Note on singularity reduction of isomonodromy systems associated with Garnier systems

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Motivation and Main Questions

Painlevé Equation and Isomonodromy Property

- The first Painlevé equation (with $\hbar$):

$$P_{\mathrm{I}} \;:\; \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t \quad \Leftrightarrow \quad \hbar \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \hbar \frac{dp}{dt} = -\frac{\partial H}{\partial p}$$

  where the Hamiltonian H is given by

$$H = \frac{p^2}{2} - 2q^3 - tq.$$

- Painlevé property: all movable singular points must be poles.
- Description as isomonodromic deformation (integrablity):

$$L_{\mathrm{I}} \;:\; \left(\hbar^2 \frac{d^2}{dx^2} - Q(x,t,\hbar)\right)\psi = 0$$

$$Q = 4x^3 + 2tx + 2H - \hbar \frac{p}{x-q} + \hbar^2 \frac{3}{4(x-q)^2}$$

  - $x = q$ is an "apparent" singular point.
  - Stokes multipliers around $x = \infty$ are t-**independent** if q satisfies P_{I}.

Laurent Seriers Solution and Painlevé Test

- For any $\alpha \in \mathbb{C}$, there exists a Laurent series solution of P_{I}:

$$q(t) = \frac{\hbar^2}{(t-\alpha)^2} - \frac{\alpha}{10\hbar^2}(t-\alpha)^2 - \frac{(t-\alpha)^3}{6\hbar^2} - \frac{\beta}{14\hbar^2}(t-\alpha)^4 + \cdots$$

  - β : **arbitrary parameter**.
  - This converges on a punctured disc around $t = \alpha$.

$\rightsquigarrow$ 2-parameter family of Laurent series solutions.
We say that P_{I} passes the "Painlevé test".

- Near the movable pole α, the τ**-function** defined by

$$\hbar^2 \frac{d}{dt} \log \tau = H$$

behaves as follows:

$$\tau(t) = (t-\alpha)\left(1 + \frac{\beta}{\hbar^2}(t-\alpha) + \cdots\right), \quad t \to \alpha$$

  - Well-known similarity between Painlevé transcendents and **elliptic functions**:

$$\begin{array}{ccc} q & \longleftrightarrow & \wp \\ \tau & \longleftrightarrow & \sigma \end{array}$$

Singularity Reduction (From Painlevé to Heun)

- The previous Laurent expansion of q leads

$$q = \frac{\hbar^2}{(t-\alpha)^2} + \cdots , \quad p = -\frac{\hbar^3}{(t-\alpha)^3} + \cdots , \quad H = \frac{\hbar^2}{t-\alpha} + \beta + \cdots$$

- Although q, p and H diverge as $t \to \alpha$, the Schrödinger potential Q has a **finite limit**!

$$\left(\hbar^2 \frac{d^2}{dx^2} - Q\right)\psi = 0 \ , \quad Q = 4x^3 + 2tx + 2H - \hbar \frac{p}{x-q} + \hbar^2 \frac{3}{4(x-q)^2}$$

$$\xrightarrow{t \to \alpha} \quad \left(\hbar^2 \frac{d^2}{dx^2} - (4x^3 + 2\alpha x + 2\beta)\right)\psi = 0$$

[Its–Novokshenov 86], [Masoero 10], ...

  - The resulting equation is called reduced tri-confluent Heun equation.
  - The parameter β is called accessory parameter (AP).

- We call the limiting procedure (restriction of isomonodromic linear ODE to the movable pole) as **singularity reduction**, following [Dubrovin–Kapaev 12].

  The singularity reduction also exists for the isomonodromic linear ODEs associated with other Painlevé equations P_{II},..., P_{VI}.

Main Questions

$$\left(\hbar^2 \frac{d^2}{dx^2} - Q\right)\psi = 0 \quad \xrightarrow{t\to\alpha} \quad \left(\hbar^2 \frac{d^2}{dx^2} - (4x^3 + 2\alpha x + 2\beta)\right)\psi = 0$$

Question 1

Does the singularity reduction also exists in higher order analogue of Painlevé equations? (C.f., [Kawakami–Nakamura–Sakai 12], [Kawakami 17])

- [Dubrovin–Kapaev 12] studied one example, and found an ODE which describes isomonodromic deformation but does **not** have Painlevé property.

