Canonical Representations of Surface Groups

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1) Goal: classify alg. sol'ns to isomonodromy diff'l eqns

$$\left\{\begin{array}{c}\text{alg. sol'ns to} \\ \text{Painlevé VI}\end{array}\right\} \overset{\text{R. Fuchs}}{\rightsquigarrow} \left\{\begin{array}{c}\text{rep'ns} \\ \pi_1(\mathbb{P}^1 \setminus \{4\text{ pts}\}) \to SL_2(\mathbb{C}) \\ \text{w/ finite orbit under } \\ MCG(\mathbb{P}^1 \setminus 4 \text{ pts}) \end{array}\right\}$$

$\Sigma_{g,n}$: orientable surface of genus g w/ n punctures

$\mathrm{Mod}_{g,n} = \pi_0(\mathrm{Homeo}^+(\Sigma_{g,n})) \text{ "=" } \pi_1(\mathcal{M}_{g,n})$

= mapping class gp of $\Sigma_{g,n}$.

$$\mathrm{Mod}_{g,n} \longrightarrow \mathrm{Out}(\pi_1(\Sigma_{g,n}))$$

$$\mathrm{Mod}_{g,n} \circlearrowright \mathrm{Hom}(\pi_1(\Sigma_{g,n}), GL_r(\mathbb{C}))/\sim$$

Goal: Classify finite orbits —
"canonical" rep'ns of $\pi_1(\Sigma_{g,n})$

Painlevé VI $\longleftrightarrow$ $\boxed{g=0 \quad n=4 \quad r=2}$

"Canonical" b/c reps construct w/out making choices have finite orbit.

<u>Ex</u> $(n=0)$ $\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod [a_i, b_i] = 1 \rangle$

Canonical $\Leftrightarrow$ finite orbit under $\mathrm{Out}(\pi_1(\Sigma_g))$

2) Examples

(1) Reps $\pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$ w/ finite image

(2) Rigid local systems (genus 0 only)

(3) $\pi_1(\Sigma_{g,n}) \trianglelefteq \mathrm{Mod}(\Sigma_{g,n},*) \cong \pi_1(\mathcal{M}_{g,n+1})$

$\subseteq \Gamma \subseteq$ finite index

Given $\rho: \Gamma \to GL_r(\mathbb{C})$,

$\rho|_{\pi_1(\Sigma_{g,n})}$ is canonical.

(i) TQFT techniques

(ii) AG techniques

$X \xrightarrow{\pi} \mathcal{M}_{g,n+1}$ sm. proper

$R^i\pi_*\underline{\mathbb{C}}$ gives rep as desired

Kodaira-Parshin trick:

$\pi^{-1}([C, x_1, \dots, x_n]) =$ disjoint union of all H-covers of C ramified over x_i

