Floer-theoretic filtration on Painlevé Hitchin systems

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Outline

- Painlevé equations $\rightsquigarrow$ Hitchin moduli spaces
- $P_I, P_{II}, \ldots, P_{VI}$, all 2-dimensional
- We prove: Only $P_I, P_{II}, P_{IV}, P_{VI}$ admit $\mathbb{C}^*$-actions
- [Ritter-Ž. '23] Hamiltonian Floer theory + $\mathbb{C}^*$-action on real-symplectic $Y \implies$ filtration $\mathscr{F}$ on $H^*(Y)$ by cup ideals
- We compute $\mathscr{F}$ for $P_{I,II,IV}$ and compare with $P = W$ on H^2
- [Ritter-Ž. '23]: the same for *parabolic* Higgs moduli $\dim \mathscr{M}_\Gamma = 2$
- Outcome: $\mathscr{F} \subset P$ for Painlevé, but $\mathscr{F} \supset P$ for parabolic
- Uniformising both families: $\mathscr{F}|_{H^2} = \mathscr{M}$, multiplicity filtration (scheme theoretic on the nilpotent core)

Very brief intro to symplectic manifolds

- **Def:** Symplectic manifold (Y, ω) is a manifold (..) with a non-degenerate $(\omega(X, \cdot) = 0 \implies X = 0)$ closed $(d\omega = 0)$ 2-form ω.
- Necessarily even-dimensional
- Simplest example: $(\mathbb{C}^n = \mathbb{R}^{2n}, \omega_{std} = \sum_i dx_i \wedge dy_i)$,
- **Darboux chart**: $d\omega = 0 \implies (Y, \omega)$ locally $\cong (\mathbb{R}^{2n}, \omega_{std})$
- Kähler manifolds (X, ω, I), $g = \omega(\cdot, I\cdot)$ is Riemannian.
- In particular, smooth (quasi)projective varieties $/\mathbb{C}$.

Very brief intro to Floer theory

- Floer theory studies Hamiltonian flows = "symplectic gradients" $\omega(\cdot, X_H) = dH$ on symplectic manifolds (Y, ω).
- Given a Hamiltonian $H : Y \to \mathbb{R}$, Floer chain complex $CF^*(H) := \mathbb{K}\langle x : S^1 \to Y \mid \dot{x} = X_H(x)\rangle$
  d := counts $\partial_s u + I(\partial_t u - X_H) = 0$.
  (Morse complex for $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$)
- **Upshot:** For closed Y, $HF^*(H) \cong H^*(Y)$.
- For open Y and a "small" H_δ still get $HF^*(H_\delta) \cong H^*(Y)$.
- for non-small H **issue: non-compactness of** Y
  $\implies$ assume (Y, ω) Liouville $= (\Sigma \times [1, \infty), d(R\alpha))$ at ∞
  $+\ H_\lambda = \lambda R$ at ∞, $\lambda > 0$ generic
  $\implies$ **symplectic cohomology** $SH^*(Y) := \lim_{\lambda\to\infty} HF^*(H_\lambda)$.

Example: $SH^*(T^*Q) \cong H_{\dim M - *}(\mathcal{L}Q)$

Symplectic $\mathbb{C}^*$-manifolds (Ritter-Ž.)

Definition

Symplectic $\mathbb{C}^*$-manifold is a connected symplectic manifold (Y, ω, I) admitting a pseudoholomorphic $\mathbb{C}^*$-action φ whose S^1-part is Hamiltonian.

- Assume $\mathbb{C}^*$-action is *contracting*, $\mathfrak{F} := Y^{\mathbb{C}^*}$ is compact and $\forall y, \exists \lim_{\mathbb{C}^* \ni t \to 0} t \cdot y \in \mathfrak{F}$.
- The other limit defines the $\mathrm{Core}(Y) := \{y \in Y \mid \exists \lim_{\mathbb{C}^* \ni t \to \infty} t \cdot y\}$.

Theorem

1. *$\mathrm{Core}(Y)$ is compact and connected.*
2. *It is deformation retract of Y when $(Y, Core(Y))$ CW-pair*
3. *$H^*(Y) \cong \bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha]$*

$\implies \exists! \mathfrak{F}_{\min}$ *minimum of H* **(minimal component)**.

