Hodge atoms and their moduli spaces

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Gromov–Witten potential

- X: smooth projective variety over $\mathbb{C}$ with $H^{2n+1}(X,\mathbb{Q}) = 0$
- $T_1, \dots, T_m$: basis of $H^2(X,\mathbb{Q})$
- $T_0 = 1, T_{m+1}, \dots, T_M$: basis of $H^{\neq 2}(X,\mathbb{Q})$
- $(\vec{t},\vec{q}) = (t_0, q_1, \dots, q_m, t_{m+1}, \dots, t_M)$: formal variables
- parameterization of $H^*(X,\mathbb{Q})$

$$t_0 T_0 + q_1 T_1 + \cdots + q_m T_m + t_{m+1} T_{m+1} + \cdots + t_M T_M$$

- $\beta \in H_2(X,\mathbb{Z})$ and $\beta_1, \dots, \beta_k \in H^*(X,\mathbb{Q})$
- $I_\beta(\beta_1, \dots, \beta_k) \in \mathbb{Q}$: genus 0 Gromov–Witten invariant (Kontsevich–Manin '94) $\rightsquigarrow$ Gromov–Witten potential

$$F(\vec{t},\vec{q}) = \sum I_\beta(T_{m+1}^{n_{m+1}}, \dots, T_M^{n_M}) q_1^{\int_\beta T_1} \cdots q_m^{\int_\beta T_m} \frac{t_{m+1}^{n_{m+1}}}{n_{m+1}!} \cdots \frac{t_M^{n_M}}{n_M!}$$

(in principle just a formal power series, but assumed to be convergent)

Quantum cohomology

- for $1 \leq i \leq m$ let $\partial_i = q_i \frac{\partial}{\partial q_i}$
- otherwise, let $\partial_i = \frac{\partial}{\partial t_i}$
- define $F_{ije} = \partial_i \partial_j \partial_e F$
- set $g_{ij} = \int_X T_i \cup T_j$
- $(g^{ef}) =$ inverse matrix of (g_{ij})
- $T_i *_{\vec{t},\vec{q}} T_j = T_i \cup T_j + F_{ije} g^{ef} T_f$ deformation of $(H^*(X, \mathbb{C}), \cup)$, parameterized by $H^*(X, \mathbb{C})$
- Witten–Dijkgraaf–Verlinde–Verlinde formula: $*_{\vec{t},\vec{q}}$ is associative

Semisimplicity

Definition

We say that X has semisimple quantum cohomology if there exists an open set $U \subset \operatorname{Spec} \mathbb{C}\{\vec{t}, \vec{q}\}$ such that for any $(\vec{t}, \vec{q}) \in U$ the algebra $(H^*(X, \mathbb{C}), *_{\vec{t},\vec{q}})$ is semisimple.

Artin–Wedderburn: $(H^*(X, \mathbb{C}), *_{\vec{t},\vec{q}})$ is isomorphic to $\mathbb{C}^{\oplus N}$.

Theorem (Bayer–Manin '04, Hertling–Manin–Teleman '09)

If X has semisimple quantum cohomology, then

- $H^{2n+1}(X, \mathbb{Q}) = 0$
- *X is of Hodge–Tate type, i.e. $h^{p,q} = 0$ unless $p = q$.*

Consequence

The quantum cohomology of many interesting varieties X is not semisimple.

Dubrovin (or quantum) connection

Dubrovin '96:

- $S = \operatorname{Spec} \mathbb{C}\{\vec{t}, \vec{q}\}[\vec{q}^{-1}]$
- z: further variable
- $S \times \operatorname{Spec} \mathbb{C}[z^{\pm 1}]$ base scheme
- $H^*(X, \mathbb{C}) \times S \times \operatorname{Spec} \mathbb{C}[z^{\pm 1}] \to S \times \operatorname{Spec} \mathbb{C}[z^{\pm 1}]$ trivial vector bundle (pull-back of TS)
- Euler vector field: $\mathrm{Eu} = c_1(T_X) + \sum (2 - \deg T_i) t_i T_i$
- ∇^X meromorphic integrable connection:

$$\begin{aligned} \nabla^X_{\partial_z} =& \frac{\partial}{\partial z} + \frac{1}{z^2} \mathrm{Eu} *_{\vec{t},\vec{q}} \\ & + \frac{1}{z} \bigoplus_d \frac{d - \dim_{\mathbb{C}} X}{2} \mathrm{I}_{H^d} \\ (\text{notation}) =& \frac{\partial}{\partial z} + K_{-2} z^{-2} + K_{-1} z^{-1} \end{aligned}$$

Hodge atoms of X

Katzarkov–Kontsevich–Pantev–Yu '26:

