Functional Identities and Geometry of Painlevé III_3 Quantum Operators

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Plan

0. Introducing some string theory/susy special functions

1. The modified Mathieu operator and Painlevé III_3

2. The McCoy-Tracy-Wu operator and Painlevé III_3

3. Relating the two operators

4. The geometrical origin of these operators and their relation

Based on 2503.21762 with M. François

1603.01174 with G.Bonelli and A.Tanzini

2304.11027 with M.Bershtein and P. Gavrylenko

Special functions

An interesting aspect of topological string theory is that it has led to the discovery of new classes of special functions that today permeate many areas of theoretical and mathematical physics: from general relativity to integrable systems.

Examples:

- Gopakumar-Vafa functions
- Nekrasov functions (in 4dim or 5dim)
- Nekrasov-Shatashvili functions (in 4dim or 5dim)

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Special functions

Example 1: the Nekrasov partition function for 4-dim SU(2) $\mathcal{N} = 2$ SYM

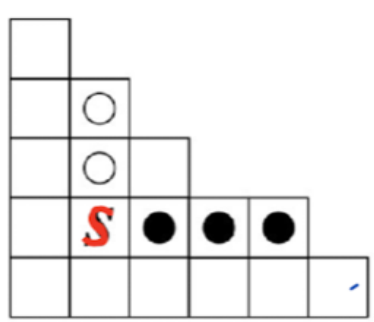
$$Z^{\text{Nek}}(\Lambda, a, \epsilon_1, \epsilon_2) = \sum_{\underbrace{Y_1,Y_2}_{\text{Young tableaux}}} \Lambda^{\overbrace{\ell(Y_1)+\ell(Y_2)}^{\text{size of Young tableaux}}} \tilde{c}_{Y_1,Y_2}(a, \epsilon_1, \epsilon_2) = 1 + \sum_{n\geq 1} c_n(a, \epsilon_1, \epsilon_2)\Lambda^n$$

$$\tilde{c}_{Y_1,Y_2}(a, \epsilon_1, \epsilon_2) = \prod_{I,J=1}^{2} \prod_{s\in Y_I} \frac{1}{\underbrace{\alpha_I - \alpha_J}_{\alpha_1 = -\alpha_2 = a} - \epsilon_1 \underline{v_{Y_J}(s)} + \epsilon_2 \left(\underline{h_{Y_I}(s)} + 1\right)} \prod_{s\in Y_J} \frac{1}{\alpha_I - \alpha_J + \epsilon_1 \left(v_{Y_I}(s) + 1\right) - \epsilon_2 h_{Y_J}(s)}$$

leg and arm of box s

nb of ● boxes

nb of ○ boxes



Thm: This is a convergent expansion [Its,Lisovyy,Tykhyy-Arnaudo, Bonelli,Tanzini]

Special functions

Example 2: the Nekrasov-Shatashvili free energy the for 4-dim SU(2) $\mathcal{N} = 2$ SYM

$\epsilon_2 \to 0$

$$F^{\mathrm{NS}}(\Lambda, a, \epsilon_1) = \lim_{\epsilon_2 \to 0} \epsilon_2 \epsilon_1 \log Z^{\mathrm{Nek}}(\Lambda, a, \epsilon_1, \epsilon_2) = 1 + \sum_{n \geq 1} \Lambda^n b_n(a, \epsilon_1)$$

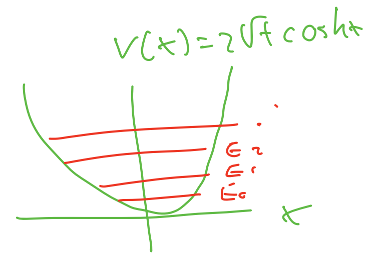
with $$b_1(a, \epsilon_1) = \frac{2}{4\epsilon_1^4 \sigma^2 - 1}, \quad b_2(a, \epsilon_1) = \frac{\left(20\sigma^2 + 7\right)}{\epsilon_1^8 \left(4\sigma^2 - 1\right)^3 \left(4\sigma^2 - 4\right)}, \quad \cdots \quad \text{where} \quad \sigma = \frac{\mathrm{a}}{\epsilon_1}$$

Comments:

- $\sum_{n \geq 1} \Lambda^n b_n(a, \epsilon_1)$ is convergent [Desiraju, Ghosal, Prokhorov 2024]
- We often define $t = \dfrac{\Lambda}{\epsilon_1^4}$, then the function just depend on σ, t

The modified Mathieu.

