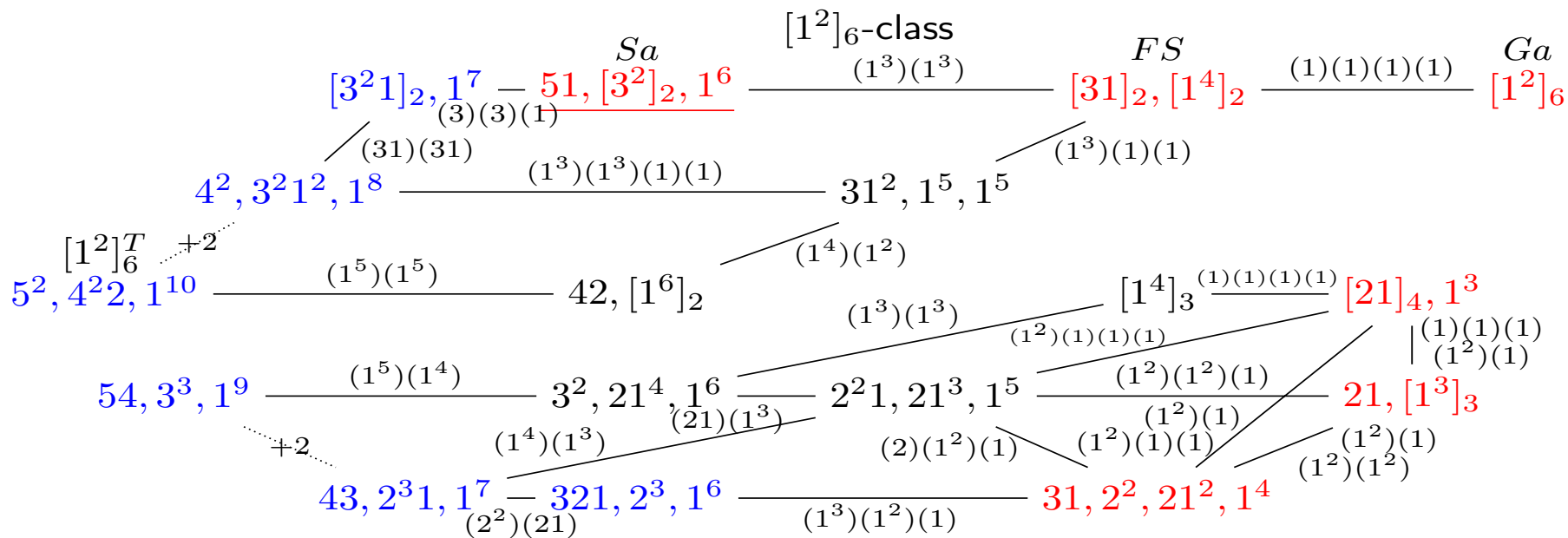
(2) $\mathbf{m}' : \text{an unfolding of Harnad dual of a confluence of } \mathbf{m} \Rightarrow \mathcal{S}^2\mathbf{m}' \underset{\text{mc}}{\sim} \mathcal{S}^2\mathbf{m}$

In particular, $\mathcal{S}^2\text{mc}_{\sigma}\mathbf{m} = s_{\beta_{\sigma}}\mathcal{S}^2\mathbf{m}$ ($\exists \beta_{\sigma} : \text{a real root}$)

(3) $\text{ord } \mathcal{RS}\mathbf{m} > \text{ord } \mathbf{m}$ ($\mathbf{m} \in \mathcal{F} \setminus \mathcal{F}^{E_8}$)