Symplectic $\mathbb{C}^*$-manifolds over a convex base

- Attempt to define $SH^*(Y)$ as for Liouville (Y almost never Liouville)
  Issue: Analysis does not work (apriori).
- Motivated by examples, impose further: there is a proper map

$$\Psi : (Y \setminus \text{compact}, I) \to (\Sigma \times [1,\infty), I_B), \ \ \Psi_* X_{S^1} = (f > 0) \cdot \mathcal{R}_B.$$

- Such Y we call **Symplectic $\mathbb{C}^*$-manifolds over a convex base.**
- **Main examples:** Equivariant projective morphisms $p : Y \to X$ to affine X with a contracting $\mathbb{C}^*$-action.
  Here equivariantly embed $X \subset \mathbb{C}^n =: B$ and compose with p to get Ψ.
- In particular: toric varieties, symplectic resolutions, weighted homogeneous singularities, quotient singularities, Higgs moduli spaces

Construction of Symplectic cohomology

Theorem (Construction of SH)

Given a symplectic $\mathbb{C}^*$-manifold over a convex base (Y, ω, I, φ), $SH(Y, \varphi) := \varinjlim_\lambda HF(F)$ is a well-defined unital ring ($F = \lambda H$ at infinity)

Considering "clean" Hamiltonians λH, for φ-generic λ, we get:

Proposition

$c_1(Y) = 0 \implies SH^*(Y, \varphi) = 0.$
(idea: support of $HF^*(\lambda H)$ shifts negatively, linearly with λ)

Application: Filtration on cohomology

- Canonical $c_\lambda^* : QH^*(Y) \cong HF^*(F_{\text{small slope}}) \to HF^*(F_\lambda)$
- Filtration $\mathscr{F}_\lambda^\varphi := \ker c_\lambda^*$ "survival time"

Proposition

$\exists$ Floer-theoretic filtration $\mathscr{F}_\lambda^\varphi(QH^*(Y))$ by ideals on the ring $QH^*(Y)$. If $SH^*(Y) = 0$, it exhausts it, otherwise define $\mathscr{F}_{+\infty}^\varphi := QH^*(Y)$

- $\mathscr{F}^\varphi$ is compatible with grading $\implies$ get filtrations $\mathscr{F}^\varphi(QH^k(Y))$.
- Although $SH^*(Y, \varphi)$ is usually φ-independent, $\mathscr{F}^\varphi$ **can** depend on φ!
- Specialise at $T = 0$ (Novikov $\mathbb{K} = \{\sum_n a_n T^{r_n} \mid \mathbb{R} \ni r_n \to +\infty\}$) $\implies$ filtration $\mathscr{F}_{\mathbb{B},\lambda}^\varphi$ on $H^*(Y, \mathbb{B})$ by cup-ideals, $\mathrm{rk}_{\mathbb{K}} \mathscr{F}_\lambda = \mathrm{rk}_{\mathbb{B}} \mathscr{F}_{\mathbb{B},\lambda}$.

Lower bounds on filtration

- Using clean Hamiltonian λH we get the energy spectral sequence

$$\oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)] \implies HF^*(\lambda H)$$

  where $2 \mid \mu_\lambda(\mathfrak{F}_\alpha)$ computable via **weights** $T_{\mathfrak{F}_\alpha} Y = \oplus \mathbb{C}_{w_i}$.
- When $H^{odd}(Y) = 0$ get

$$\oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)] \cong HF^*(\lambda H)$$

- The continuation maps $c^*_\lambda : \oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha] \to \oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)]$

Proposition

$$rk(\mathscr{F}_\lambda(H^k(Y)) \geq \sum_\alpha b_{k-\mu_\alpha}(\mathfrak{F}_\alpha) - b_{k-\mu_\lambda(\mathfrak{F}_\alpha)}(\mathfrak{F}_\alpha).$$

Survival of the minimal component

- Assuming that $H^{odd}(Y) = 0$, recall the continuation map becomes: $c_\lambda^* : \oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha] \to \oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)]$

Proposition

Assume $H^{odd}(Y) = 0$ and $\lambda < 1/(max\ absolute\ weight\ of\ \mathfrak{F}_{\min})$.

$$c_\lambda^*|_{H^*(\mathfrak{F}_{min})} = \mathrm{id}_{H^*(\mathfrak{F}_{\min})[-\mu_\lambda(\mathfrak{F}_{\min})]} + (T^{>0}\text{-}terms),$$

hence

$$\mathscr{F}^{\varphi}_{\mathbb{B},\lambda} \subset \bigoplus_{\alpha \neq \min} H^*(\mathfrak{F}_\alpha; \mathbb{B})[-\mu_\alpha].$$

Spectral sequence reads the filtration

- On $\mathbb{C}^n$ (Liouville), use a convex Hamiltonian H_λ that is linear at infinity, and the action functional $\mathcal{A}_H$ to filter $CF^*(H_\lambda)$.

Theorem

Projecting via $\Psi : Y \to B$, can use the modification of [McLean–Ritter'18] filtration on B to get filtration on $CF^*(H_\lambda)$, that follows the value of moment map H, such that the continuation maps $CF^*(H_\lambda) \subset CF^*(H_{\lambda'})$.

