

# Compactification of the fifth Painlevé Foliation

Web-seminar on Painlevé Equations and related topics

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# Meromorphic Connections

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# MEROMORPHIC CONNECTIONS ON $\mathbb{P}^1$

## Definition

A rank 2 meromorphic connection  $(E, D, \nabla)$  on  $\mathbb{P}^1$  is the data of:

- a rank 2 holomorphic vector bundle  $E \rightarrow \mathbb{P}^1$ ,
- a effective divisor  $D$  of  $\mathbb{P}^1$  called the polar divisor,
- a morphism  $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^1(D)$ .

## Remark

The sheaf  $\Omega^1(D) := \Omega^1 \otimes \mathcal{O}(D)$  is the sheaf of meromorphic 1-forms with poles as prescribed by  $D$ .

## Leibniz Rule

The operator  $\nabla$  is required to satisfy

$$\nabla(f\sigma) = df \cdot \sigma + f\nabla\sigma$$

# LOCAL EXPRESSION

## Trivialisation

On a trivializing open set  $U$ , a connection  $(E, D, \nabla)$  has the form

$$\nabla|_U = d + \Omega_U, \quad \text{for } \Omega_U \in \mathfrak{gl}_2(\Omega^1(D \cap U)).$$

## Gluing conditions

Any connection  $(E, D, \nabla)$  on  $\mathbb{P}^1$  is then identified with the data

$$\begin{cases} d + \Omega_0 & \text{on } U_0 := \mathbb{P}^1 \setminus \{\infty\} \\ d + \Omega_\infty & \text{on } U_\infty := \mathbb{P}^1 \setminus \{0\} \\ g_{0,\infty} & \text{cocycle of } E \end{cases}$$

with gluing condition on the overlap

$$\Omega_0 = g_{0,\infty}^{-1} \cdot \Omega_\infty \cdot g_{0,\infty} + g_{0,\infty}^{-1} \cdot dg_{0,\infty}.$$

# EQUIVALENCE OF CONNECTIONS

## Gauge equivalence of connections

We say that  $(E, D, \nabla) \sim (E', D', \nabla')$  if there exists  $\Phi \in \text{Mor}(E, E')$  such that

$$\Phi^* \nabla' = \nabla.$$

## Fact

$$(E, D, \nabla) \sim (E', D', \nabla') \implies M_{\nabla} \sim M_{\nabla'}$$

## Local equivalence

The connections matrices on a trivialising open set  $U$  are locally related by:

$$\Omega_U = \Phi^{-1} \Omega'_U \Phi + \Phi^{-1} d\Phi$$

# EQUIVALENCE OF CONNECTIONS

## Example

The action of the meromorphic gauge transformation

$$\Phi(x) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x-q} \end{pmatrix}$$

on the connection  $(\mathcal{O} \oplus \mathcal{O}, D, d + \Omega)$  is

$$(\mathcal{O} \oplus \mathcal{O}(1), D + [q], d + \Omega'),$$

where

$$\Omega' := \begin{pmatrix} \omega_{1,1} & (x-q)\omega_{1,2} \\ \frac{1}{x-q}\omega_{2,1} & \omega_{2,2} - \frac{dx}{x-q} \end{pmatrix}$$

# **Painlevé V Connections**

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# CONNEXIONS DU TYPE PV

## Definition

Meromorphic connections  $(E, D, \nabla)$  such that

$$D^{\min} = [0] + 2[1] + [\infty],$$

where  $D^{\min}$  is the *minimal* polar divisor w.r.t. the gauge equivalence class of  $(E, D, \nabla)$ .

## Consequence

In  $D^{\min}$  only appear poles

- that cannot be eliminated nor reduced via a gauge transformation,
- with minimal Poincaré rank,
- with non-trivial monodromy.

