Canonical 2-form on the moduli space of ramified connections

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This talk is based on the paper

> "Moduli space of factorized ramified connections and generalized isomonodromic deformation."
> SIGMA Symmetry Integrability Geom. Methods Appl. 19 (2023), Paper No. 013.

Plan of talk:

1. On ramified connections in the paper
2. Symplectic form on the moduli space
3. Construction of generalized isomonodromic deformation

Works on moduli spaces of connections

1. Regular case: A part of the Simpson's framework connecting Betti, de Rham and Dolbeault moduli spaces.
2. Logarighmic case:

   Non-parabolic case by Nitsure.

   Parabolic case: works with Iwasaki and Saito.
3. Unramified irregular singular case:

   On the trivial bundle over $\mathbb{P}^1$: by Boalch.

   Higher genus case: work with Saito.
4. Ramified irregular singular case:

   Over the trivial bundle on $\mathbb{P}^1$: by Bremer and Sage.

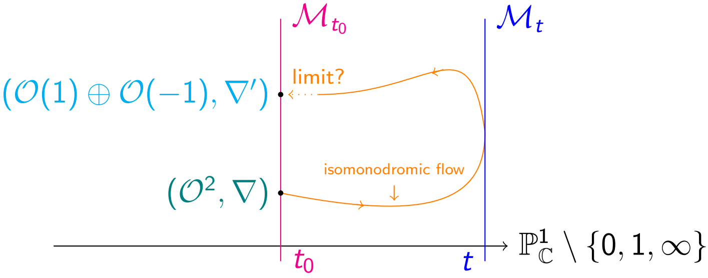
The corresponding monodromy space:

Wild character variety constructed by Boalch.

Construction with explicit descriptions in the case of Painlevé equations by van der Put and Saito.

Today we consider the moduli space of ramified connections in a higher genus case.
Even in the case of $\mathbb{P}^1$, we don't want to restrict to the connections on the trivial bundle.

In the following picture, $\mathcal{M}_t$ is the moduli space of rank 2 logarithmic connections with 4 singular points $0, 1, \infty, t$.



Ramified connections

C: smooth projective curve $/\mathbb{C}$.

D: effective divisor on C.

For $t \in D$, take a uniformaizing parameter $z \in \mathcal{O}_{C,t}$. Then $\widehat{\mathcal{O}}_{C,t} \cong \mathbb{C}[[z]]$. Consider

E: an algebraic vector bundle of rank r on C

$\nabla : E \longrightarrow E \otimes \Omega_C(D)$ rational connection.

Let w be a variable satisfying

$$w^r = z. \qquad (w \text{ is a ramifying parameter})$$

Then

$$\mathbb{C}[[w]] = \mathbb{C}[[z]]1 \oplus \mathbb{C}[[z]]w \oplus \cdots \oplus \mathbb{C}[[z]]w^{r-1}$$

is a free $\mathbb{C}[[z]]$-module of rank r.

$$m := \mathrm{mult}_t(D).$$

Definition

$(E,\nabla)\otimes\widehat{\mathcal{O}}_{C,t}$ is a generic ν-ramified connection if

$$(E,\nabla)\otimes\widehat{\mathcal{O}}_{C,t} \;\cong\; (\mathbb{C}[[w]],\nabla_\nu)$$

where

$$\begin{array}{rccc} \nabla_\nu : & \mathbb{C}[[w]] & \longrightarrow & \mathbb{C}[[w]]\otimes\dfrac{dz}{z^m} \\ & f(w) & \mapsto & df(w)+f(w)\nu(w) \end{array}$$

and $\nu(w)=\nu_0(z)+\nu_1(z)w+\cdots+\nu_{r-1}(z)w^{r-1}$ with

$$\nu_0(z)\in\sum_{l=0}^{m-1}\mathbb{C}z^l\frac{dz}{z^m},\qquad \nu_1(z)\in\mathbb{C}^\times dz+\sum_{l=1}^{m-2}\mathbb{C}z^l\frac{dz}{z^m}$$

$$\nu_k(z)\in\sum_{l=0}^{m-2}\mathbb{C}z^l\frac{dz}{z^m}\quad(2\le k\le r-1).$$

For the construction or the investigation of the moduli space, we want to replace the condition of formal isomorphism to ∇_ν with an alternative condition on the restriction $(E,\nabla)|_{mt}$.