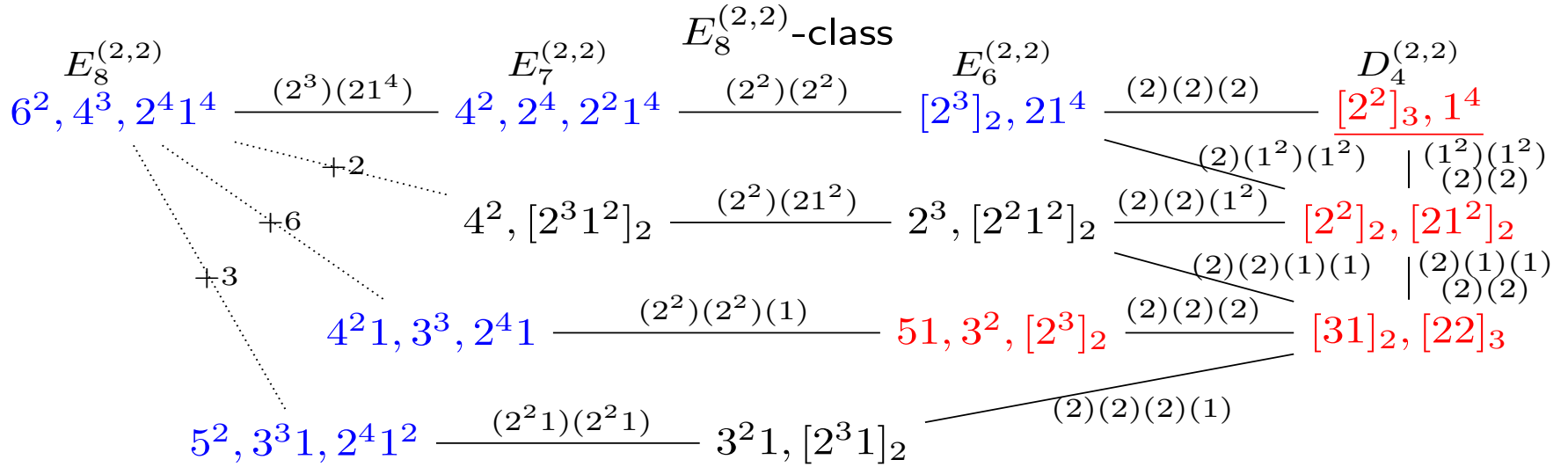
idx=-4 (Pidx=3), 37 basic (fundamental) tuples

| | | | |
|-----------------------------------|-------------------------------|------------------------------|--------------------------------|
| 10: 55,442,1111111111 $[11]_6^T$ | 8: 44,3311,11111111 | 7: 331,331,1111111 | 6: 42,111111,111111 |
| 6: 51,33,33,111111 <i>Sa</i> | 5: 311,11111,11111 | 4: 31,31,1111,1111 <i>FS</i> | 2: 11,11,11,11,11,11 $[11]_6$ |
| 18: 99,666,3333321 $E_8^{(3)}$ | 12: 66,3333,33321 $E_7^{(3)}$ | 9: 333,333,3321 $E_6^{(3)}$ | 6: 33,33,33,321 $D_4^{(3)}$ |
| 9: 54,333,111111111 | 7: 43,2221,11111111 | 6: 321,222,111111 | 6: 33,21111,11111 |
| 5: 221,2111,11111 | 4: 1111,1111,1111 | 4: 31,22,211,1111 | 3: 21,111,111,111 |
| 3: 21,21,21,21,111 | | | |
| 12: 66,444,22221111 $E_8^{(2,2)}$ | 10: 55,3331,222211 | 9: 441,333,22221 | 8: 44,22211,22211 |
| 8: 44,2222,221111 $E_7^{(2,2)}$ | 7: 331,2221,2221 | 6: 222,2211,2211 | 6: 222,222,21111 $E_6^{(2,2)}$ |
| 6: 51,33,222,222 | 4: 22,22,211,211 | 4: 31,31,22,22,22 | 4: 22,22,22,1111 $D_4^{(2,2)}$ |
| 14: 77,554,2222222 | 10: 55,3322,22222 | 8: 332,332,2222 | 6: 42,33,33,222 |