Question 2 (Not today)

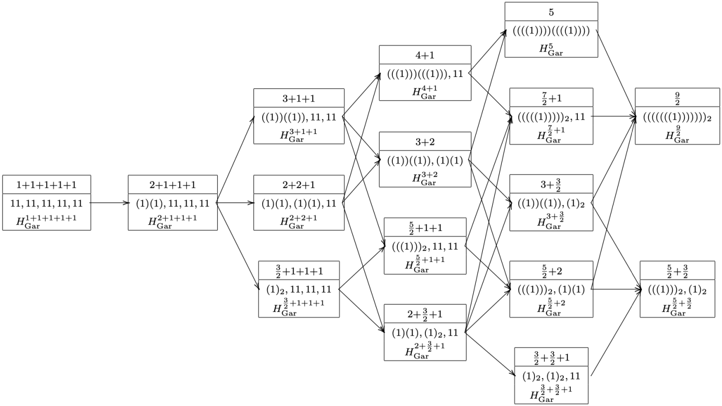
Can we describe a relation between (α, β) and Stokes multipliers?
In particular, can we reproduce the AP? (Riemann–Hilbert problem)

- Exact WKB method ([Voros 83]) allows us to describe the Stokes multipliers by the Voros periods (VP) on the classical limit $y^2 = 4x^3 + 2\alpha x + 2\beta$.
  So, the above question is related to: Can we reproduce the AP from VP?
- Relation to classical irregular conformal blocks?
  Zamolodchikov-type conjecture is studied in [Lisovyy–Naidiuk 21] in the presence of **irregular singular points**.

Result 1

Singularity Reduction of
some Degenerate Garnier Systems
(4th Order Painlevé Equations)

Confluence Diagram of Garnier Systems



[Kawakami 17] (c.f., [Kimura 89], [Kawakami–Nakamura–Sakai 12])

- We study two examples: $\mathrm{Gar}_{9/2}$ and $\mathrm{Gar}_{5/2+3/2}$.
- They describe the isomonodromic deformation of 2×2 linear systems, and there are two isomonodromic times t_1, t_2.
- We will study the singularity reduction of the ODE with respect to t_1 which is obtained as the restriction of the Garnier systems on $\{t_2 = \text{constant}\}$.

Isomonodromy System for $\mathrm{Gar}_{9/2}$

$$\hbar \frac{\partial \Phi}{\partial x} = L(x, t_1, t_2)\Phi, \quad \hbar \frac{\partial \Phi}{\partial t_j} = M_j(x, t_1, t_2)\Phi \qquad (j = 1, 2)$$

$$\begin{aligned} L &= L_{\mathrm{Gar}_{9/2}} &&= L_0\, x^3 + L_1\, x^2 + L_2\, x + L_3, \\ M_1 &= M_{\mathrm{Gar}_{9/2},1} &&= -L_0\, x + M_{10}, \\ M_2 &= M_{\mathrm{Gar}_{9/2},2} &&= L_0\, x^2 + L_1\, x + M_{20}, \end{aligned}$$

$$L_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} 0 & p_1 \\ 1 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} q_2 & p_1^2 + p_2 + 2t_2 \\ -p_1 & -q_2 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} q_1 - p_1 q_2 & p_1^3 + 2p_1 p_2 - q_2^2 + t_2 p_1 - t_1 \\ -p_2 + t_2 & -q_1 + p_1 q_2 \end{pmatrix},$$

$$M_{10} = \begin{pmatrix} 0 & -2p_1 \\ -1 & 0 \end{pmatrix}, \qquad M_{20} = \begin{pmatrix} q_2 & p_1^2 + 2p_2 + t_2 \\ -p_1 & -q_2 \end{pmatrix}.$$

c.f., [Kimura 89], [Kawakami 17]