$$(E,\nabla)\otimes\widehat{\mathcal{O}}_{C,t}\cong(\mathbb{C}[[w]],\nabla_\nu)$$

$$\rightsquigarrow\quad (E|_{mt},\nabla|_{mt})\cong(\mathbb{C}[w]/(w^{mr}),\nabla_\nu\otimes\mathbb{C}[z]/(z^m)).$$

$\nabla|_{mt}\colon E|_{mt}\longrightarrow E|_{mt}\otimes\Omega_C(D)|_{mt}$ is represented by

$$\begin{pmatrix} \nu_0(z) & z\nu_{r-1}(z) & \cdots & z\nu_1(z) \\ \nu_1(z) & \nu_0(z)+\frac{1}{r}\frac{dz}{z} & \cdots & z\nu_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{r-1}(z) & \nu_{r-2}(z) & \cdots & \nu_0(z)+\frac{r-1}{r}\frac{dz}{z} \end{pmatrix}$$

with respect to the basis $\overline{1},\overline{w},\ldots,\overline{w}^{r-1}$.
However, this condition is too strict.

If $\nabla|_{mt}$ has a representation matrix

$$\begin{pmatrix} \nu_0(z) & z\nu_{r-1}(z) & \cdots & z\nu_1(z) \\ \nu_1(z) + a_{21}\frac{dz}{z} & \nu_0(z) + \frac{1}{r}\frac{dz}{z} & \cdots & z\nu_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{r-1}(z) + a_{r,1}\frac{dz}{z} & \nu_{r-2}(z) + a_{r,2}\frac{dz}{z} & \cdots & \nu_0(z) + \frac{r-1}{r}\frac{dz}{z} \end{pmatrix} \quad (*)$$

with $a_{ij} \in \mathbb{C}$ for $i > j$, then we have

$$(E, \nabla) \otimes \widehat{\mathcal{O}}_{C,t} \cong (\mathbb{C}[[w]], \nabla_\nu). \qquad (\dagger)$$

In the paper, I introduce the notion of

ν-ramified structure on $(E, \nabla)|_{mt}$

which reflects the ambiguities in the matrix $(*)$ such that

$$\exists \nu\text{-ramified structure} \Leftrightarrow \exists \text{formal isomorphism } (\dagger).$$

We omit the precise definition of ν-ramified structure here.

Moduli space of connections on curves

$$\mathcal{M}_{C,D}(\lambda,\mu,\nu) := \{(E,\nabla) \text{ satisfying the following}\} / \cong$$

(i) E: algebraic vector bundle of rank r and degree d

(ii) $\nabla\colon E \longrightarrow E \otimes \Omega_C(D)$: rational connection

which is endowed with

(a) λ-parabolic structure at logarithmic points $t \in D$:
$E|_t = l_0^t \supset \cdots \supset l_{r-1}^t \supset l_r^t = 0$ such that
$(\mathrm{res}_t(\nabla) - \lambda_j^t \mathrm{id})(l_j^t) \subset l_{j+1}^t$ for $0 \leq j \leq r-1$,

(b) μ-unramified parabolic structure
at unramifed irregular singular points $t \in D$:
$E|_{mt} = \ell_0^t \supset \cdots \supset \ell_{r-1}^t \supset \ell_r^t = 0$ such that
$(\nabla|_{mt} - \mu_j^t \mathrm{id})(l_j^t) \subset l_{j+1}^t \otimes \Omega_C(D)|_{mt}$ for $0 \leq j \leq r-1$,

(c) ν-ramified structure at ramified irregular singular points
$t \in D$.