# NORMAL FORM

## Normal Form on $\mathcal{O} \oplus \mathcal{O}(2)$ (Diarra, Loray 2019)

$$\nabla|_0 = d + \Omega_0 =$$

$$d + \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \frac{dx}{(x-1)^2} + \begin{pmatrix} 0 & -1 \\ 0 & \kappa_1 \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} 0 & 1 \\ 0 & -\kappa_0 \end{pmatrix} \frac{dx}{x} \\ + \begin{pmatrix} 0 & 0 \\ \kappa_\infty & 0 \end{pmatrix} xdx + \begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix} \frac{dx}{x-q} + \begin{pmatrix} 0 & 0 \\ \hat{K} & 0 \end{pmatrix} dx,$$

## Fixed Parameters

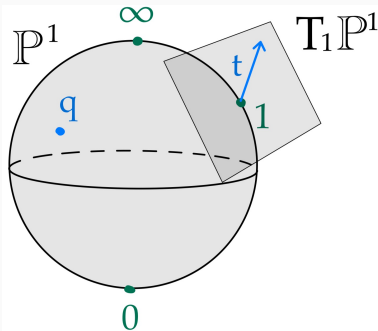
$$\Theta := \{\kappa_0, \kappa_1, \kappa_\infty\} \subseteq \mathbb{C}$$

# GEOMETRICAL INTERPRETATION OF $t$ AND $q$

## Action of $f \in \text{Aut}(\mathbb{P}^1)$

Let  $f \in \text{Aut}(\mathbb{P}^1)$ , then:

- $q \mapsto f(q)$ ,
- $t \mapsto Df(1) \cdot t$ .



# RATIONAL IRREGULAR CURVES

## Definition

A rational irregular curve is  $(\mathbb{P}^1, D, J)$ , where

- $D = \sum_{i \in I} n_i [p_i]$  is an effective divisor,
- $J = \{j^{n_i-1}(p_i)\}_{i \in I}$  is a collection of jets.

## Example

$(\mathbb{P}^1, D, J)$  with

- $D = [0] + 2[1] + [\infty] + [q]$
- $J = \{j^0(0), j^1(1) = (1, t), j^0(\infty), j^0(q)\}$ .

## Proposition

$[ (\nabla, E, D) ]$  "generic" of PV type  $\iff$  a rational irregular curve and  $p \in \mathbb{C}$ .

# **Moduli Space and Compactification**

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# MODULI SPACE

## Proposition

$$[(\nabla, E, D)] \text{ "generic" of PV type} \iff \begin{cases} (\mathbb{P}^1, D, J) \text{ , and} \\ p \in \mathbb{C} \end{cases}$$

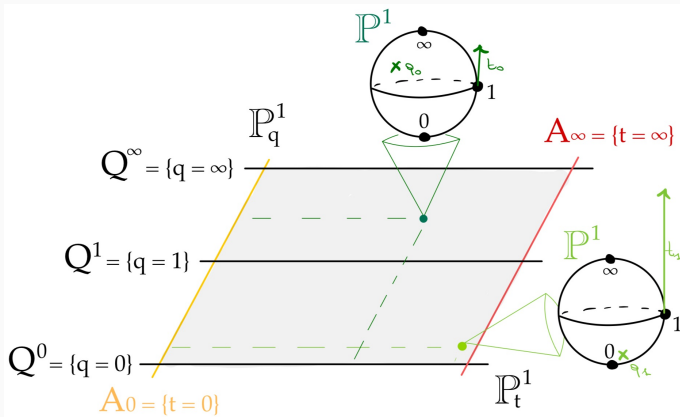
## Moduli space of rational irregular curves $\mathcal{M}$

$$\mathcal{M} := \left\{ q \in \mathbb{P}^1 \setminus \{0, 1, \infty\}; t \in T_1\mathbb{P}^1 \setminus \{0\} \right\}.$$

## Moduli space of PV connections

$$\text{Con}_{\Theta}^V \supseteq \mathcal{M} \times \mathbb{C}$$

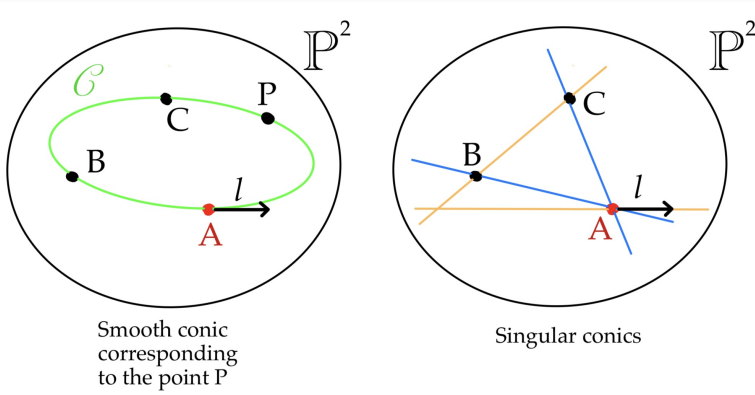
# MODULI SPACE $\mathcal{M} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$



## Remark

The moduli space  $\mathcal{M}$  is a generalization of  $\mathcal{M}_{0,5}$  where the fifth marked point is replaced by a tangent vector.