$\mathcal{H} = L^2(\mathbb{R})$ $\qquad O_M = -\partial_x^2 + 2\sqrt{t}\cosh x$



$$-\partial_x^2 \psi(x) + 2\sqrt{t}\cosh x\, \psi(x) = E\,\psi(x)$$

Thm $t \in \mathbb{R}_+$, O_M has a discrete spectrum $\{E_n^M\}_{n\geq 0}$. The inverse $\rho_M : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is of trace class $\mathrm{Tr}\,\rho_M^\ell = \sum_{n\geq 0} \frac{1}{E_n^\ell} < \infty \quad \forall \ell >$

In the context of Painlevé (Novokshenov '80, ...)

$P\mathrm{III}_3$ eq:
$$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} - \frac{2}{t}$$

The associated 2 x 2 linear system is (see Fokas, Its, Kapaev, Novokshenov)

$$\frac{d\psi}{dz} = A(z)\psi, \qquad A(z) = z^{-2}\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} + z^{-1}\begin{pmatrix} -p/q & t/q \\ -1 & p/q \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where $q = q(t,\sigma,\eta)$ solves PIII_3 and $p = p(t,\sigma,\eta) = \frac{t}{2}\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{q}{2}$

From 2x2 linear → scalar equation

Start from a 2 x 2 linear system

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} Y_1(z) \\ Y_2(z) \end{pmatrix} = \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix}\begin{pmatrix} Y_1(z) \\ Y_2(z) \end{pmatrix} \quad \begin{matrix} \to & Y_2 = f_1\left(Y_1, \frac{\mathrm{d}}{\mathrm{d}z}Y_1\right) \\ \to & \frac{\mathrm{d}}{\mathrm{d}z}Y_2 = f_2\left(Y_1, Y_2\right) \end{matrix} \Bigg\} \text{ 2nd order equation for } Y_1$$

Define $\tilde{Y}_1(z) = Y_1(z)/\sqrt{A_{12}(z)}$ → $\left(-\partial_z^2 + W(z)\right)\tilde{Y}_1(z) = 0$

where $W(z) = \left(-\det A + A_{11}' - \frac{A_{11}A_{12}'}{A_{12}} - \frac{2A_{12}A_{12}'' - 3(A_{12}')^2}{4A_{12}^2}\right)$

Suppose $A_{12}(z)$ has zero at $z = z_0$ → Additional singularity at $z = z_0$

We want to avoid this (singularity matching condition). → more in ArXiv: 2105.00985

↓

In Painlevé context this can be done by expanding around specific q ~ typically poles

$W(z) \longrightarrow V(x)$

For Painlevé III_3 :

The potential is $(z = \mathrm{e}^x)$:

$$W(x) = \frac{p^2 + pq - q\left(t + q^2\right)}{q^2} + \frac{t(p+q)}{q(q\mathrm{e}^x - t)} + \frac{3t^2}{4(t - q\mathrm{e}^x)^2} + \frac{t}{\mathrm{e}^x} + \mathrm{e}^x + \frac{1}{4}$$

By expanding around $q = \infty$ $\left(Z\left(t, \sigma + \frac{1}{2}, \Lambda\right) = 0 \right)$

$$\left(-\partial_z^2 + W(z)\right)\tilde{Y}_1(z) = 0 \longrightarrow \left(-\partial_x^2 + \sqrt{t}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) + E(t)\right)\tilde{Y}_1(x) = 0$$

mod. mathieu

$E(t) = t \frac{d}{dt} \log Z$

↑ painlevé III_3 Z function

$q \sim -1/\partial_{\log t}^2 Z(t)$

The McCoy-Tracy-Wu operator

$\rho_{MTW} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ with kernel

$$\rho_{MTW}(x,y) = \frac{e^{-2t^{1/4}(\cosh x + \cosh y)}}{\cosh\left(\frac{x-y}{2}\right)}$$