- $\widetilde{S} = \operatorname{Spec} \mathbb{C}\{\vec{t}, \vec{q}\}[\vec{q}^{-1}, \zeta]/(\det(\mathrm{Eu} *_{\vec{t},\vec{q}} - \zeta)) \to S$ spectral cover (finite)
- $S^{Hod} = (\oplus_i H^{i,i}(X)) \times \operatorname{Spec} \mathbb{C}[z^{\pm 1}] \subset S$
- $U \subset S^{Hod}$ open subset where $\deg(\widetilde{S}_{reduced}|S)$ is maximal
- $\widetilde{U} = \widetilde{S}_{reduced} \times_{S^{Hod}} U$ unramified part of spectral cover
- Hodge atoms of X: $\pi_0(\widetilde{U})$, irreducible components of $\widetilde{S}$
- Operations on varieties: union, $\mathrm{Bl}_Z X$, projective bundles $\rightsquigarrow$ equivalence relation $\sim$ on atoms of different varieties
- Iritani's blow-up theorem ('23): detailed understanding of $\sim$
- $\Rightarrow$ spectacular applications to Birational Geometry

Stack of Non-Semisimple Quantum Connections

- All known applications have $\deg(\widetilde{S}_{reduced}|S) > 1$.
- We say that $*_{\vec{t},\vec{q}}$ is maximally non-semisimple if $\mathrm{Eu} *_{\vec{t},\vec{q}}$ has only one eigenvalue λ.
- What to do if $*_{\vec{t},\vec{q}}$ is maximally non-semisimple?
- Fix $r \geq 2$ and $\vec{c} = (c_1, \ldots, c_r) \in (\mathbb{C}^\times)^r \subset \mathrm{GL}(r, \mathbb{C})$ diagonal maximal torus
- Aim: introduce a stack $NSQC_{\vec{c}}$ of Non-Semisimple Quantum Connections, where Hodge atoms of projective varieties X with maximally non-semisimple quantum cohomology live.

First observations

- Assume that $*_{\vec{t},\vec{q}}$ is maximally non-semisimple
- Up to $\mathrm{Ad}(e^{\frac{\lambda}{z}})$, may assume $\lambda = 0$
- So, we have

$$\nabla^X = \mathrm{d} + (K_{-2}z^{-2} + K_{-1}z^{-1})\,\mathrm{d}z$$

  with K_{-2} nilpotent
- $B \subset \mathrm{GL}(r,\mathbb{C})$ upper triangular Borel subgroup
- $\mathfrak{b} \subset \mathfrak{gl}_r(\mathbb{C})$ Lie algebra of B, $\mathfrak{n} \lhd \mathfrak{b}$ nilpotent radical
- there exists $g_0 \in \mathrm{GL}(r,\mathbb{C})$ such that $g_0^{-1}K_{-2}g_0 \in \mathfrak{n}$

Proposition

Assume that K_{-2} is regular nilpotent. TFAE:

1. ∇ *is of unramified Hukuhara–Turrittin type*
2. $g_0^{-1}K_{-1}g_0 \in \mathfrak{b}$
3. ∇^X *has regular singularity at* 0.

Objects of $NSQC_{\vec{c}}$

Fix $\varepsilon > 0$, set $\Delta = \{|z| < 1 + \varepsilon\}$, $\mathcal{O} = \mathcal{O}_\Delta, \Omega = \Omega^1_\Delta$
Objects: triples $(E, \nabla, (\mathbf{v}_1(1), \ldots, \mathbf{v}_r(1)))$, with

- E free $\mathcal{O}$-module of rank r,
- $\nabla \colon E \to E \otimes \Omega(2 \cdot \{0\})$ meromorphic connection
- $(\mathbf{v}_1(1), \ldots, \mathbf{v}_r(1))$ basis of $E\big|_1$

satisfying the conditions:

- ∇ has regular singularity at 0
- the flag of $E\big|_1$ determined by $(\mathbf{v}_1(1), \ldots, \mathbf{v}_r(1))$ is preserved by the monodromy transformation T_∇
- T_∇ has c_i as generalized eigenvalue associated to $\mathbf{v}_i(1)$.

Morphisms of $NSQC_{\vec{c}}$

Given by the group

$$\mathrm{Aut}(E) = \{g \colon \Delta \to \mathrm{GL}(r, \mathbb{C}) \text{ analytic}\}$$

acting on the right by gauge transformations

$$\nabla \mapsto g^{-1} \circ \nabla \circ g$$

and by the defining representation

$$(\mathbf{v}_1(1), \ldots, \mathbf{v}_r(1)) \mapsto (\mathbf{v}_1(1), \ldots, \mathbf{v}_r(1))g(1).$$

Main result

Theorem (Kontsevich–Sz. '23 (= 3 B.A.I.), Moduli of Atoms of Complex Projective Varieties, to appear in: Advances in Complex Lagrangians, Integrable Systems and Quantisation, World Scientific Studies in Pure Mathematics: Volume 1 (2026))

The connected components of the coarse moduli space of the stack $NSQC_{\vec{c}}$ *are isomorphic to* $\mathfrak{n} \oplus \mathfrak{n}$.