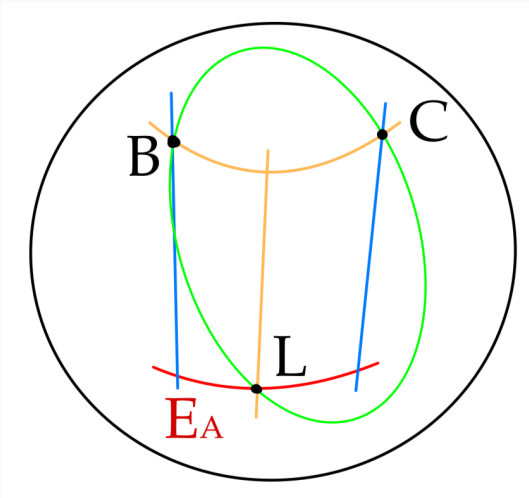
# MODULI SPACE OF CONICS



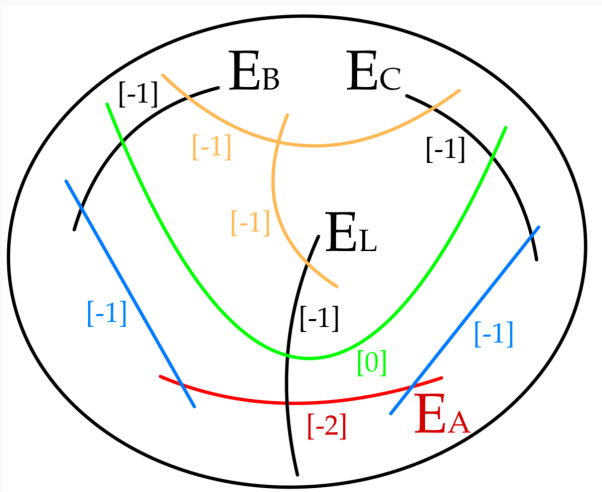
## Moduli Space of Smooth Conics

$$\mathbb{P}^2 \setminus (\overline{AB}, \overline{AC}, \overline{BC}, l) \cong \mathcal{M}$$

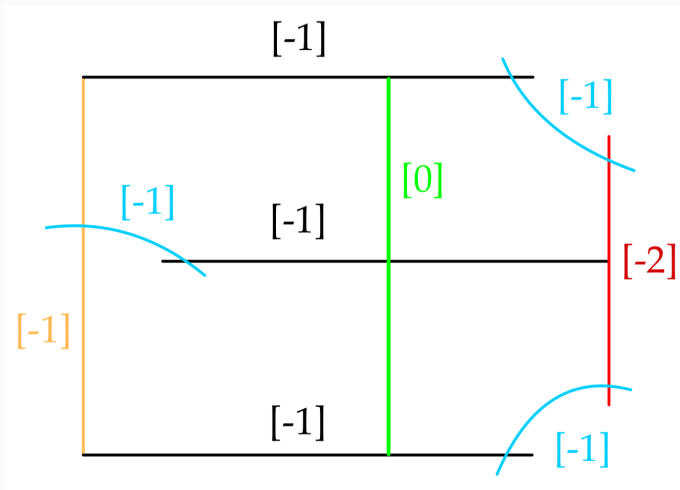
# BLOW-UP OF $A$



# BLOW-UP OF $B$ , $C$ AND $L$



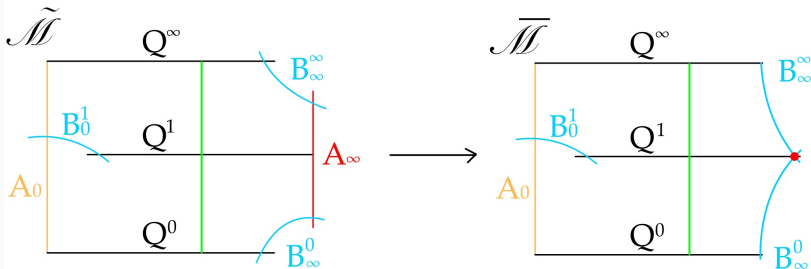
# COMPACTIFICATION $\overline{\mathcal{M}}$