Isomonodromy System for $\mathrm{Gar}_{3/2+5/2}$

$$\hbar \frac{\partial \Phi}{\partial x} = L(x,t_1,t_2)\Phi, \quad \hbar \frac{\partial \Phi}{\partial t_j} = M_j(x,t_1,t_2)\Phi \qquad (j=1,2)$$

$$\begin{aligned} L &= L_{\mathrm{Gar}_{5/2+3/2}} &&= L_0\, x + L_1 + \frac{L_2}{x} + \frac{L_3}{x^2}, \\ M_1 &= M_{\mathrm{Gar}_{5/2+3/2},1} &&= M_{10}\, x + M_{11}, \\ M_2 &= M_{\mathrm{Gar}_{5/2+3/2},2} &&= M_{20} - \frac{L_3}{t_2 x}, \end{aligned}$$

$$L_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} q_1 & p_1 - q_1^2 - t_1 \\ 1 & -q_1 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} p_2 q_2 & q_2 \\ -p_1 & -p_2 q_2 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 0 \\ t_2/q_2 & 0 \end{pmatrix},$$

$$M_{10} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad M_{11} = \begin{pmatrix} -q_1 & 0 \\ -1 & q_1 \end{pmatrix}, \quad M_{20} = \begin{pmatrix} 0 & -q_2/t_2 \\ 0 & 0 \end{pmatrix}.$$

c.f., [Kawakami 17]

Restriction of Garnier Systems

- Isomonodromy condition with respect to t_1 can be written as a Hamiltonian systems for (q_1, q_2, p_1, p_2):

$$\hbar\left(\frac{\partial L}{\partial t_1} - \frac{\partial M_1}{\partial x}\right) + [L, M_1] = 0 \quad \Leftrightarrow \quad \hbar\frac{dq_i}{dt_1} = \frac{\partial H_1}{\partial p_i}, \quad \hbar\frac{dp_i}{dt_1} = -\frac{\partial H_1}{\partial q_i} \quad (i = 1, 2)$$

  - The first example $\mathrm{Gar}_{9/2}$ is a 4th order analogue of P_{I}:

$$H_1 = H_1^{\mathrm{Gar}_{9/2}} = {p_1}^4 + 3{p_1}^2 p_2 + p_1 {q_2}^2 - 2q_1 q_2 + {p_2}^2 - t_1 p_1 + t_2 p_2$$

    This example was studied by [Dubrovin–Kapaev 12].

  - The second example $\mathrm{Gar}_{5/2+3/2}$ is a 4th order analogue of $P_{\mathrm{III}}(D_8)$:

$$H_1 = H_1^{\mathrm{Gar}_{5/2+3/2}} = p_1^2 - (q_1^2 + t_1)p_1 - 2p_2 q_1 q_2 - q_2 - \frac{t_2}{q_2}$$

- In fact, there exists another Hamiltonian H_2 such that

$$\hbar\frac{dq_i}{dt_2} = \frac{\partial H_2}{\partial p_i}, \quad \hbar\frac{dp_i}{dt_2} = -\frac{\partial H_2}{\partial q_i} \quad (i = 1, 2)$$

  is compatible with the above Hamiltonian system[1].

[1] We follow the approach of [Nakamura 17] and swap the labels of the isomonodromic times of the Garnier systems of type $9/2$ in [Kawakami 16].

4-Parameter Laurent Solution of $\mathrm{Gar}_{9/2}$

Proposition ([Shimomura 00] (c.f., [Nakamura 17]))

There exists a convergent Laurent series solution of $\mathrm{Gar}_{9/2}$ the form:

$$
\begin{aligned}
q_1(t_1) &= -\frac{\hbar^5}{(t_1-\alpha)^5} + \frac{\beta\hbar^3}{(t_1-\alpha)^3} + \gamma + \frac{(10\alpha - 18t_2\beta - 35\beta^3)(t_1-\alpha)}{70\hbar} \\
&\quad - \frac{3(50\beta\gamma - 3\hbar)(t_1-\alpha)^2}{20\hbar^2} + \frac{\delta(t_1-\alpha)^3}{\hbar^3} + O((t_1-\alpha)^5), \\
q_2(t_1) &= -\frac{\hbar^3}{(t_2-\alpha)^3} - \frac{3(4t_2+5\beta^2)(t_1-\alpha)}{20\hbar} - \frac{6\gamma(t_2-\alpha)^2}{\hbar^2} \\
&\quad + \frac{(4\alpha - 24t_2\beta - 35\beta^3)(t_1-\alpha)^3}{14\hbar^3} + \frac{(-30\beta\gamma+\hbar)(t_1-\alpha)^4}{4\hbar^4} + O((t_1-\alpha)^6), \\
p_1(t_1) &= \frac{\hbar^2}{(t_1-\alpha)^2} + \frac{\beta}{2} - \frac{3(4t_2+5\beta^2)(t_1-\alpha)^2}{20\hbar^2} - \frac{4\gamma(t_1-\alpha)^3}{\hbar^3} \\
&\quad + \frac{(4\alpha - 24t_2\beta - 35\beta^3)(t_1-\alpha)^4}{28\hbar^4} + \frac{(-30\beta\gamma+\hbar)(t_1-\alpha)^5}{10\hbar^5} 15400\hbar^6 + O((t_1-\alpha)^7), \\
p_2(t_1) &= -\frac{3\beta\hbar^2}{2(t_1-\alpha)^2} + \left(t_2 + \frac{3\beta^2}{2}\right) + \frac{6\gamma(t_1-\alpha)}{\hbar} + \frac{9(4t_2\beta+5\beta^3)(t_1-\alpha)^2}{40\hbar^2} \\
&\quad + \frac{(t_1-\alpha)^3}{5\hbar^2} - \frac{3(1008t_2^2 + 400\alpha\beta + 120t_2\beta^2 - 1925\beta^4 - 1680\delta)(t_1-\alpha)^4}{12320\hbar^4} + O((t_1-\alpha)^6),
\end{aligned}
$$