Theorem

Moduli space $\mathcal{M}_{C,D}(\lambda,\mu,\nu)$ *is smooth and quasi-projective.*

$\dim \mathcal{M}_{C,D}(\lambda,\mu,\nu) = 2r^2(g-1) + r(r-1)\deg D + 2$ *(if* $\mathcal{M}_{C,D}(\lambda,\mu,\nu) \neq \emptyset$*) where* g *is the genus of* C*.*

$^{\exists}\omega_{\mathcal{M}_{C,D}(\lambda,\mu,\nu)}$*: algebraic symplectic form on* $\mathcal{M}_{C,D}(\lambda,\mu,\nu)$*.*

Today we will see how to construct $\omega_{\mathcal{M}_{C,D}(\lambda,\mu,\nu)}$.

In the case of logarithmic connections or unramified irregular singular connections, parabolic structure produces a good duality on the tangent space.

For ramified connections, we introduce the notion of factorized ν-ramified structure on $(E,\nabla)|_{mt}$ which induces a duality on the tangent space and

factorized ν-ramified structure

$\Leftrightarrow$ ν-ramified structure.

Works on the canonical 2-form (irregular case)

1. Hamiltonian description of isomonodromy equations by Jimbo, Miwa and Ueno (on the trivial bundle over $\mathbb{P}^1$).
2. Krichever's construction of canonical 2-form using formal solutions (in higher genus case).
3. Boalch's construction of 2-form on the wild character variety (including ramifed case with Yamakawa).
4. Dubrovin and Mazzocco proved the coincidence of Krichever's 2-form with the 2-form by Jimbo, Miwa and Ueno (on the trivial bundle over $\mathbb{P}^1$).
5. With Saito: Construction of algebraic symplectic form (unramified irregular singular case).
6. Bremer–Sage's symplectic structure on the moduli space of ramified connections on the trivial bundle over $\mathbb{P}^1$.

Idea of factorized ramified structure

Recall $(E,\nabla)|_{mt} \;\cong\; (\mathbb{C}[w]/(w^{mr}), \nabla_\nu|_{z^m=0})$.

$${}^{\exists}N \colon E|_{mt} \longrightarrow E|_{mt} \quad \leftrightarrow \quad \mathbb{C}[w]/(w^{mr}) \xrightarrow{w} \mathbb{C}[w]/(w^{mr}).$$

Consider the $\mathcal{O}_{mt}[T]$-module structure on $E|_{mt}$ defined by

$$P(T)v := P(N)v \qquad (\text{for } v \in E|_{mt},\ P(T) \in \mathcal{O}_{mt}[T]).$$

Then we have

$$E|_{mt} \cong \mathcal{O}_{mt}[T]/(T^r - z). \tag{$\dagger$}$$

$E|_{mt}^{\vee}$ also has an $\mathcal{O}_{mt}[T]$-module structure defined by ${}^tN$ and

$$E|_{mt}^{\vee} \cong \mathcal{O}_{mt}[T]/(T^r - z). \tag{$\ddagger$}$$

Composing the isomorphisms (†), (‡), we get an isomorphism

$$\theta\colon E|_{mt}^{\vee} \xrightarrow{\sim} \mathcal{O}_{mt}[T]/(T^r - z) \xrightarrow{\sim} E|_{mt}$$

of $\mathcal{O}_{mt}[T]$-modules. Set $\kappa := \theta^{-1}N\colon E|_{mt} \longrightarrow E|_{mt}^{\vee}$.
Then θ, κ corresponds to

$$\vartheta\colon E|_{mt}^{\vee} \times E|_{mt}^{\vee} \longrightarrow \mathcal{O}_{mt}$$ symmetric perfect pairing

$$\varkappa\colon E|_{mt} \times E|_{mt} \longrightarrow \mathcal{O}_{mt}$$ symmetric pairing.