$$55, 442, 111111111 \xrightarrow[+2]{\text{mc}} 1111111 \cdot 311 | 75, \underline{444} \xrightarrow{\text{Hd}} 44, \underline{11111111}, \underline{3311}$$

$$54, 333, 111111111 \xrightarrow[+2]{\text{mc}} 1^6 \cdot 221 | 65, \underline{443} \xrightarrow{\text{Hd}} 43, \underline{1111111}, \underline{2221}$$



$$66, 444, 22221111 \xrightarrow[+2]{\text{mc}} 2221 \cdot 2221 | 77, \underline{644} \xrightarrow{\text{Hd}} 44, \underline{12221}, \underline{12221}$$

$$66, 444, 22221111 \xrightarrow[+6]{\text{mc}} 2221 \cdot 33 \cdot 41 | 765, \underline{99} \xrightarrow{\text{Hd}} \underline{22221}, \underline{333}, \underline{441}$$

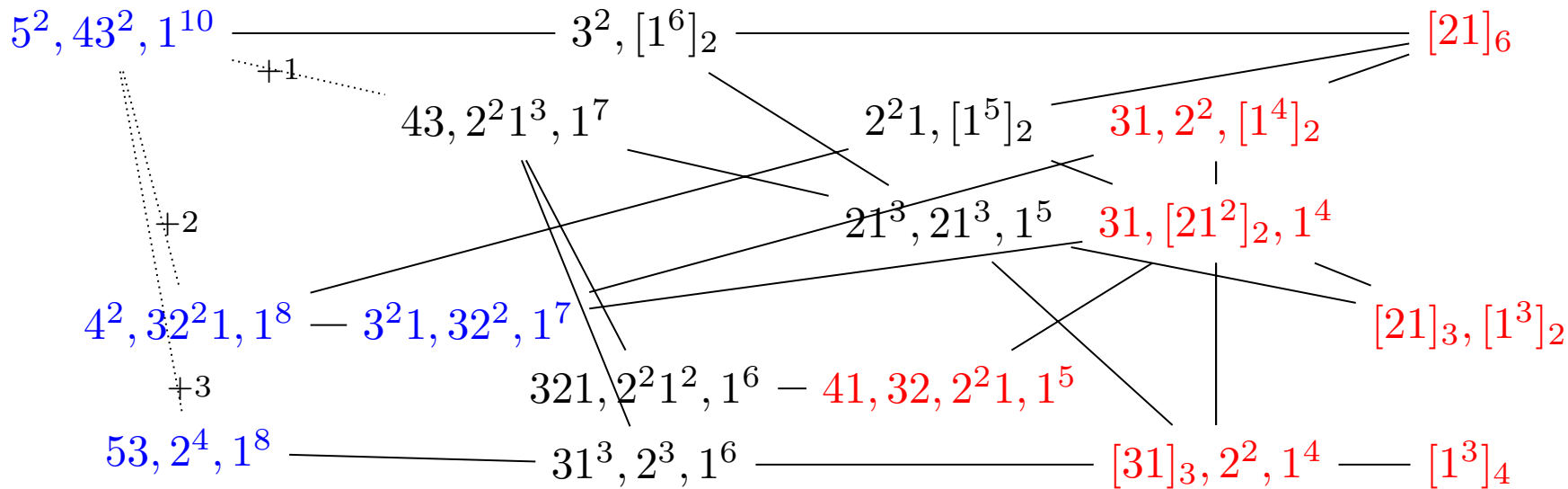
$$66, 444, 22221111 \xrightarrow[+3]{\text{mc}} 22211 \cdot 331 | 87, \underline{555} \xrightarrow{\text{Hd}} 55, \underline{222211}, \underline{3331}$$

$$77, 554, 2222222 \xrightarrow[+1]{\text{mc}} 2^4 \cdot 32^2 | 87, 5^3 \xrightarrow{\text{Hd}} 5^2, \underline{22^4}, \underline{332^2} = 55, 3322, 22222$$