Here $(\alpha, \beta, \gamma, \delta)$ are free parameters which are independent of t_1.

4-Parameter Laurent Solution of $\mathrm{Gar}_{5/2+3/2}$

Proposition

There exists a convergent Laurent series solution of $\mathrm{Gar}_{5/2+3/2}$ the form:

$$q_1(t_1) = \frac{\hbar}{t_1-\alpha} - \frac{(\alpha-2\beta)(t_1-\alpha)}{3\hbar} + \frac{\gamma(t_1-\alpha)^2}{\hbar^2} + \frac{\delta(t_1-\alpha)^3}{\hbar^3} - \frac{(3\alpha\gamma-10\beta\gamma+\alpha\hbar-2\beta\hbar)(t_1-\alpha)^4}{9\hbar^4} + O((t_1-\alpha)^5),$$

$$q_2(t_1) = \frac{\beta\hbar^2}{(t_1-\alpha)^2} + \frac{1}{3}(\alpha\beta-2\beta^2) - \frac{2\beta\gamma(t_1-\alpha)}{3\hbar} + \frac{(\alpha^2\beta-4\alpha\beta^2+4\beta^3-9\beta\delta)(t_1-\alpha)^2}{18\hbar^2} + \frac{2(-2\alpha\beta\gamma+\alpha\beta\hbar-2\beta^2\hbar)(t_1-\alpha)^3}{45\hbar^3} + O((t_1-\alpha)^4),$$

$$p_1(t_1) = \beta + \left(\frac{1}{2}+\frac{2\gamma}{\hbar}\right)(t_1-\alpha) + \frac{(\alpha^2-4\alpha\beta+4\beta^2+45\delta)(t_1-\alpha)^2}{18\hbar^2} - \frac{(4\alpha\gamma-12\beta\gamma+\alpha\hbar-2\beta\hbar)(t_1-\alpha)^3}{3\hbar^3} + O((t_1-\alpha)^4),$$

$$p_2(t_1) = -\frac{(t_1-\alpha)}{\hbar} - \frac{(4\gamma+\hbar)(t_1-\alpha)^2}{4\beta\hbar^2} + \frac{2(\alpha-2\beta)(t_1-\alpha)^3}{3\hbar^3} + \frac{(4\alpha\gamma-20\beta\gamma+\alpha\hbar-2\beta\hbar)(t_1-\alpha)^4}{12\beta\hbar^4} + O((t_1-\alpha)^5),$$

Here $(\alpha, \beta\neq 0, \gamma, \delta)$ are free parameters which are independent of t_1.

Schrödinger Form

- For each $j = 1, 2$, the isomonodromic linear system

$$\hbar \frac{\partial \Phi}{\partial x} = L(x, t_1, t_2)\Phi, \quad \hbar \frac{\partial \Phi}{\partial t_j} = M_j(x, t_1, t_2)\Phi$$

can be reduced to the following scalar equation:

$$\begin{cases} \left(\hbar^2 \dfrac{\partial^2}{\partial x^2} - Q(x, t_1, t_2)\right)\psi = 0, \\ \dfrac{\partial \psi}{\partial t_j} = \left(A_j(x, t_1, t_2)\dfrac{\partial}{\partial x} - \dfrac{1}{2}\dfrac{\partial A_j}{\partial x}(x, t_1, t_2)\right)\psi \end{cases}$$

We call the system the Schrödinger form of the isomonodromy system.