<u>Thm</u> Let $t \in \mathbb{R}_+$. Then ρ_{MTW} is self-adjoint with a discrete spectrum $\{E_n^{MTW}\}_{n\geq 0}$ and it is of trace class $\Longrightarrow$ its Fredholm det.

$\det(1 + E\,\rho_{MTW})$ is an entire function of E.

<u>Inverse</u> The inverse of this operator is associated with the following diff. eq

$$\phi(x+i\pi) + \phi(x-i\pi) = \frac{1}{E} e^{-8t^{1/4}\cosh x}\,\phi(x)$$

<u>Relation to Painlevé</u> <u>Thm</u> (Mc-T-W, W)

$$\tau(t,\sigma) = e^{-4\sqrt{t}} \det\left(1 + \frac{\cos(2\pi\sigma)}{2\pi}\,\rho_{MTW}^{(t)}\right)$$

computes Pain III_3 τ function with initial conditions specified by σ

PIII3 :

$$\frac{d^2w}{d^2t} = \frac{1}{w}\left(\frac{dw}{dt}\right)^2 - \frac{1}{t}\frac{dw}{dt} + \frac{2w^2}{t^2} - \frac{2}{t}.$$

$$w = -\partial^2_{\log t}\, \mathcal{T}(t,\sigma)$$

Relating the two operators [MF-AG]

Let φ_n be a basis of $L^2(\mathbb{R})$ st

$$\mathcal{S}_{MTW}\, \varphi_n = E_n^{MTW}\, \varphi_n$$

Let $P_n = |\varphi_n\rangle\langle\varphi_n|$ the ort. proj on φ_n.

We define $G(x) = \left(t\,\partial_t\, F^{NS}\!\left(\frac{1}{2\pi}\,\mathrm{Arcsech}\left(\frac{x}{2\pi}\right), t\right)\right)^{-1}$

Nek-Shat. introduced earlier.

inv. $1/\cosh$

By definition

$$G(\mathcal{S}_{MTW}) := \sum_{n \geq 0} G(E_n^{MTW}) \cdot P_n$$

Then we find

$$\boxed{\mathcal{S}_M = G(\mathcal{S}_{MTW})}$$

inv. mathieu

Idea of proof

1. Mathieu and MTW op commute

$$\Rightarrow \rho_M = \sum_{n \geq 0} \frac{1}{E_n^{math}} P_n$$

2. Need to show that

$$E_n^{Math} = t\, \partial_t F^{NS}\left(\frac{1}{2\pi} \text{Arcsech}\left(\frac{E_n^{MTW}}{2\pi}\right), t\right)$$

a) AGT correspondence

b) MTW-W theorem above

c) Kyiv formula for Painlevé τ-function

d) NY Blow up relation.

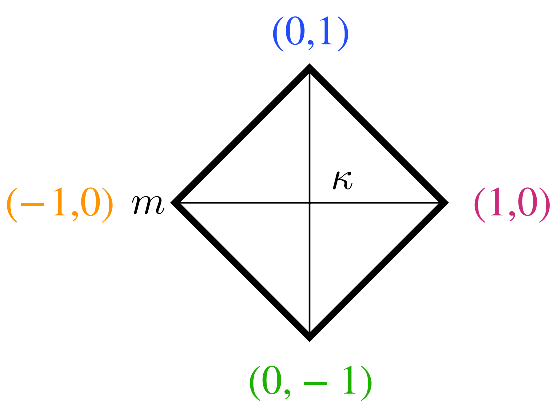
Quantum Mirror Curves

Toric Calabi–Yau threefolds can be classified in terms of two-dimensional polytopes.