So we get $(\vartheta, \varkappa)$ (mod $\mathcal{O}_{mt}[N]^{\times}$) (which characterizes the orbit of N with respect to the adjoint action of $\mathrm{Aut}(E|_{mt})$).

Factorized ν-ramified structure is precisely defined as a modified data of $(\vartheta, \varkappa)$ considering the ambiguity $(*)$ before.

Tangent space of the moduli space

$$\mathcal{G}^0 = \left\{ u \in \mathcal{E}nd(E) \;\middle|\; \begin{array}{l} u|_t(l^t_k) \subset l^t_k,\; u|_{mt}(\ell^t_k) \subset \ell^t_k, \\ u|_{mt}(\mathrm{Im}\, N^k) \subset \mathrm{Im}\, N^k \;\; (0 \le k \le r-1) \end{array} \right\}$$

$$\mathcal{G}^1 = \left\{ v \in \mathcal{E}nd(E) \otimes \Omega^1_C(D) \;\middle|\; \begin{array}{l} v|_t(l^t_k) \subset l^t_{k+1} \otimes \Omega^1(D)|_t, \\ v|_{mt}(\ell^t_k) \subset \ell^t_{k+1} \otimes \Omega^1(D)|_{mt}, \\ v|_{mt}(\mathrm{Im}\, N^k) \subset \mathrm{Im}\, N^k \otimes \Omega^1(D)|_{mt} \\ \text{for } 0 \le k \le r-1 \end{array} \right\}$$

S = 'modification' of $\mathrm{Sym}^2(E|^\vee_{D_{\mathrm{ram}}})$ (1st order deformation of ϑ)

T = 'modification' of $\mathrm{Sym}^2(E|_{D_{\mathrm{ram}}})$ (1st order deformation of $\varkappa$)

A^0 = 'modification' of $\mathcal{O}_{D_{\mathrm{ram}}}[T]/(w^r - z)$

A^1 = 'modification' of $\mathcal{H}om(\mathcal{O}_{D_{\mathrm{ram}}}[T]/(w^r - z), \mathcal{O}_{D_{\mathrm{ram}}})$

G^1 = 'modification' of $\mathcal{G}^1|_{D_{\mathrm{ram}}}$.

Define a complex

$$\mathcal{F}^\bullet \;:\; \mathcal{G}^0 \oplus A^0 \xrightarrow{d^0_{\mathcal{F}^\bullet}} \mathcal{G}^1 \oplus S \oplus T \xrightarrow{d^1_{\mathcal{F}^\bullet}} G^1 \oplus A^1$$

of sheaves on C by

$$d^0_{\mathcal{F}^\bullet}(u,0) = (\nabla u - u\nabla,\ u\theta + \theta\,{}^t u,\ -\kappa u - {}^t u\kappa)$$
$$d^0(0,(P(T))) = (0,\ \theta P(N),\ -P({}^t N)\kappa)$$
$$d^1_{\mathcal{F}^\bullet}(v,0) = (v|_D, 0)$$
$$d^1_{\mathcal{F}^\bullet}(0,\tau,\xi) = \left(\sum \nu_j(z) N^{j-l}(\theta\xi + \tau\kappa)N^{l-1}, \mathrm{Tr}\left(\ -\ ,\theta\xi + \tau\kappa\right)\right).$$

Proposition

Tangent space of the moduli space is

$$T_{\mathcal{M}_{C,D}(\lambda,\mu,\nu)} \ \cong\ \mathbb{H}^1(\mathcal{F}^\bullet).$$

- There are dualities $S \overset{dual}{\leftrightarrow} T$, $A^0 \overset{dual}{\leftrightarrow} A^1$ and
- $(\mathcal{G}^0)^\vee \otimes \Omega^1_C \overset{\text{quasi-isom}}{\cong} [\mathcal{G}^1 \to G^1]$ (inducing Serre duality).