Corollary

There is a Morse–Bott–Floer spectral sequence
$\bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha] \oplus \bigoplus H^*(B_{p,\beta})[-\mu_{p,\beta}] \Rightarrow SH^*(Y,\varphi)$, *where*
$\sqcup_\beta B_{p,\beta} = \{H = H_p\} \cap Y^{\mathbb{Z}/m}$, $c'(H_p) =: T_p = \frac{2\pi k}{m}$, $(k,m) = 1$.

Proposition (Spectral sequence reads the filtration)

$x \in \mathscr{F}_\lambda^\varphi \Leftrightarrow$ *the columns having* $T_p \leq \lambda$ *kill* $x \in E_1^{0,q} = H^*(Y)$.

$\mathbb{C}^*$-action on Painlevé moduli spaces

- Ordinary Higgs moduli $\mathcal{M} := \{[E, \theta] \mid \text{stable pairs}\}/\text{gauge}$ have natural $\mathbb{C}^*$-action induced from the Higgs field $t \cdot (E, \theta) = (E, t\theta)$. It acts $t \cdot \Omega_I = t\Omega_I$ on holo-symplectic form.
- Painlevé spaces are irregular Higgs moduli on $\mathbb{CP}^1$ with Higgs poles at $|D| = 4$. Choices of partition of 4 and linear algebra at poles distinguish $PI, \ldots, PVI$
- Irregular Higgs moduli boundary conditions at poles need not to be $\mathbb{C}^*$-equivariant $\implies$ no $\mathbb{C}^*$-action apriori.

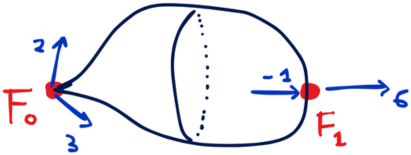
Theorem

For PI, PII, PIV, PVI, with a degenerate choice of parameters (residue not regular semi-simple), there is equivariant $\mathbb{C}^*$-action on $h^{PX} : \mathcal{M}^{PX} \to B^{PX} \cong \mathbb{C}$ Ω_I is weight-1 only in PVI.

- Other Painlevé (PIII, PV) have more than one singular Hitchin fibre, so no equivariant $\mathbb{C}^*$-action on $h^{PX} : \mathcal{M}^{PX} \to \mathbb{C}$.

Computing $\mathscr{F}$ on Painlevé I

- $\mathrm{Core}(\mathcal{M}^{PI}) =$ cuspidal$(x^2 = y^3)$ curve of genus 0.
- $(\mathcal{M}^{PI})^{\mathbb{C}^*} = F_0(\text{cusp}) \sqcup F_1, \; F_i \cong *$
- $T_{F_0} = \mathbb{C}_2 \oplus \mathbb{C}_3, \, T_{F_1} = \mathbb{C}_{-1} \oplus \mathbb{C}_6$

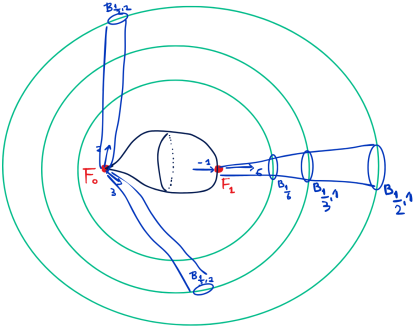


- Method [Ritter-Ž. I] gives complete description:
  1. Lower bounds $\mathscr{F}_{1/6} \supset H^2, \mathscr{F}_{1/3} = H^*$,
  2. Unit survival $1 \notin \mathscr{F}_{1/3^-}$

  $\implies \mathscr{F}_{1/6} = H^2 \subset \mathscr{F}_{1/3} = H^*$,

Computing $\mathscr{F}$ on Painlevé I, via spectral sequence

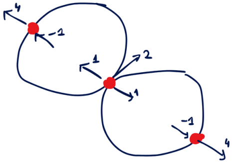
- Can also use the method from [Ritter-Ž. II], i.e. spectral sequence
- $B_{1/6} = S^1$, $B_{1/3} = S^1 \sqcup S^1$, $B_{1/2} = S^1 \sqcup S^1$.



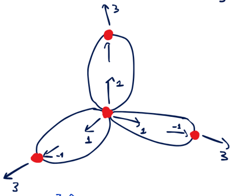
p+q\p	$H^*(M^{PI})$	$H^*(B_{1/6})[0]$	$H^*(B_{1/3})[2]$	$H^*(B_{1/2})[4]$	$H^*(B_{2/3})[6]$	$H^*(B_{5/6})[8]$	$H^*(B_1)[10]$
2	•						
1		•					
0	•	•					
-1			• •				
-2			• •				
-3				• •			
-4				• •			
-5					• •		
-6					• •		
-7						•	•
-8						•	•
-9							•
-10							•

$\implies \mathscr{F}_{1/6} = H^2 \subset \mathscr{F}_{1/3} = H^*$ again.