(iii) Group theory techniques

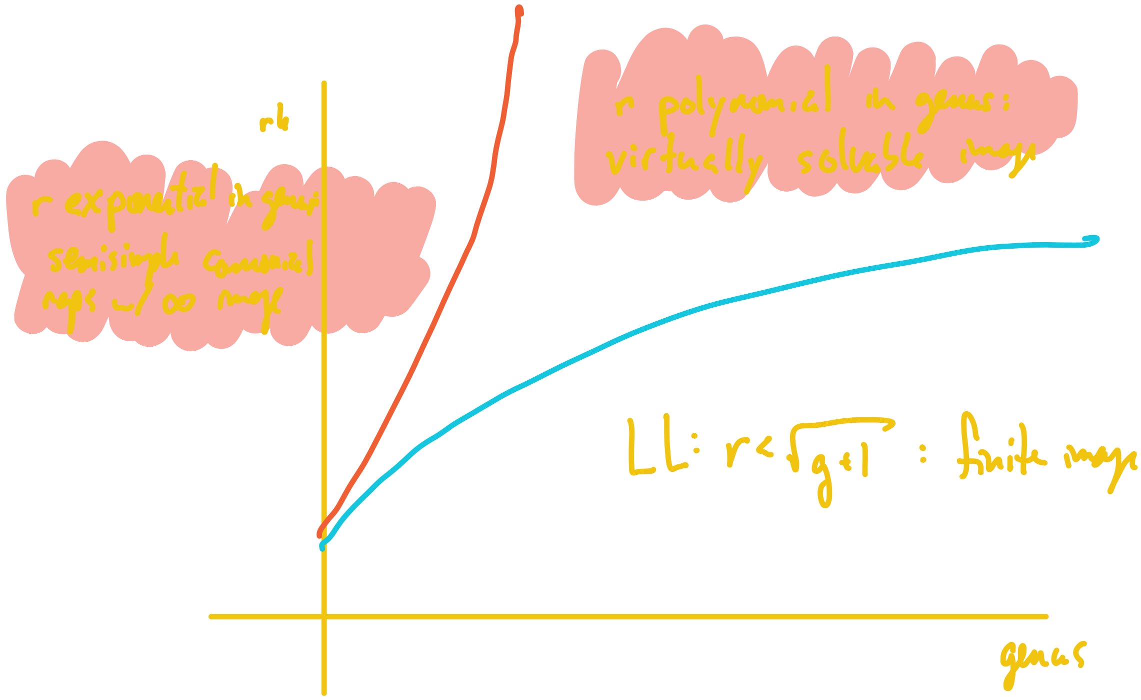
3) History:

(i) Painlevé VI: algebraic sol'ns found by Hitchin, Dubrovin-Mazzocco, Doran, Boalch, ...
Classification completed by Lisovyy-Tykhyy

(ii) Q (Whang, Kisin)
For $g \gg r$, do all canonical rep'ns $\pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$ have finite image?

(Biswas, Gupta, Mj, Whang + Cousin-Heu) Yes for $r=2$.

Thm (Landesman-L—) Suppose $r < \sqrt{g+1}$.
Then all canonical rep'ns $\pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$ have finite image.

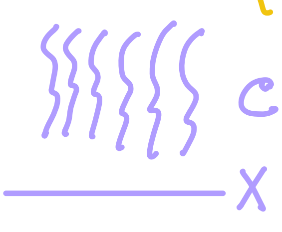


Q. How sharp is the bound?
(Cannot be improved beyond $2g+1$) .

What about in the semisimple case?

4) Sketch of proof

Inputs: (Non-abelian + mixed) Hodge theory
+ ε Langlands



(i) $\rho : \pi_1(\Sigma_{g,n}) \longrightarrow GL_r(\mathbb{C})$
canonical, irreducible

(.

local system $\mathbb{V}_\rho$ on universal curve $\pi: C \to X$ over étale cover X of $\mathcal{M}_{g,n}$.

Unitary case:

(ii) ρ unitary + $\dim \rho < \sqrt{g+1}$ $\Rightarrow$ $\mathbb{V}_\rho$ "cohomologically rigid" (*)

<u>Pf idea</u> Enough to show

$$H^1(C, \mathrm{ad}\,\mathbb{V}_\rho) = H^1(X, R^1\pi_* \mathrm{ad}\,\mathbb{V}_\rho) = 0.$$

underlies a $\mathbb{C}$-PVHS

+ Analysis of period map. $\square$

(iii) Cohomological rigidity + quasi-unipotent monodromy at ∞

$\Downarrow$ Esnault-Groechenig, Klevdal-Patrikis

unitary ρ defined over $\mathcal{O}_K$ for some #field K

(iv) Enough to show that for each $\iota: \mathcal{O}_K \hookrightarrow \mathbb{C}$