$$\omega_{\mathcal{M}_{C,D}(\lambda,\mu,\nu)} : \mathbb{H}^1(\mathcal{F}^\bullet) \times \mathbb{H}^1(\mathcal{F}^\bullet) \longrightarrow \mathbb{H}^2(\mathcal{O}_C \xrightarrow{d} \Omega^1_C(D_{\mathrm{ram}}) \to \Omega^1_C(D_{\mathrm{ram}})|_{D_{\mathrm{ram}}}) \cong \mathbb{C}$$

is defined by

$$\omega_{\mathcal{M}_{C,D}(\lambda,\mu,\nu)}([\{u_{\alpha\beta}, v_\alpha, \eta_\alpha\}], [\{u'_{\alpha\beta}, v'_\alpha, \eta'_\alpha\}]) = [\{\mathrm{Tr}(u_{\alpha\beta}u'_{\beta\gamma}), -\mathrm{Tr}(u_{\alpha\beta}v'_\beta - v_\alpha u'_{\alpha\beta}), \Xi(\eta_\alpha, \eta'_\alpha)\}]$$

for $\{u_{\alpha\beta}\}, \{u'_{\alpha\beta}\} \in C^1(\mathcal{G}^0)$, $\{v_\alpha\}, \{v'_\alpha\} \in C^0(\mathcal{G}^1)$,
$\eta_\alpha = (\tau_\alpha, \xi_\alpha), \eta'_\alpha = (\tau'_\alpha, \xi'_\alpha)$ and

$$\Xi(\eta_\alpha, \eta'_\alpha) = \sum_{p=1}^{r-1} \sum_{j=1}^{p} \frac{\nu_p(z)}{2} \mathrm{Tr}(\tau'_\alpha \, {}^tN^{p-j} \xi_\alpha N^{j-1} - N^{p-j} \tau_\alpha \, {}^tN^{j-1} \xi'_\alpha).$$

Idea of Ξ comes from the symplectic form on an adjoint orbit.

How to prove $d\omega = 0$

We construct a family of moduli spaces

$$\pi\colon \mathcal{M}' \longrightarrow U (\subset \mathbb{A}^1\text{: open})$$

such that

$\pi^{-1}(0) = \mathcal{M}_{C,D}(\lambda, \mu, \nu)$ and

$\pi^{-1}(u) =$ moduli space of logarithmic parabolic connections for $u \neq 0$.

We can also construct a relative 2-form $\omega' \in H^0(\mathcal{M}', \Omega^1_{\mathcal{M}'/U})$ such that

$\omega'|_{\pi^{-1}(0)} = \omega$ and

$\omega'|_{\pi^{-1}(u)}$ coincides with the symplectic form on the moduli space of logarithmic connections for $u \neq 0$.

Since $d\omega'|_{\pi^{-1}(u)} = 0$ (which was already proved), we have $d\omega' = 0 \Rightarrow d\omega = 0$.

Family of moduli spaces

$n = n_{\log} + n_{\mathrm{un}} + n_{\mathrm{ram}}$.

$\mathcal{M}_{g,n} :=$ moduli stack of n-pointed curves of genus g.

Take a universal family $(\mathcal{C}, \tilde{t}_1, \ldots, \tilde{t}_n)$ over $\widetilde{\mathcal{M}_{g,n}} \xrightarrow{\text{étale}} \mathcal{M}_{g,n}$.

$$\mathcal{D}_{\log} := \sum_{i=1}^{n_{\log}} \tilde{t}_i, \quad \mathcal{D}_{\mathrm{un}} := \sum_{i=n_{\log}+1}^{n_{\log}+n_{\mathrm{un}}} m_i \tilde{t}_i \quad (m_i \geq 2)$$

$$\mathcal{D}_{\mathrm{ram}} := \sum_{i=n_{\log}+n_{\mathrm{n}}+1}^{n} m_i \tilde{t}_i, \quad \mathcal{D} := \mathcal{D}_{\log} + \mathcal{D}_{\mathrm{un}} + \mathcal{D}_{\mathrm{ram}}.$$