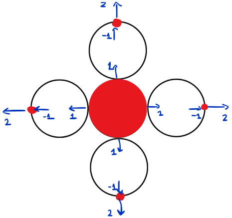
Computing $\mathscr{F}$ on Painlevé II, IV, VI



p+q\p	$H^*(M^{PII})$	$H^*(B_{1/4})[0]$	$H^*(B_{1/2})[2]$	$H^*(B_{3/4})[4]$	$H^*(B_1)[6]$
2	• •				
1		• •			
0	•	• •			
-1			• • •		
-2			• • •		
-3				• •	•
-4				• •	•
-5					•
-6					•



p+q\p	$H^*(M^{PIV})$	$H^*(B_{1/3})[0]$	$H^*(B_{2/3})[2]$	$H^*(B_1)[4]$
2	• • •			
1		• • •		
0	•	• • •		
-1			• • •	•
-2			• • •	•
-3				•
-4				•

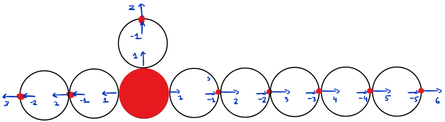


p+q\p	$H^*(M^{PVI})$	$H^*(B_{1/2})[0]$	$H^*(B_1)[2]$	$H^*(B_{3/2})[2]$	$H^*(B_2)[4]$
2	• • • • •				
1		• • • •	•		
0	•	• • • •	•		
-1			•	• • • •	•
-2			•	• • • •	•
-3					•
-4					•

Upshot: $\mathscr{F}$ is refined by $P = W$

$\mathscr{F}$ on parabolic Higgs moduli $\mathcal{M}_\Gamma$ (Ritter–Ž.)

- Parabolic Higgs moduli of dim $= 2$ are
  $\mathcal{M}_\Gamma = T^*E/\Gamma, \;\; \Gamma \in \{0, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6\}$
- projection $T^*E \cong E \times \mathbb{C} \to \mathbb{C}$ yields the Hitchin map $\mathcal{M}_\Gamma \to \mathbb{C}$, and $\mathbb{C}^*$-action from fibre-dilation on T^*E makes it equivariant.
- In [Szabo–Ž.] describe this in the Higgs moduli language.
- $\mathrm{Core}(\mathcal{M}_\Gamma) = Q_\Gamma$-tree of curves, where $Q_\Gamma = \widetilde{A}_0, \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$

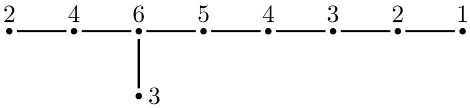


p+q\p	$H^*(\mathcal{M}_{\mathbb{Z}/6})$	$H^*(B_{1/6})[0]$	$H^*(B_{1/3})[0]$	$H^*(B_{1/2})[0]$	$H^*(B_{2/3})[4]$	$H^*(B_{5/6})[0]$	$H^*(B_1)[0]$	$H^*(B_{7/6})[0]$	$H^*(B_{4/3})[2]$
2	• • • • • • • • •								
1		•	• •	• •	• •	•	•		
0	•	•	• •	• •	• •	•	•		
-1							•	•	• •
-2							•	•	• •

- Upshot: $\mathscr{F}$ is a refinement of $P = W$, for all $\mathcal{M}_\Gamma$.

Comparison with multiplicity fitration

- Noticed that $\mathscr{F}$=imaginary root labels



- Fact: Imaginary root of Q_Γ depicts the multiplicities of the components of $\mathrm{Core}(\mathcal{M}_\Gamma)$
- $h : \mathcal{M} \to \mathbb{C}^n \implies \mathrm{Core}(\mathcal{M}) = h^{-1}(0) = \cup_i m_i E_i$ (scheme)
- $\mathrm{Core}(\mathcal{M})$ is Lagrangian $\implies E_i$ equidimensional
  $\implies [E_i]$ is a base on $H^{mid}(\mathcal{M})$
  $\implies$ filtration $\mathscr{M}_k := \{[E_i] \mid m_i \leq k\}$ on $H^{\mathrm{mid}}(\mathscr{M})$.

Theorem

$\mathscr{F}_{\mathbb{B}}(H^{mid}(\mathcal{M})) = \mathscr{M}$ *(rank-wise), for *all** $\dim_{\mathbb{C}} = 2$ *Higgs moduli* $\mathcal{M}$.

- Higher dimensions? ($\mathrm{Hilb}^n(\mathcal{M}_\Gamma)$, in progress with S. Minets)

The end

Thank you for listening.