Choosing a logarithmic lattice of $E \otimes_{\mathcal{O}} \mathcal{O}[z^{-1}]$

Let $A \in \mathfrak{gl}_r(\mathbb{C})$ satisfy

$$\exp(2\pi\sqrt{-1}A) = T_\nabla$$

(a discrete choice).
Deligne: $\exists$ a $\mathcal{O}[z^{-1}]$-basis $(\mathbf{v}'_1, \mathbf{v}'_2 \ldots, \mathbf{v}'_r)$ of $E \otimes_{\mathcal{O}} \mathcal{O}[z^{-1}]$ such that

- the connection matrix of ∇ is $A\,\mathrm{d}z/z$
- $\mathbf{v}'_i(1) = \mathbf{v}_i(1)$.

Construction of a new trivialization of E

For all $1 \leq i \leq r$, recursively set

$$n_i = \min\{n \in \mathbb{Z} \colon\ z^n \mathbf{v}'_i \in E + \mathcal{O}[z^{-1}]\mathbf{v}'_{i-1} + \cdots + \mathcal{O}[z^{-1}]\mathbf{v}'_1\}.$$

Proposition

There exist unique

- $b_{1i}, \ldots, b_{i-1,i} \in z^{-1}\mathbb{C}[z^{-1}]$ *for each* $1 \leq i \leq r$
- $\mathcal{O}$*-basis* $\mathbf{w}_1, \ldots, \mathbf{w}_r$ *of* E

such that

$$z^{n_i}\mathbf{v}'_i = \mathbf{w}_i - b_{i-1,i}\mathbf{w}_{i-1} - \cdots - b_{1i}\mathbf{w}_1.$$

Generalized Levelt normal form

Define $g \in \mathrm{Aut}(E)$ by

$$(\mathbf{w}_1, \ldots, \mathbf{w}_r) \cdot g = (\mathbf{e}_1, \ldots, \mathbf{e}_r), \text{ and write}$$

$$\nabla \cdot g = \mathrm{d} + D\,\mathrm{d}z = \mathrm{d} + \sum_{n=-2}^{\infty} D_n z^n\,\mathrm{d}z$$

Denote by d_{ii} the i'th diagonal entry of D and $\mu_i = \mathrm{res}_0\, d_{ii}$.

Proposition

- $\exists N' \in \mathbb{N}$ *such that* $D_n = 0$ *for all* $n > N'$
- $D_{-2} \in \mathfrak{n}$ *and* D_{-2} *is conjugate to* K_{-2}
- $D_n \in \mathfrak{b}$ *for every* $-1 \leq n \leq N'$
- $T_{\nabla \cdot g}$ *leaves invariant the flag defined by* $\mathbf{e}_1(1), \ldots, \mathbf{e}_r(1)$,
- *the* i*'th eigenvalue of* $T_{\nabla \cdot g}$ *is equal to* $\exp(2\pi\sqrt{-1}\mu_i)$.

Coarse moduli space

Proposition

The coarse moduli space of $NSQC_{\vec{c}}$ *is given by*

$$\mathcal{A}/\operatorname{Aut}_0(E)$$

with

$$\mathcal{A} = \left\{ \mathrm{d} + \sum_{n=-2}^{N'} D_n z^n \,\mathrm{d}z \middle| \quad D_{-2} \in \mathfrak{n}, D_{-1}, \ldots, D_{N'} \in \mathfrak{b} \right\}$$

and

$$\operatorname{Aut}_0(E) = \{\ell \in \operatorname{Aut}(E) \mid \quad \ell \in B,\, \ell(1) = I_r\}.$$

Normal form of polynomial objects

For every $1 \leq j < i \leq r$, let us set

$$m(j,i) = \begin{cases} \mu_j - \mu_i & \text{if } \mu_j - \mu_i \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition

Given any $\nabla \in \mathcal{A}$, there exist a unique polynomial $\ell \in \mathrm{Aut}_0(E)$ and unique $\alpha_{ji}, \beta_{ji} \in \mathbb{C}$ such that for all $1 \leq j < i \leq r$, the (ji)-entry of $\nabla \cdot \ell$ is equal to

$$\alpha_{ji} z^{-2} + \beta_{ji} z^{m(j,i)-1}.$$

Corollary

The moduli space $NSQC_{\vec{c}}$ is parameterized by $(\alpha_{ji}), (\beta_{ji}) \in \mathfrak{n} \oplus \mathfrak{n}$.

Key technical lemma

For fixed $\mu \in \mathbb{C}$ define

$$L_\mu \colon \{P \in \mathbb{C}[z] \colon P(1) = 0\} \to z^{-2}\mathbb{C}[z]$$
$$P \mapsto z^\mu \partial_z \left(z^{-\mu} P\right).$$

Lemma

- *For* $\mu \notin \mathbb{N}$, $\operatorname{coker}(L_\mu)$ *is generated by* $\mathbb{C} \cdot z^{-2}, \mathbb{C} \cdot z^{-1}$.
- *For* $\mu \in \mathbb{N}$, $\operatorname{coker}(L_\mu)$ *is generated by* $\mathbb{C} \cdot z^{-2}, \mathbb{C} \cdot z^{\mu-1}$.

Time for questions

Thank you for your attention!