# COMPACTIFICATION $\overline{\mathcal{M}}$

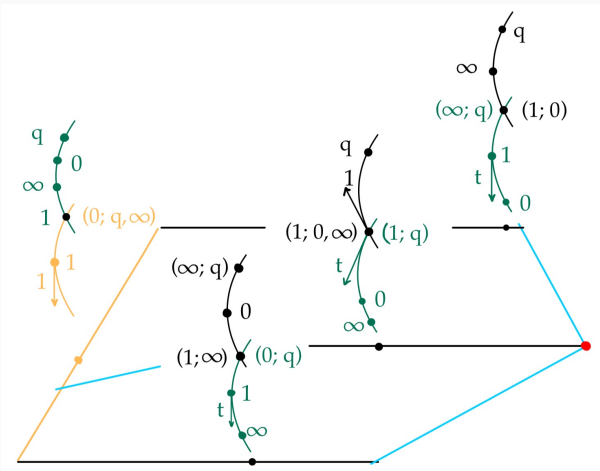
## Compactification (M., 2025)

Following, *mutatis mutandis*, a similar idea that Kapranov used to compactify the spaces  $\mathcal{M}_{0,n}$ , we get:



where  $\tilde{\mathcal{M}}$  is the weak del Pezzo surface of degree five.

# UNIVERSAL CURVE



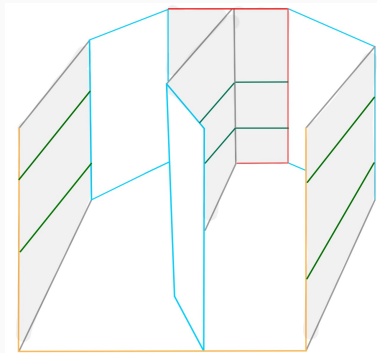
# COMPACTIFICATION $\overline{\text{Con}}_V^\ominus$

## Line Bundle Extension (M., 2025)

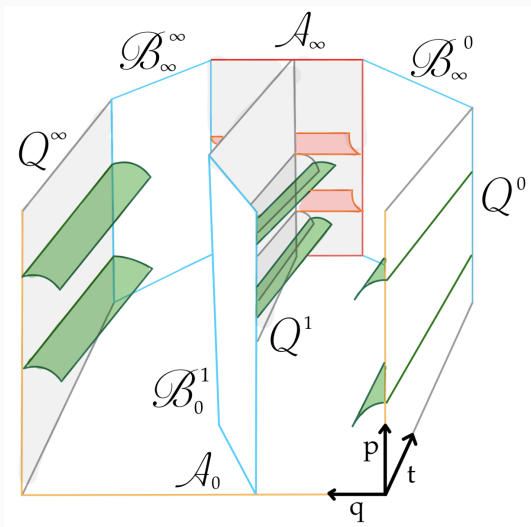
The trivial bundle  $\mathcal{M} \times \mathbb{C}$  extends to

$$\mathcal{O}_{\widetilde{\mathcal{M}}}(A_0 + 2Q^1 + 2B_0^1 - B_\infty^0 - B_\infty^\infty).$$

Moreover, it is trivial when restricted to  $Q^0$ ,  $Q^\infty$  and  $Q^1 \cup A_\infty$ .



# COMPACTIFICATION $\overline{\text{Con}}_V^\ominus$

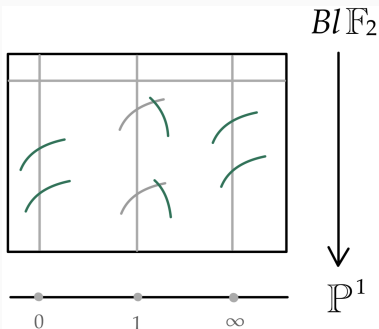


# COMPACTIFICATION $\overline{\mathcal{C}on_V^\ominus}$

## Theorem

The compactified moduli space  $\overline{\mathcal{C}on_V^\ominus}$  comes with a fibration  $\pi: \overline{\mathcal{C}on_V^\ominus} \xrightarrow{t} \mathbb{P}^1$  such that:

- For any  $t \in \mathbb{C}^*$ , the fiber  $\pi^{-1}(t)$  is an Okamoto space, that is a 8-blow-up of the second Hirzebruch surface  $\mathbb{F}_2$ .