  - ψ is a gauge transform of the first entry of Φ.
  - Q is written by L_{ij}'s in a complicated way, and $A_j = (M_j)_{12}/L_{12}$.
  - They have poles at zeros of L_{12}, which are called apparent singular points. Our examples have two or three apparent singular points.

- The compatibility (i.e., isomonodromy) condition is given by

$$2\frac{\partial Q}{\partial t_j} + \hbar^2 \frac{\partial^3 A_j}{\partial x^3} - 4Q\frac{\partial A_j}{\partial x} - 2A_j \frac{\partial Q}{\partial x} = 0$$

Existence of Singularity Reduction ($\mathrm{Gar}_{9/2}$)

Theorem (c.f., Theorem 3.1 in [Dubrovin–Kapaev 12])

$Q = Q_{\mathrm{Gar}_{9/2}}$, with (q_1, q_2, p_1, p_2) substituted by the previous Laurent series solution of $\mathrm{Gar}_{9/2}$, has finite limit as $t_1 \to \alpha$:

$$\lim_{t_1 \to \alpha} Q_{\mathrm{Gar}_{9/2}}(x) = R_{\mathrm{Gar}_{9/2}}(x)$$

where $R_{\mathrm{Gar}_{9/2}}(x) = R_{\mathrm{Gar}_{9/2},0}(x) + \hbar\, R_{\mathrm{Gar}_{9/2},1}(x) + \hbar^2\, R_{\mathrm{Gar}_{9/2},2}(x)$ is given by

$$R_{\mathrm{Gar}_{9/2},0} = x^5 + 3t_2 x^3 - \alpha x^2 + \frac{3(9072t_2^2 + 8000\alpha\beta - 34560t_2\beta^2 - 51975\beta^4 - 15120\delta)}{12320}\, x \\ - \frac{9(9072t_2^2\beta + 1840\alpha\beta^2 - 6840t_2\beta^3 - 31185\beta^5 - 221760\gamma^2 - 15120\beta\delta)}{24640},$$

$$R_{\mathrm{Gar}_{9/2},1} = \frac{18\gamma}{2x - 3\beta}, \qquad R_{\mathrm{Gar}_{9/2},2} = \frac{3}{(2x-3\beta)^2}.$$

- The above result implies the existence of the singularity reduction

$$\mathrm{SR}_{\mathrm{Gar}_{9/2}} \ : \ \left(\hbar^2 \frac{d^2}{dx^2} - R_{\mathrm{Gar}_{9/2}}(x)\right)\psi = 0$$

- One can check that $x = 3\beta/2$ is an **apparent singular point** of $\mathrm{SR}_{\mathrm{Gar}_{9/2}}$.

Existence of Singularity Reduction ($\mathrm{Gar}_{5/2+3/2}$)

Theorem

$Q = Q_{\mathrm{Gar}_{5/2+3/2}}$, with (q_1, q_2, p_1, p_2) substituted by the previous Laurent series solution of $\mathrm{Gar}_{5/2+3/2}$, has finite limit as $t_1 \to \alpha$:

$$\lim_{t_1 \to \alpha} Q_{\mathrm{Gar}_{5/2+3/2}}(x) = R_{\mathrm{Gar}_{5/2+3/2}}(x)$$

where $R_{\mathrm{Gar}_{5/2+3/2}}(x) = R_{\mathrm{Gar}_{5/2+3/2},0}(x) + \hbar\, R_{\mathrm{Gar}_{5/2+3/2},1}(x) + \hbar^2\, R_{\mathrm{Gar}_{5/2+3/2},2}(x)$ is given by

$$R_{\mathrm{Gar}_{5/2+3/2},0} = x - \alpha + \frac{\alpha^2 + 32\alpha\beta - 50\beta^2 + 45\delta}{18x} - \frac{18t_2 + \beta(\alpha^2\beta + 14\alpha\beta^2 - 32\beta^3 - 18\gamma^2 + 45\beta\delta)}{18\beta x^2} + \frac{t_2}{x^3},$$

$$R_{\mathrm{Gar}_{5/2+3/2},1} = \frac{(3x-\beta)\gamma}{2x^2(x-\beta)}, \qquad R_{\mathrm{Gar}_{5/2+3/2},2} = \frac{5x^2 + 10x\beta - 3\beta^2}{16x^2(x-\beta)^2}.$$