To each such polytope, we associate a curve known as the mirror curve.

[Batyrev , Chiang-Klemm-Yau-Zaslow, Hori-Vafa - …].

Example: **local** $\mathbb{P}_1 \times \mathbb{P}_1$.



$$\xrightarrow{\text{mirror curve}} \qquad \mathrm{e}^{\mathrm{y}} + \mathrm{e}^{\mathrm{x}} + m\mathrm{e}^{-\mathrm{x}} + \mathrm{e}^{-\mathrm{y}} - \kappa = 0$$

m, κ: CY complex moduli

$$(a, b) \text{ vertex} \longrightarrow e^{ax+by}$$

Quantum Mirror Curves

One can quantize such mirror curves by using Weyl's prescription

$$e^{ax+by} \xrightarrow[\text{quantization}]{} e^{a\hat{x}+b\hat{p}}$$

where $\hat{x}$, $\hat{p}$ are momentum and position operators on $L^2(\mathbb{R})$ in one dimensional quantum mechanics: $\left[\hat{x}, \hat{p}\right] = \mathrm{i}\hbar$

$$e^{\hat{x}}\phi(x) = e^{x}\phi(x) \qquad\qquad e^{\hat{p}}\phi(x) = \phi(x - i\hbar)$$

[Aganagic-Dijkgraaf-Klemm-Mariño-Vafa, Aganagic-Dijkgraaf-Cheng-Krefl-Vafa, Mironov-Morozov, Nekrasov-Shatashvili,…]

Quantum Mirror Curves

Example: local $\mathbb{P}_1 \times \mathbb{P}_1$

The quantization leads to $\mathcal{O} = \mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}} + m\mathrm{e}^{-\hat{x}} + \mathrm{e}^{\hat{x}}$ whose eigenvalue equation is

$$\phi(x - \mathrm{i}\hbar) + \phi(x + \mathrm{i}\hbar) + (m\mathrm{e}^{-x} + \mathrm{e}^{x} + \kappa)\phi(x) = 0$$

Theorem: The operator $\rho = \mathcal{O}^{-1}$ has a discrete spectrum $\{E_n^{-1}\}_{n\geq 0}$ and it is of trace class on $L^2(\mathbb{R})$

$$\mathrm{Tr}\rho^N = \sum_{n\geq 0} E_n^{-N} < \infty$$

[AG-Hatsuda-Mariño, Kashaev-Mariño, Laptev-Schwimmer-Takhtajan]

Quantum Mirror Curves

Example: local $\mathbb{P}_1 \times \mathbb{P}_1$ (genus one mirror curve)

The kernel of the operator ρ is $\rho(x,y) = \dfrac{e^{-u(x,m,\hbar)-u(y,m,\hbar)}}{4\pi\cosh\left(\frac{x-y}{2}\right)}$

[Kashaev-Marino-Zakany]

$$u(x,m,\hbar) = \pi x \mathrm{b}/2 + \log\left|\frac{\phi_{\mathrm{b}}(x-\frac{1}{4\pi b}\log m + ib/4)}{\phi_{\mathrm{b}}(x+\frac{1}{4\pi b}\log m - ib/4)}\right| + \frac{1}{8}\log m \qquad \hbar = \pi b^2$$

where ϕ_{b} is the Faddeev quantum dilogarithm

If Im(b)>0 it reduces to

$$\Phi_b(x) = \frac{(e^{2\pi \mathrm{b}(x+c_b)}, e^{2i\pi b^2})_\infty}{(e^{2\pi \mathrm{b}^{-1}(x+c_b)}, e^{-2i\pi b^{-2}})_\infty}$$

$$2c_b = i(b+b^{-1})$$

Two limits

Let us consider the case of local $\mathbb{P}^1 \times \mathbb{P}^1$, we have

$$\det\left(1+\kappa\rho\right)$$

[Katz,Klemm,Vafa 1996]