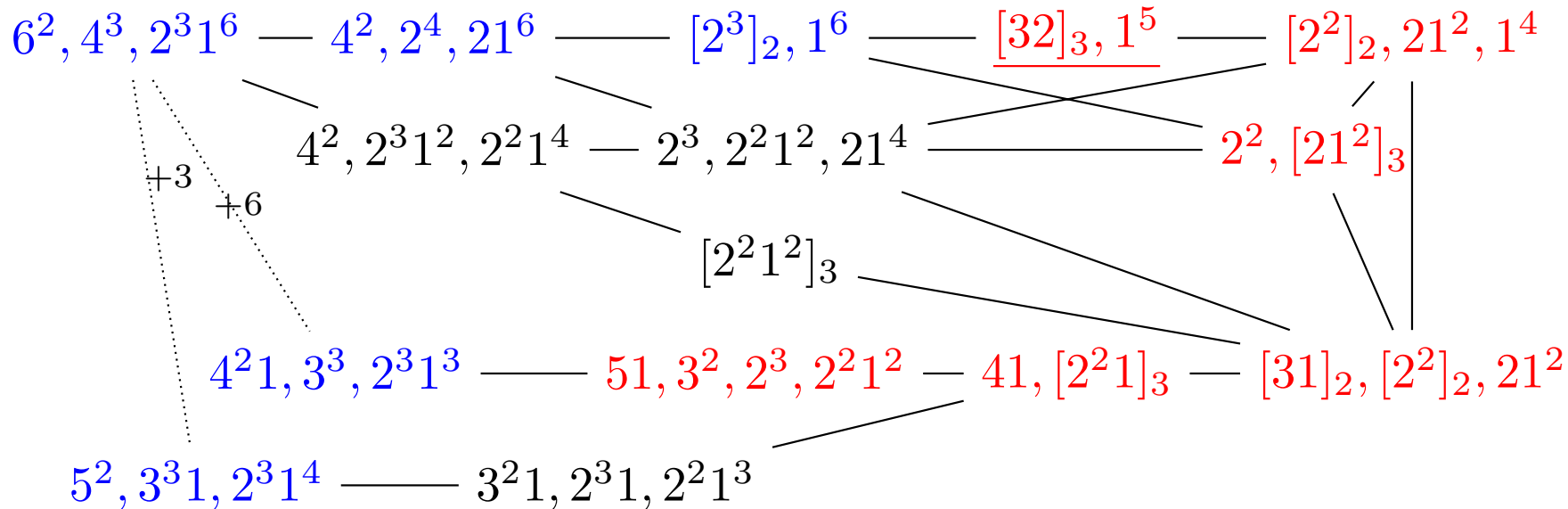
$$\xrightarrow{\text{conf.}} 32 \cdot 32 | 55, \underline{22222} \xrightarrow{\text{Hd}} \underline{2222}, \underline{332}, \underline{332} \xrightarrow{\text{conf.}} 3 \cdot 3 \cdot 2 | 332, \underline{2222} \xrightarrow{\text{Hd}} \underline{222}, \underline{33}, \underline{33}, \underline{42}$$