- In [Dubrovin–Kapaev 12], it was conjectured that **any** isomonodromy system admits a singularity reduction.
- [Dubrovin–Kapaev 12] worked with Linear 2×2 matrix linear system for $\mathrm{Gar}_{9/2}$, and found an **appropriate gauge** to have finite limit as $t_1 \to \alpha$.
- Would the Schrödinger form always provide a gauge choice for the singularity reduction of all Garnier systems?

Results 2

Secondary Isomonodromy Property
and quasi-Painlevé property

Secondary Isomonodromy Property

We have seen that

- Both $\mathrm{Gar}_{9/2}$ and $\mathrm{Gar}_{5/2+3/2}$ admit a 4-parameter family of Laurent series solutions, and they provide the singularity reduction

$$\left(\hbar^2 \frac{\partial^2}{\partial x^2} - Q\right)\psi = 0 \quad \xrightarrow{t_1 \to \alpha} \quad \mathrm{SR} \ : \ \left(\hbar^2 \frac{d^2}{dx^2} - R\right)\psi = 0$$

- $R = R(x, \hbar)$ is rational in x, and polynomial in $\hbar$: $R = R_0(x) + \hbar\, R_1(x) + \hbar^2\, R_2(x)$.
  - In the case $\mathrm{Gar}_{9/2}$, R_1, R_2 has a pole at $x = 3\beta/2$.
  - In the case $\mathrm{Gar}_{5/2+3/2}$, R_1, R_2 has a pole at $x = \beta$.

  The pole is an apparent singularity of SR.

- On the other hand, both $\mathrm{Gar}_{9/2}$ and $\mathrm{Gar}_{5/2+3/2}$ were originally PDEs in t_1, t_2. The original (i.e., before taking SR) Schrödinger form is also **isomonodromic in** t_2.

Proposition

The function $A_2 = (M_2)_{12}/L_{12}$, which appeared in deformation equation w.r.t. t_2, also has a finite limit:

$$B(x) := \lim_{t_1 \to \alpha} A_2(x) = \begin{cases} \dfrac{2}{2x - 3\beta} & \text{for } \mathrm{Gar}_{9/2} \\[2ex] \dfrac{\beta x}{t_2(x-\beta)} & \text{for } \mathrm{Gar}_{5/2+3/2} \end{cases}$$

Secondary Isomonodromic Property (cont)

Theorem (for $\mathrm{Gar}_{5/2+3/2}$)

If $(\alpha, \beta, \gamma, \delta)$ are functions of t_2 satisfying the system of ODEs

$$(*) \quad : \quad \begin{cases} \hbar \dfrac{d\alpha}{dt_2} & = -\dfrac{\beta\hbar}{t_2}, \\ \hbar \dfrac{d\beta}{dt_2} & = -\dfrac{\beta(4\gamma + \hbar)}{2t_2}, \\ \hbar \dfrac{d\gamma}{dt_2} & = \dfrac{18t_2 - \alpha^2\beta^2 + 4\alpha\beta^3 - 4\beta^4 - 45\beta^2\delta}{18t_2\beta}, \\ \hbar \dfrac{d\delta}{dt_2} & = \dfrac{2(32\alpha\beta\gamma - 100\beta^2\gamma + 9\alpha\beta\hbar - 18\beta^2\hbar)}{45t_2}, \end{cases}$$

then $R = R_{\mathrm{Gar}_{5/2+3/2}}$ and $B = B_{\mathrm{Gar}_{5/2+3/2}}$ satisfies the compatibility condition

$$2\frac{\partial R}{\partial t_2} + \hbar^2 \frac{\partial^3 B}{\partial x^3} - 4R\frac{\partial B}{\partial x} - 2B\frac{\partial R}{\partial x} = 0$$

of

$$\left(\hbar^2 \frac{d^2}{dx^2} - R\right)\psi = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial t_2} = \left(B\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial B}{\partial x}\right)\psi$$

Namely, $(*)$ **describes the isomonodromic deformation of** $\mathrm{SR}_{\mathrm{Gar}_{5/2+3/2}}$ **w.r.t.** t_2.