Take tuples of complex numbers

$$\lambda = (\lambda^i_j)^{1 \leq i \leq n_{\log}}_{0 \leq j \leq r-1}, \; c^{\mathrm{un}} = (c^i_j)^{n_{\log}+1 \leq i \leq n_{\log}+n_{\mathrm{un}}}_{0 \leq j \leq r-1},$$
$$c^{\mathrm{ram}} = (c^i)_{n_{\log}+n_{\mathrm{un}}+1 \leq i \leq n}. \quad \text{(residue parts of exponents)}$$

Assume the equality (Fuches relation)

$$d + \sum_{i=1}^{n_{\log}} \sum_{j=0}^{r-1} \lambda_j^i + \sum_{i=n_{\log}+1}^{n_{\log}+n_{\mathrm{un}}} \sum_{j=0}^{r-1} c_j^i + \sum_{i=n_{\log}+n_{\mathrm{un}}+1}^{n} \left(rc^i + \frac{r-1}{2} \right) = 0$$

(recall $d = \deg E$). Consider the parameter space

$$\mathcal{T} = \{(\mu, \overline{z}, \nu) : \text{satisfying the following}\} \longrightarrow \widetilde{\mathcal{M}}_{g,n}$$

- $\mu = (\mu_j^i)_{0 \le j \le r-1}^{n_{\log}+1 \le i \le n_{\log}+n_{\mathrm{un}}}$ are unramified exponents, where $\mu_j^i \in \Omega_C(D)|_{m_i t_i}$ and $\mathrm{res}_{t_i}(\mu_j^i) = c_j^i$,
- $\nu = (\nu_i)_{n_{\log}+n_{\mathrm{un}}+1 \le i \le n}$ are ramified exponents with respect to $\overline{z} \in \mathfrak{m}_{t_i}/\mathfrak{m}_{t_i}^{m_i+1}$ satisfying $\mathrm{res}_{\overline{w}=0}(\nu_i) = rc^i$.

We regard $\mathcal{T}$ as a space of independent variables.

$$\mathcal{M}_{\mathcal{C},\mathcal{D}} := \coprod_{(C,D,\overline{z},\mu,\nu) \in \tilde{\mathcal{T}}} M_{C,D}(\lambda, \mu, \nu) \xrightarrow{\pi_{\mathcal{T}}} \mathcal{T}$$

(family of moduli spaces of connections)

Take a universal family $(\tilde{E}, \tilde{\nabla})$ on $\mathcal{C}_{\mathcal{M}_{\mathcal{C},\mathcal{D}}}$. So

$$\tilde{\nabla}\colon \tilde{E} \longrightarrow \tilde{E} \otimes \Omega_{\mathcal{C}_{\mathcal{M}_{\mathcal{C},\mathcal{D}}}/\mathcal{M}_{\mathcal{C},\mathcal{D}}}(\mathcal{D}_{\mathcal{M}_{\mathcal{C},\mathcal{D}}})$$

is a relative connection. Take a germ of submanifold $\mathcal{L} \subset \mathcal{M}_{\mathcal{C},\mathcal{D}}$ and an analytic open $U \subset \mathcal{C}_{\mathcal{L}}$ such that

$$U \cong \Delta \times \mathcal{L}$$

for a unit disk $\Delta = \{z \in \mathbb{C} | |z| < 1\}$. We can consider a family of Stokes data corresponding to $(\tilde{E}|_U, \tilde{\nabla}|_U)$.

Principle of Jimbo–Miwa–Ueno's Theorem

The following two conditions are equivalent.