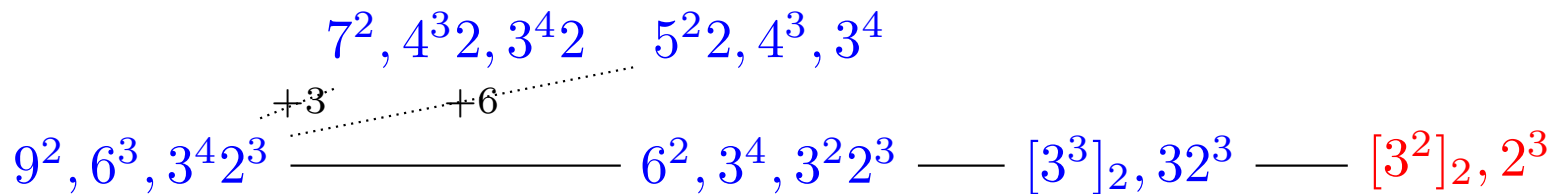
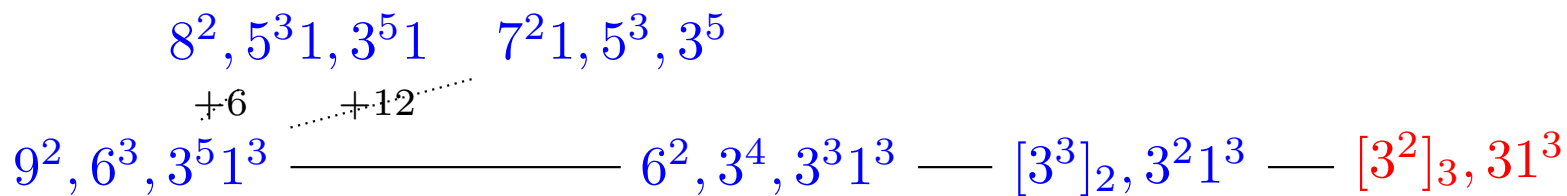
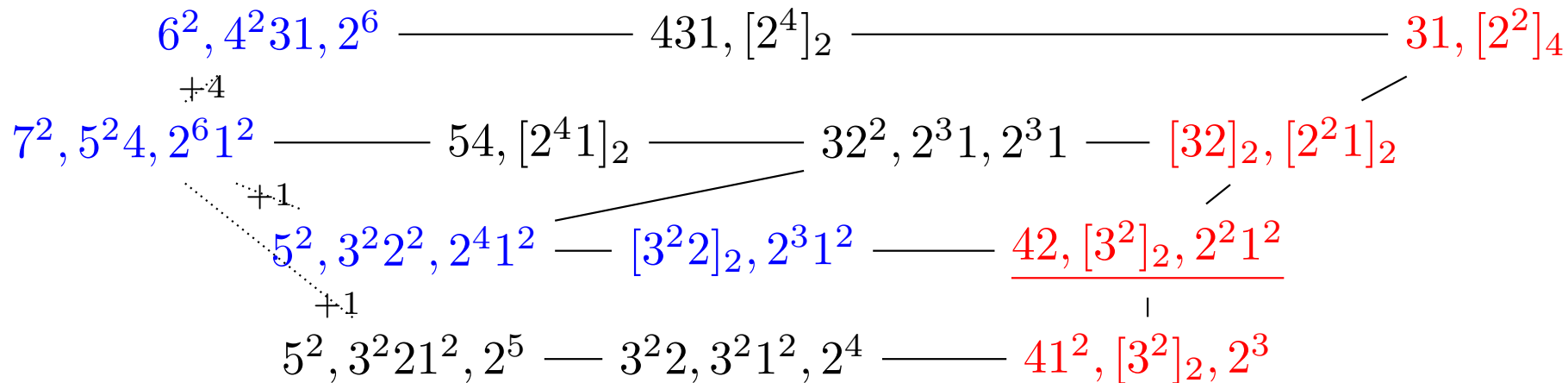
Fundamental class of 69 tuples with $\text{idx}=-6$ ($\text{Pidx}=4$)