- The above system $(*)$ can be found from Hamiltonian system w.r.t. t_2.
- [Dubrovin–Kapaev 12] gave a similar result for $\mathrm{Gar}_{9/2}$.

Isomonodromy without Painlevé Property

Observation

- The system $(*)$ of ODEs implies that $\alpha = \alpha(t_2)$ satisfies

$$(**)\ :\ \hbar^2\left(\alpha^{(4)} + \frac{2\alpha^{(3)}}{t_2} - \frac{4\alpha''\alpha^{(3)}}{\alpha'} - \frac{3(\alpha'')^2}{t_2\alpha'} + \frac{3(\alpha'')^3}{(\alpha')^2}\right)$$
$$+ \frac{4\alpha''}{t_2^2\alpha'} + 12t_2(\alpha')^3\alpha'' + 4\alpha\,(\alpha')^2\alpha'' + \frac{4\alpha\,(\alpha')^3}{t_2} + 14(\alpha')^4 + \frac{2}{t_2^3}$$

  This ODE describes the isomonodromic deformation of $\mathrm{SR}_{\mathrm{Gar}_{5/2+3/2}}$.

- We have observed that there is **no** Laurent series solution of $(**)$, but there exists a 4-parameter family of **Puiseux series solutions** of the following form:

$$\alpha(t_2) = c_1 - \frac{3^{1/3}\hbar^{2/3}(t_2-b)^{1/3}}{b^{1/3}} - \frac{c_1(t_2-b)}{5b} + \frac{5\hbar^{2/3}(t_2-b)^{4/3}}{4\cdot 3^{2/3}b^{4/3}} + \frac{c_2(t_2-b)^{5/3}}{3^{1/3}b^{5/3}\hbar^{2/3}}$$
$$+ \frac{3c_1(t_2-b)^2}{10b^2} + \frac{c_3(t_2-b)^{7/3}}{3^{2/3}b^{7/3}\hbar^{4/3}} + \frac{(81c_1^2 - 3275c_2)(t_2-b)^{8/3}}{1500\cdot 3^{1/3}b^{8/3}\hbar^{2/3}}$$
$$+ \frac{(3375b - 27c_1^4 - 1575c_1^2c_2 - 12500c_2^2 + 450c_1(7c_3 - 5\hbar^2))(t_2-b)^3}{12375b^3\hbar^2} + O((t_2-b)^{10/3}),$$

  where $(b \neq 0, c_1, c_2, c_3)$ can be taken as free parameters.

- Therefore, $(**)$ does **not** possess the Painlevé property.
  These observations are parallel with those of [Dubrovin–Kapaev 12] for $\mathrm{Gar}_{9/2}$.

Observation by [Dubrovin–Kapaev 12]

Observation ([Dubrovin–Kapaev 12])

- A similar analysis for $\mathrm{Gar}_{9/2}$ yields

$$\hbar^2\alpha^{(4)} + 40(\alpha')^3\alpha'' + 36t_2\alpha'\,\alpha'' + 4\alpha\,\alpha'' + 20\,(\alpha')^2 + 6t_2 = 0$$

  This ODE describes the isomonodromic deformation of $\mathrm{SR}_{\mathrm{Gar}_{9/2}}$.

- There exist 4-parameter family of **Puiseux series solutions** of the following form:

$$\begin{aligned}\alpha(t_2) = c_1 &- 3^{1/3}\hbar^{2/3}(t_2-b)^{1/3} + \frac{9\cdot 3^{2/3}b(t_2-b)^{5/3}}{35\hbar^{2/3}} - \frac{3^{1/3}c_1(t_2-b)^{7/3}}{7\hbar^{4/3}} \\ &+ \frac{9\cdot 3^{2/3}(t_2-b)^{8/3}}{20\hbar^{2/3}} + \frac{c_2(t_2-b)^3}{\hbar^2} + \frac{c_3(t_2-b)^{11/3}}{3^{1/3}\hbar^{8/3}} - \frac{729b(t_2-b)^4}{980\hbar^2} \\ &- \frac{3^{1/3}(729b^3+1372c_1^2+23814bc_2)(t_2-b)^{13/3}}{22295\hbar^{10/3}} + O((t_2-b)^{16/3}),\end{aligned}$$

  where (b, c_1, c_2, c_3) can be taken as free parameters.