$\rho \otimes_{\mathcal{O}_{K,\iota}} \mathbb{C}$ is unitary

$\mathbb{V}_\rho$ Rigid $\xRightarrow{\text{NAHT}}$ Underlies $\mathbb{C}$-PVHS

$\xRightarrow[\text{[Landesman-L-]}]{\text{low rank}}$ unitary

Irreducible Case: (v) ρ irreducible, $\dim \rho < \sqrt{g+1}$

NAHT: $\mathbb{V}_\rho$ deformation equivalent to $\mathbb{C}$-PVHS $\mathbb{V}'$ on $\begin{matrix} \mathcal{C} \\ \downarrow \pi \\ X \end{matrix}$

has finite monodromy

$\mathbb{V}'|_{\pi^{-1}(x)}$ unitary by [Landesman-L-]

$\Rightarrow \mathbb{V}'$ rigid

$\Rightarrow \mathbb{V}_\rho|_{\pi^{-1}(x)} = \mathbb{V}'|_{\pi^{-1}(x)}$

General case (vi) ρ^{ss} canonical $\Rightarrow \rho^{ss}$ has finite image

$\mathrm{Ext}^1(\rho_1, \rho_2)^{\mathrm{Mod}_{g,n}} = 0$

Follows from analysis of period maps as before $\square$

5) Questions

(1) New examples of canonical rep'ns?

$$\begin{array}{c} X \\ \exists ?\ \pi \big\downarrow \text{sm. proper} \\ C \end{array}$$

C - Riemann surface of genus g w/ n marked pts

$\rho \subseteq R^i \pi_* \underline{\mathbb{C}}$

(2) Classification in general: for $g>2$, are all canonical rep'ns "of geometric origin"?

(3) Are all irreps of $\mathrm{Mod}_{g,n}$, $g>2$, rigid?

Why believe (3)? Analogy b/w $\mathrm{Mod}_{g,n}$, $g>2$ and $SL_r(\mathbb{Z})$ for $r>2$.

<u>Rmk</u> $\mathrm{Mod}_{g,n}$ admits non-semi-simple rep'ns hence admits non-rigid rep'ns

Conj (Putman-Wieland) $\Sigma_{g,n}$ surface

$$\pi_1(\Sigma_{g,n}) \twoheadrightarrow H$$

$$\Sigma_{g'} \to \Sigma_g \text{ ramified over } n \text{ pts}$$

$\Gamma \underset{\text{f.i.}}{<} \mathrm{Mod}_{g,n}$ $\quad \Gamma \curvearrowright H_1(\Sigma_{g'})$

If $g > 2$, no non-zero elt of $H_1(\Sigma_{g'})$ has finite orbit under Γ.

Thm (Landesman-L–) True if $g > \sqrt{\#H}$.

Idea: $\begin{array}{c} \mathcal{C} \curvearrowleft \mathbb{V}_\rho \\ \downarrow \pi \\ \mathcal{M}_{g,n} \end{array}$ $\quad \rho : H \to GL_r(\mathbb{C})$

Thm If $\dim \rho < g$, then $H^0(\mathcal{M}_{g,n}, R^1\pi_* \mathbb{V}_\rho) = 0$

Pf idea (1) By theorem of the fixed part, non-zero $\Rightarrow$ constant sub-VHS of $R^1\pi_*\mathbb{V}_\rho$.

(2) (Assume $n=0$, C Riemann surface of genus g, E_ρ be stable vector bundle on C associated to ρ)

Derivative of period map:

$$H^0(E\otimes\omega)\otimes H^0(E^\vee\otimes\omega)\to H^0(\omega^{\otimes 2})$$

$$\Downarrow$$

$$H^0(E\otimes\omega)\to \mathrm{Hom}(H^0(E^\vee\otimes\omega), H^0(\omega^{\otimes 2}))$$

$\Rightarrow$ map has non-zero kernel

$\Rightarrow$ $E^\vee\otimes\omega$ not gen glob. generated

$\Downarrow$ impossible if $\mathrm{rk}\, E < g$ by Clifford theory for v.b.'s.