| | | | |
|------------------------------------|--------------------------------|---------------------------------|----------------------------------|
| 12: 66,552,111111111111 $[11]_7^T$ | 10: 55,4411,1111111111 | 9: 441,441,111111111 | 8: 71,44,44,11111111 <i>Sa</i> |
| 7: 52,1111111,1111111 | 6: 411,111111,111111 | 5: 41,41,11111,11111 <i>FS</i> | 2: 11,11,11,11,11,11,11 |
| 24: cc,888,4444431 $E_8^{(4)}$ | 16: 88,4444,44431 $E_7^{(4)}$ | 12: 444,444,4431 $E_6^{(4)}$ | 8: 44,44,44,431 $D_4^{(4)}$ |
| 10: 55,433,1111111111 | 8: 53,2222,11111111 | 8: 44,3221,11111111 | 7: 43,22111,1111111 |
| 7: 331,322,1111111 | 6: 33,111111,111111 | 6: 321,2211,111111 | 6: 3111,222,111111 |
| 5: 221,11111,11111 | 5: 2111,2111,11111 | 5: 41,32,221,11111 | 4: 31,22,1111,1111 |
| 4: 31,211,211,1111 | 4: 31,31,31,22,1111 | 3: 111,111,111,111 | 3: 21,21,21,111,111 |
| 3: 21,21,21,21,21,21 | | | |
| 12: 66,444,222111111 $E_8^{(2,3)}$ | 10: 55,3331,2221111 | 9: 441,333,222111 | 8: 44,2222,2111111 $E_7^{(2,3)}$ |
| 8: 44,22211,221111 | 7: 331,2221,22111 | 6: 222,222,111111 $E_6^{(2,3)}$ | 6: 222,2211,21111 |
| 6: 2211,2211,2211 | 6: 51,33,222,2211 | 5: 41,221,221,221 | 5: 32,32,32,11111 |
| 4: 22,22,211,1111 $D_4^{(2,3)}$ | 4: 22,211,211,211 | 4: 31,31,22,22,211 | |
| 14: 77,554,22222211 | 12: 66,4431,222222 | 10: 55,3322,222211 | 10: 55,33211,22222 |
| 9: 54,22221,22221 | 8: 431,2222,2222 | 8: 332,332,22211 | 8: 332,3311,2222 |
| 7: 322,2221,2221 | 6: 42,33,33,2211 | 6: 411,33,33,222 | 5: 32,32,221,221 |
| 4: 31,22,22,22,22 | | | |
| 18: 99,666,33333111 $E_8^{(3,1)}$ | 16: 88,5551,333331 | 15: 771,555,33333 | 12: 66,3333,333111 $E_7^{(3,1)}$ |
| 9: 333,333,33111 $E_6^{(3,1)}$ | 6: 33,33,33,3111 $D_4^{(3,1)}$ | | |
| 18: 99,666,3333222 $E_8^{(3,2)}$ | 14: 77,4442,33332 | 12: 66,3333,33222 $E_7^{(3,2)}$ | 12: 552,444,3333 |
| 9: 333,333,3222 $E_6^{(3,2)}$ | 6: 33,33,33,222 $D_4^{(3,2)}$ | | |



Here, $[1^6]_2 = 111111, 111111$ $[21]_6 = 21, 21, 21, 21, 21, 21$ etc.