(i) Stokes data of $\tilde{\nabla}|_U$ is 'constant' on $\mathcal{L}$

(ii) there is an integrable meromorphic connection

$$\nabla^{\mathrm{flat}}\colon \tilde{E}|_U \longrightarrow \tilde{E}|_U \otimes \Omega_{\mathcal{C}_{\mathcal{L}}}(\mathcal{D}_{\mathcal{L}})|_U$$

which induces the relative connection $\tilde{\nabla}|_{\tilde{E}|_U}$.

Global generalized isomonodromic deformation

$\pi_{\mathcal{T}} \colon \mathcal{M}_{\mathcal{C},\mathcal{D}} \longrightarrow \mathcal{T}$ induces a surjective map

$$T_{\mathcal{M}_{\mathcal{C},\mathcal{D}}} \xrightarrow{d\pi_{\mathcal{T}}} \pi_{\mathcal{T}}^{*} T_{\mathcal{T}} \longrightarrow 0 \qquad (\sharp)$$

between tangent bundles.

Theorem

We can construct a generalized isomonodromic splitting

$$\Phi \colon \pi_{\mathcal{T}}^{*} T_{\mathcal{T}} \longrightarrow T_{\mathcal{M}_{\mathcal{C},\mathcal{D}}} \quad \textit{(algebraic map)} \qquad (\heartsuit)$$

of $d\pi_{\mathcal{T}}$ in $(\sharp)$ which satisfies

$$[\Phi(v_1), \Phi(v_2)] = \Phi([v_1, v_2])$$

for vector fields $v_1, v_2 \in T_{\mathcal{T}}$.

The subbundle $\mathrm{Im}(\Phi) \subset T_{\mathcal{M}_{\mathcal{C},\mathcal{D}}}$ (called generalized isomonodromic subbundle) satisfies

$$[\mathrm{Im}\,\Phi, \mathrm{Im}\,\Phi] \subset \mathrm{Im}\,\Phi. \quad \text{(integrability condition)}$$

So it determines a foliation $\mathcal{F}^{\mathrm{GID}}_{\mathcal{M}_{\mathcal{C},\mathcal{D}}}$ on $\mathcal{M}_{\mathcal{C},\mathcal{D}}$ (called generalized isomonodromic foliation).

How to construct the splitting Φ?

Take $\forall v \in T_{\mathcal{T}}$: vector field on $\mathcal{T}' \subset \mathcal{T}$ (Zariski open).

$$v \in T_{\mathcal{T}'} \;\Leftrightarrow\; \mathcal{T}'[v] := \mathcal{T}' \times \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2) \xrightarrow{I_v} \mathcal{T}'.$$

Write $\mathcal{M}' = \mathcal{M}_{\mathcal{C},\mathcal{D}} \times_{\mathcal{T}} \mathcal{T}'$.
Consider the fiber products (twisted by I_v):

$$\mathcal{M}'[v] = \mathcal{M}_{\mathcal{C},D} \times_{\mathcal{T}} \mathcal{T}'[v] \longrightarrow \mathcal{T}'[v] \xrightarrow{I_v} \mathcal{T}' \subset \mathcal{T}$$
$$\mathcal{C}_{\mathcal{M}'[v]} = \mathcal{C} \times_{\mathcal{T}} \mathcal{M}'[v].$$

$(\tilde{E}, \tilde{\nabla})$: universal family on $\mathcal{C}_{\mathcal{M}'}$.

horizontal lift

$\exists!\,(\mathcal{E}^v, \nabla^v)$: (called a horizontal lift of $(\tilde{E}, \tilde{\nabla})$) such that

- $\mathcal{E}^v$ is a vector bundle on $\mathcal{C}_{\mathcal{M}'[v]}$
- $\nabla^v : \mathcal{E}^v \longrightarrow \mathcal{E}^v \otimes \Omega^1_{\mathcal{C}_{\mathcal{M}'[v]}/\mathcal{M}'}(\mathcal{D}_{\mathcal{M}'[v]})$ is an 'integrable connection'.
- $(\mathcal{E}^v, \nabla^v) \otimes \mathcal{O}_{\mathcal{M}'}/(\epsilon) \cong (\tilde{E}, \tilde{\nabla})$