- **Remark 1**: The above 4th order ODE and its Puiseux solution have already found in [Shimomura 01] (but the secondary isomonodromy property was not mentioned).

- **Remark 2**: The Puiseux solution also provides further singularity reduction:

$$\lim_{t_2\to b} R_{\mathrm{Gar}_{9/2}}(x) = x^5 + 3bx^3 - c_1x^2 + \frac{8748b^2+14553c_2}{3969}x - \frac{4347bc_1-7007c_3}{3969}$$

Open Questions

- We have studied only two examples: $\mathrm{Gar}_{9/2}$ and $\mathrm{Gar}_{5/2+3/2}$.
  Would a similar analysis be possible for **other Garnier systems** as well?
  - Both examples have **ramified** irregular singular point. Is it essential?
  - The pole orders of Laurent series solution is related to **homogenious degrees** of the Garnier systems (c.f., [Chiba 20]).

- Can we find a change of unknown functions $(\alpha, \beta, \gamma, \delta)$ of the system $(*)$ so that the new system possess the Painlevé property?

- Comparison with the algebro-geometric approaches by Inaba, Iwasaki, Saito?

- Relation to **classical conformal blocks**?

  In the on-going work [I–Nagoya–Shukuta 2?], we have

$$\oint_\gamma S(x,\hbar)\,dx = 2\pi i \nu \quad \Rightarrow \quad \mathcal{E} = 2t^{1/2} - 2\nu t^{1/4} + \left(\frac{\nu^2}{8} + \frac{9\hbar^2}{32}\right) + \left(\frac{\nu^3}{128} + \frac{3\nu\hbar^2}{512}\right)t^{-1/4} + \cdots (t \to \infty)$$

  where the period integral is the Voros period, and $\mathcal{E}$ is the accessory parameter of Heun III_3 (Mathieu) equation. We have checked first several terms agrees with the irregular classical conformal block proposed in [Bonelli-Shchechkin-Tanzini 25].

Thank you for your attention!

Appendix: Proof of convergence (in the case of $\mathrm{Gar}_{5/2+3/2}$)

If we take new unknown functions as

$$(\xi_1, \xi_2, \xi_3, \xi_4) = \left(\frac{1}{q_1}, \frac{q_2}{q_1^2}, 4q_2p_2 + \frac{4q_2}{q_1}, p_1q_1^2 + q_2 + 2p_2q_1q_2\right),$$

then $\mathrm{Gar}_{5/2+3/2}$ is transformed to

$$\begin{aligned}
\hbar\frac{\partial\xi_1}{\partial t_1} &= 1 + t_1\xi_1^2 - 2\xi_1^2\xi_2 + \xi_1^3\xi_3 - 2\xi_1^4\xi_4,\\
\hbar\frac{\partial\xi_2}{\partial t_1} &= 2t_1\xi_1\xi_2 - 4\xi_1\xi_2^2 + 2\xi_1^2\xi_2\xi_3 - 4\xi_1^3\xi_2\xi_4,\\
\hbar\frac{\partial\xi_3}{\partial t_1} &= 4t_1\xi_2 - 8\xi_2^2 + 4\xi_1\xi_2\xi_3 - 8\xi_1^2\xi_2\xi_4 - \frac{4t_2\xi_1^2}{\xi_2},\\
\hbar\frac{\partial\xi_4}{\partial t_1} &= \frac{t_1\xi_3}{2} + \frac{\xi_1\xi_3^2}{2} - \xi_2\xi_3 - 2t_1\xi_1\xi_4 + 4\xi_1\xi_2\xi_4 - 3\xi_1^2\xi_3\xi_4 + 4\xi_1^3\xi_4^2 - \frac{2t_2\xi_1}{\xi_2}.
\end{aligned}$$

The Laurent series solution corresponds to the solution of this system at $t_1 = \alpha$ with the initial value

$$(\xi_1(\alpha), \xi_2(\alpha), \xi_3(\alpha), \xi_4(\alpha)) = \left(0, \beta, -4\gamma - \hbar, \frac{\alpha^2 + 14\alpha\beta - 32\beta^2 + 45\delta}{18}\right).$$

Since we assumed $\beta \neq 0$, we can apply the Cauchy existence theorem.