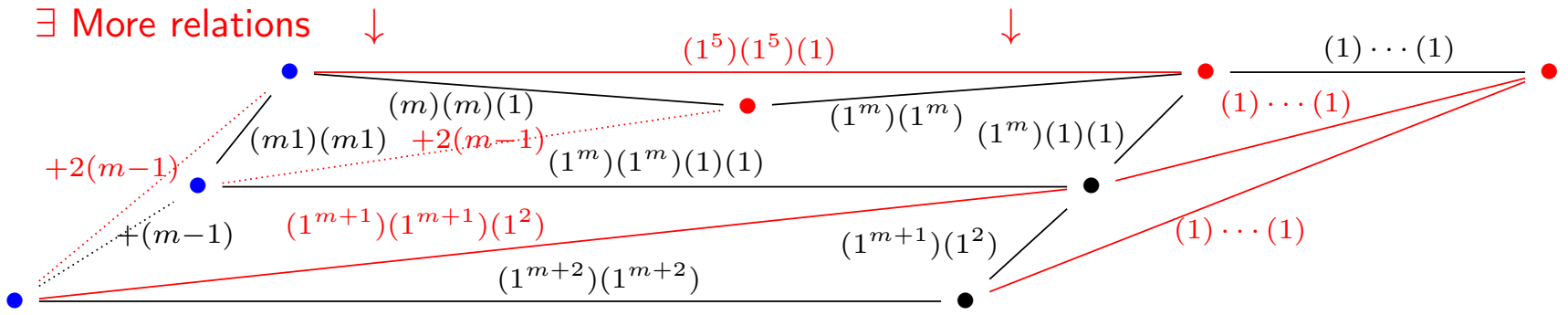




$$\text{idx} = 2(m - 1)$$

$$\begin{array}{c}
 [1^2]_{m+3}\text{-class} \\
 \begin{array}{c}
 \text{Sa} \\
 \text{FS} \\
 \text{Ga}
 \end{array} \\
 \frac{[m^2 1]_2, 1^{2m+1} \frac{(m)(m)(1)}{(m1)(m1)} (2m-1) 1, [m^2]_2, 1^{2m} \frac{(1^m)(1^m)}{(1^m)(1)(1)} [m 1]_2, [1^{m+1}]_2 \frac{(1) \cdots (1)}{(1^m)(1)(1)} [1^2]_{m+3}}{(m+1)^2, m^2 1^2, 1^{2m+2} \frac{(1^m)(1^m)(1)(1)}{(1^{m+2})(1^{m+2})} m 1^2, 1^{m+2}, 1^{m+2} \frac{(1^{m+1})(1^2)}{(1^{m+2})(1^{m+2})} (m+1) 2, [1^{m+3}]_2}
 \end{array}$$

∃ More relations



$$1^{2m+4}, (m+2)^2, (m+1)^2 2 \xleftarrow{\frac{mc}{m-1}} 1^{2m+1} \cdot m 1^2 | (2m+1)(m+2), (m+1)^3 \xrightarrow{\text{Hd}} (m+1)^2, \underline{1} 1^{2m+1}, \underline{m} m 1^2$$

$$\begin{array}{c}
 E_8^{(m)} \\
 (3m)^2, (2m)^3, m^5(m-1) 1 \xrightarrow{(m^3)(m^2 m - 11)} (2m)^2, m^4, m^3(m-1) 1 \xrightarrow{E_7^{(m)} \frac{(m^2)(m^2)}{[m^3]_2, m^2(m-1) 1}} [m^3]_2, m^2(m-1) 1 \xrightarrow{E_6^{(m)} \frac{(m^3)}{[m^2]_3, m(m-1) 1}} [m^2]_3, m(m-1) 1 \xrightarrow{D_4^{(m)}}
 \end{array}$$

$D_4^{(m)} - E_8^{(m)}$ -class

$$\text{idx} = -2 : D_4^{(2)} - E_8^{(2)}\text{-class contains } 5^3, 3^3 1, 2^5$$

Fundamental class of 113 tuples with idx=-8 (Pidx=5)