$(\mathcal{E}^v, \nabla^v)$ induces a morphism $\mathcal{M}'[v] \xrightarrow{I_{\Phi(v)}} \mathcal{M}'$ which corresponds to a vector field $\Phi(v) \in T_{\mathcal{M}_{\mathcal{C},D}}$. □

For a germ of a leaf $\mathcal{L}$ in $\mathcal{F}^{\mathrm{GID}}_{\mathcal{M}_{\mathcal{C},\mathcal{D}}}$, a horizontal lift induces an integrable connection $\nabla^{\mathrm{flat}} : \tilde{E}_{\mathcal{L}} \longrightarrow \tilde{E}_{\mathcal{L}} \otimes \Omega^1_{\mathcal{C}_{\mathcal{L}}}(\mathcal{D}_{\mathcal{L}})$ which is a lift of the relative connection $\tilde{\nabla}_{\mathcal{L}}$.

Generalized isomonodromic 2-form

The relative symplectic form $\omega_{\mathcal{M}_{\mathcal{C},D}} \in H^0(\mathcal{M}_{\mathcal{C},D}, \Omega^2_{\mathcal{M}_{\mathcal{C},D}/\mathcal{T}})$ can be lifted to a total 2-form

$$\omega^{GIM}_{\mathcal{M}_{\mathcal{C},D}} \in H^0(\mathcal{M}_{\mathcal{C},D}, \Omega^2_{\mathcal{M}_{\mathcal{C},D}})$$

defined by

$$\omega^{GIM}_{\mathcal{M}_{\mathcal{C},D}}(v_1, v_2) := \omega_{\mathcal{M}_{\mathcal{C},D}}(v_1 - \Phi(d\pi_{\mathcal{T}}(v_1)), v_2 - \Phi(d\pi_{\mathcal{T}}(v_2)))$$

for vector fields $v_1, v_2 \in T_{\mathcal{M}_{\mathcal{C},D}}$.

(It is first defined by Komyo in the logarithmic case).

For a vector field $v \in T_{\mathcal{M}_{\mathcal{C},D}}$:

$$v \in \mathrm{Im}\,\Phi \iff \omega^{GIM}_{\mathcal{M}_{\mathcal{C},D}}(v, w) = 0 \text{ for any } w \in T_{\mathcal{M}_{\mathcal{C},D}}.$$

Theorem

$$d\omega^{GIM}_{\mathcal{M}_{\mathcal{C},D}} = 0.$$

Idea of proof: The foliation $\mathcal{F}^{\mathrm{GID}}_{\mathcal{M}_{\mathcal{C},\mathcal{D}}}$ induces

$$\mathcal{T}' \times \mathcal{M}'_{t_0} \quad \cong \quad \mathcal{M}' \quad \text{(local analytic isomorphism)}$$

where $\mathcal{T}' \subset \mathcal{T}$, $\mathcal{M}'_{t_0} \subset \pi^{-1}_{\mathcal{T}'}(t_0)$ and $\mathcal{M}' \subset \mathcal{M}_{\mathcal{C},\mathcal{D}}$ are analytic open subsets. We prove

$$\omega^{GIM}_{\mathcal{M}_{\mathcal{C},D}}|_{\mathcal{M}'} = p_2^*(\omega_{\mathcal{M}_{\mathcal{C},D}}|_{\mathcal{M}'_{t_0}})$$

where $p_2 \colon \mathcal{T}' \times \mathcal{M}'_{t_0} \to \mathcal{M}'_{t_0}$ is the second projection. □

What have not been done.

- Construction of generalized Riemann–Hilbert map to the wild character variety.
- Comparison of the canonical two form with those by Krichever, Boalch, Dubrovin–Mazzocco, ...
- Explicit description in the case corresponding to Painlevé equations (Working in progress with Komyo by constructing compactification of the moduli spaces).