[Bonelli,AG,Tanzini 2016]

$$\kappa = -\frac{2}{\sqrt{t}\beta^2} + \frac{E}{\sqrt{t}}$$

$$m^{-1} = \sqrt{t}\beta^2$$

$$\hbar \to \epsilon\beta$$

$$\beta \;\to\; 0$$

$$\kappa = \log\left(1+\mathrm{e}^{4\pi\mathrm{i}\sigma}\right) - \frac{4\pi}{\beta}\log\left(\beta^4 t\right)$$

$$\log m = -\frac{2\pi}{\beta}\log \beta^4 t$$

$$\hbar \to \frac{\epsilon}{\beta}$$

Mod. Mathieu operator

McCoy-Tracy-Wu operator

On Painlevé equations

Let us consider the case of local $\mathbb{P}^1 \times \mathbb{P}^1$, we have

$$\det\left(1+\kappa\rho\right) \qquad (1)$$

McCoy-Tracy-Wu operator

$$\det\left(1+E\rho_{\mathrm{MTW}}\right) \sim \text{Painlevé } \mathrm{III}_3 \text{ tau function}$$

(one parameter family)

Question 1: what does (1) mean in the context of Painlevé equations?

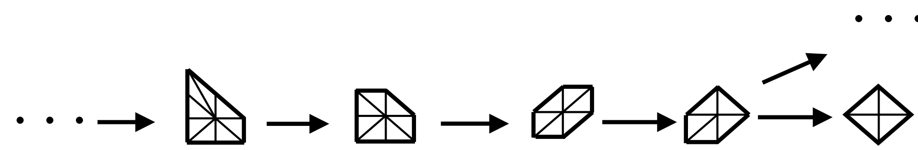
It computes the tau function of the *q*-deformed Painlevé equation with specific initial conditions [Bonelli, AG, Tanzini 2017].

Question 2: can we generalise this to other toric CY3?

Quantum curves and q-Painleve

Yes, this holds more in general

$$\begin{tikzcd} & & & & & \cdots \\ \cdots \arrow[r] & \mathrm{qP}_{\mathrm{VI}} \arrow[r] & \mathrm{qP}_{\mathrm{V}} \arrow[r] & \mathrm{qP}_{\mathrm{III}_1} \arrow[r] & \mathrm{qP}_{\mathrm{III}_2} \arrow[r] \arrow[ur] & \mathrm{qP}_{\mathrm{III}_3} \end{tikzcd}$$



Fredholm determinant of quantum mirror curves computes the tau function of *q*-Painlevé equations with specific initial conditions [Bonelli,AG,Tanzini 17+…].

The construction also extends to higher genus geometries [Gavrylenko, AG, Hao 23].

… all this is related the topological string/spectral theory correspondence and its interplay with Kyiv formula …

Thank you!

old slides

Example: $\mathrm{qP}_{\mathrm{III}_3}$ equation corresponding to local CP1

$$G(qZ)G(q^{-1}Z) = \left(\frac{G(Z)-Z}{G(Z)-1}\right)^2$$

If you rescale $G(Z) = \epsilon^2 w(t)$, $Z = \epsilon^4 t$, $q = \mathrm{e}^{\epsilon}$ in the limit $\epsilon \to 0$ the above equation reduces to Painleve III_3

$$\frac{d^2w}{dt^2} = \frac{1}{w}\left(\frac{dw}{dt}\right)^2 - \frac{1}{t}\frac{dw}{dt} + \frac{2w^2}{t^2} - \frac{2}{t}$$

Recall: we can write the solution to PIII_3 as

$$w(t,\sigma,\eta) = \sqrt{t}\mathrm{e}^{-2\pi i\eta}\left(\frac{\tau(t,\sigma,\eta)}{\tau(t,\sigma+\frac{1}{2},\eta)}\right)^2$$

tau function

Likewise we write the solution to $q\mathrm{PIII}_3$ as

$$G(Z,q,u,s) = Z^{1/2}\left(\frac{\tau(Z,q,u,s)}{\tau(Zq,q,u,s)}\right)^2$$

q-tau function

Tsuda 2006 - Bershtein, Shchechkin 2016