| | | | |
|---|--|--|---|
| 14: 77,662,11111111111111 8: 62,11111111,11111111 | 12: 66,5511,111111111111 7: 511,1111111,1111111 | 11: 551,551,111111111111 6: 51,51,111111,111111 | 10: 91,55,55,1111111111 2: 11,11,11,11,11,11,11,11 |
| 30: ff,aaa,5555541 | 20: aa,5555,55541 | 15: 555,555,5541 | 10: 55,55,55,541 |
| 12: 66,444,221111111111 8: 44,22211,2111111 7: 322,322,1111111 6: 51,33,222,21111 5: 32,32,311,11111 4: 31,31,22,22,1111 | 10: 55,3331,22111111 8: 44,221111,221111 6: 222,2211,111111 6: 51,33,2211,2211 4: 22,22,1111,1111 4: 31,31,22,211,211 | 9: 441,333,2211111 7: 331,2221,211111 6: 222,21111,21111 6: 42,42,33,111111 4: 22,211,211,1111 4: 31,31,31,31,22,22 | 8: 44,2222,11111111 7: 331,22111,22111 6: 2211,2211,21111 5: 41,221,221,2111 4: 211,211,211,211 |
| 14: 77,554,2222211111 10: 55,33211,222211 8: 431,2222,22211 8: 71,44,332,2222 6: 42,33,33,21111 5: 32,311,221,221 | 12: 66,4431,2222211 10: 55,331111,22222 8: 332,332,221111 7: 322,2221,22111 6: 411,33,33,2211 4: 31,22,22,22,211 | 11: 551,443,222221 9: 54,22221,222111 8: 332,3311,22211 7: 3211,2221,2221 6: 51,51,33,33,222 | 10: 55,3322,2221111 9: 441,3321,22221 8: 3311,3311,2222 6: 51,321,222,222 5: 32,32,221,2111 |
| 11: 65,443,111111111111 8: 431,332,11111111 6: 321,21111,111111 5: 2111,11111,11111 4: 31,31,31,211,1111 | 9: 54,3321,11111111 7: 43,211111,111111 6: 3111,2211,111111 5: 41,32,2111,11111 3: 21,21,111,111,111 | 8: 53,22211,11111111 7: 421,2221,1111111 6: 51,42,222,111111 5: 41,311,221,11111 3: 21,21,21,21,21,111 | 8: 44,32111,11111111 7: 331,3211,1111111 6: 51,33,321,111111 4: 31,211,1111,1111 |
| 18: 99,666,33332211 12: 66,3333,332211 10: 442,3331,3331 6: 33,33,321,321 | 16: 88,5551,333322 12: 66,33321,33321 9: 333,333,32211 6: 51,42,33,33,33 | 14: 77,4442,333311 12: 5511,444,3333 9: 333,3321,3321 | 14: 77,44411,33332 11: 551,3332,3332 6: 33,33,33,2211 |
| 15: 87,555,22222221 8: 332,3221,2222 | 11: 65,3332,222221 6: 42,33,321,222 | 10: 55,32221,22222 | 9: 432,333,22221 |
| 20: aa,776,3333332 | 14: 77,4433,33332 | 11: 443,443,3332 | 8: 53,44,44,332 |
| 24: cc,888,4444422 8: 44,44,44,422 | 20: aa,6662,44444 | 16: 88,4444,44422 | 12: 444,444,4422 |
| 16: 88,664,22222222 8: 422,2222,2222 | 12: 66,4422,222222 8: 62,44,44,2222 | 10: 64,22222,22222 6: 42,42,222,222 | 10: 442,442,22222 4: 22,22,22,22,22 |

The tables of the classification are calculated for $\text{idx} \geq -100$ (cf. [O4]) by `spfundclass()` in [O3], 2026.2.8.

`spbasic()` : get required irreducibly realizable tuples (idx , ord , $\#\text{partitions}$)

`spharnadcv()` : calculate mc_σ , Hd_σ

`spdiagram()` : draw a diagram of tuples with relations defined by mc_σ , Hd_σ

`spfundauto()` : get the automorphism group of a tuple defined by mc_σ

`sptab()` : draw table of classified tuples

`refinements()` : get all confluences of a tuple

Classification of fundamental tuples using Risa/Asir

| idx | 0 | -2 | -4 | -6 | -8 | -10 | -12 | -14 | -16 | -18 | -20 |
|---------------|------|------|------|------|------|------|------|------|------|------|------|
| run time(sec) | 0.02 | 0.03 | 0.05 | 0.09 | 0.13 | 0.25 | 0.44 | 0.72 | 1.89 | 2.77 | 3.00 |
| # fund | 4 | 13 | 37 | 69 | 113 | 198 | 291 | 415 | 647 | 884 | 1186 |
| # class | 1 | 2 | 5 | 7 | 10 | 15 | 18 | 21 | 33 | 36 | 40 |

`spDEfromE8()` : get D_4 , E_6 , E_7 -f. equivalent to a given E_8 -fundamental

Thank you for your attention!

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