# COMPACTIFICATION $\overline{\mathcal{C}on_V^\Theta}$

## Theorem (M, 2025)

The compactified moduli space  $\overline{\mathcal{C}on_V^\Theta}$  comes with a fibration  $\pi: \overline{\mathcal{C}on_V^\Theta} \xrightarrow{t} \mathbb{P}^1$  such that:

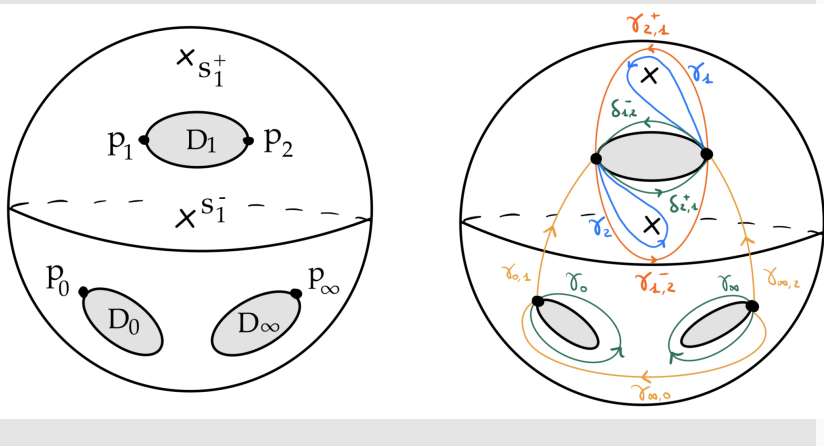
- The surface  $\pi^{-1}(0)$  is given by  $\mathcal{A}_0 \cup \mathcal{B}_0^1$ . Both these components are isomorphic to  $\mathbb{F}_1$ .
- Let us denote by  $\mathcal{F}_\infty^+$  and  $\mathcal{F}_\infty^-$  the blow up of the two special sections in  $\mathcal{A}_\infty$ . The surface  $\pi^{-1}(\infty)$  is given by  $\mathcal{B}_0^\infty \cup \mathcal{B}_\infty^\infty \cup \mathcal{F}_\infty^+ \cup \mathcal{F}_\infty^-$ . The surfaces  $\mathcal{B}_0^\infty$  and  $\mathcal{B}_\infty^\infty$  are isomorphic to  $\mathbb{F}_1$ .

# Painlevé V Foliation

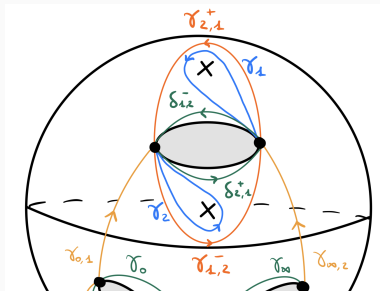
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# MONODROMY OF A PV CONNECTION

## Wild fundamental groupoid (E. Paul, J-P. Ramis, 2023)



# MONODROMY OF A PV CONNECTION



## Fact

The monodromy along  $\gamma_{1,2}^+ \star \gamma_{1,2}^-$  is given by

$$M^\gamma = S_0 S_1 D$$

with  $S_0$  upper triangular,  $S_1$  lower triangular (Stokes matrices) and  $D$  diagonal.

# PAINLEVÉ V EQUATION

## The Hamiltonian Function

$$H^V := \frac{q(q-1)^2 p^2 - (\kappa_0 (q-1)^2 + \kappa_1 q (q-1) - tq) p + \kappa_\infty (q-1)}{t}$$

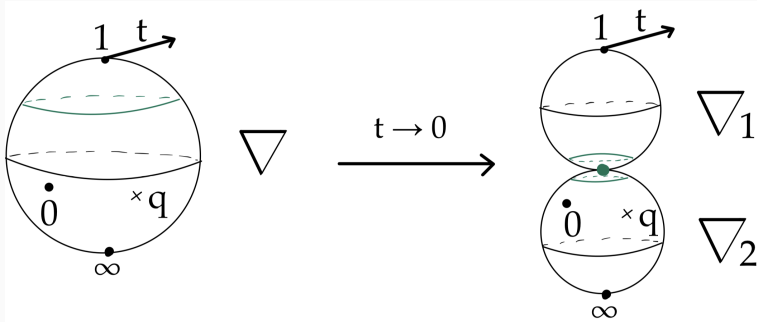
## The Hamiltonian System

$$\begin{cases} \frac{\partial H^V}{\partial p} = \frac{dq}{dt} \\ \frac{\partial H^V}{\partial q} = -\frac{dp}{dt} \end{cases}$$

# HOW TO FIND FIRST INTEGRALS

## Deformation of a PV connection

A PV connection splits into a connection over any smooth component. Both connections have a new pole at the node.



# HOW TO FIND FIRST INTEGRALS

## Proposition

The residual spectral datum in the node is a first integral for the hamiltonian vector field in restriction to the respective boundary component.

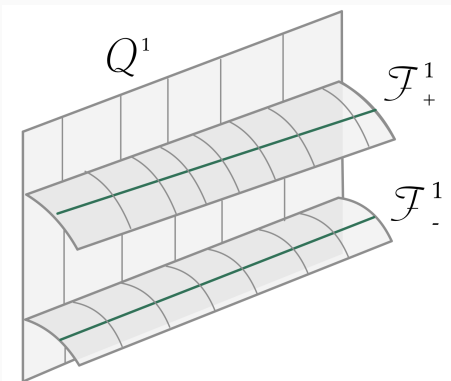
## Idea

Since we are following an isomonodromic leaf, the monodromy along the loop  $\gamma$  must be the same as the one of  $\gamma_1$  and  $\gamma_2$ , corresponding to the monodromy of the new poles at the node. In particular the residual spectral datum in the node must be preserved during the deformation.

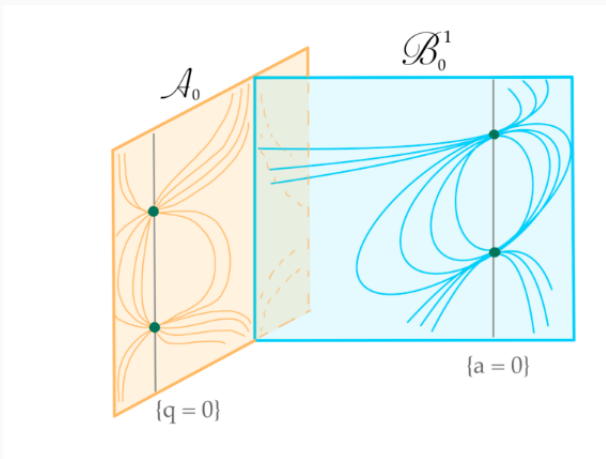
# ISOMONODROMIC FOLIATION ON THE OKAMOTO DIVISOR

## Proposition

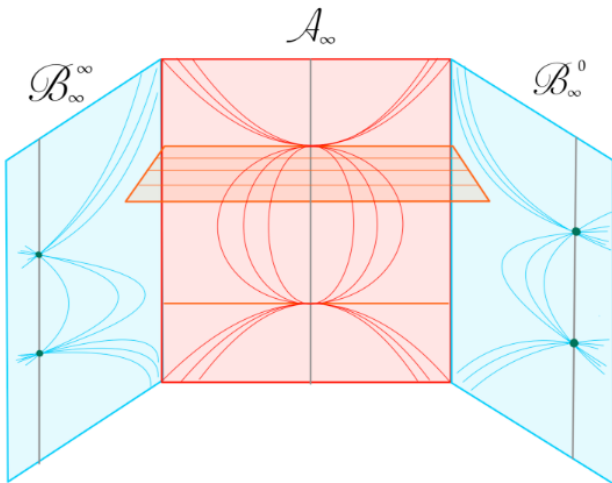
The isomonodromic foliation is stationary along the Okamoto divisor.



# FIRST INTEGRALS AROUND $t = 0$



# FIRST INTEGRALS AROUND $t = \infty$



THANKS :)

Thanks for your attention !!