Classical Solutions of the Fifth Painlevé Equation

Peter Clarkson
School of Engineering, Mathematics and Physics
University of Kent, Canterbury, CT2 7NF, UK
P.A.Clarkson@kent.ac.uk

Web-seminar on Painlevé Equations and related topics
11 February 2026

Joint work with Clare Dunning (University of Leeds)
and Ben Mitchell (University of Kent)

Earlier contributions from former PhD students:
Sanjida Islam and Bryn Thomas

Outline

Fifth Painlevé equation ($\mathrm{P_V}$)

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}$$

with α, β, γ and $\delta \neq 0$ arbitrary constants

1. Introduction
2. Special function solutions
   - Kummer function solutions
   - Bessel function solutions
3. Rational solutions
   - Generalised Laguerre polynomials
   - Generalised Umemura polynomials
   - Non-unique rational solutions
   - Application to random matrices
4. Bäcklund transformations
5. Associated discrete Painlevé equations
6. Application to quantum minimal surfaces
7. Rational solutions of discrete Painlevé equations

Painlevé Equations

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z \qquad \mathrm{P_I}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \qquad \mathrm{P_{III}}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \qquad \mathrm{P_V}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\} \qquad \mathrm{P_{VI}}$$

with α, β, γ and δ arbitrary constants.

Some Properties of the Painlevé Equations

- The **general solutions** of the Painlevé equations are **transcendental**, for all values of the parameters.
- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **special function solutions** expressed in terms of the classical special functions [Airy, Bessel, parabolic cylinder, Kummer, hypergeometric] for certain values of the parameters, which have one arbitrary constant.
- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **rational solutions** for certain values of the parameters, which have no arbitrary constants.
- $\mathrm{P_{III}}$, $\mathrm{P_{V}}$ and $\mathrm{P_{VI}}$ have **algebraic solutions** for certain values of the parameters, which have no arbitrary constants.
- Rational, algebraic and special function solutions of $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ can often be written as **determinants**, usually as **Wronskians**.
- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **Bäcklund transformations** which map solutions of a given Painlevé equation to solutions of the same Painlevé equation, with different values of the parameters, or another equation.
- $\mathrm{P_{I}}$–$\mathrm{P_{VI}}$ can be written as a (non-autonomous) **Hamiltonian system**.

Classical Solutions of the Fifth Painlevé Equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} - \frac{w(w+1)}{2(w-1)}$$

where we have set $\delta = -\frac{1}{2}$, without loss of generality.

Definition

A ***classical solution*** of a Painlevé equation is either a special function solution, a rational solution, an algebraic solution or a solution in terms of elementary functions.

Special Function Solutions of P_V

$$\frac{d^2w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2(\alpha w^2+\beta)}{z^2 w} + \frac{\gamma w}{z} - \frac{w(w+1)}{2(w-1)}$$

Theorem

P_V *with* $\delta = -\frac{1}{2}$, *has solutions in terms of* ***Kummer functions*** $M(a,b,z)$ *and* $U(a,b,z)$, *or equivalently* ***Whittaker functions*** $M_{\kappa,\mu}(z)$ *and* $W_{\kappa,\mu}(z)$, *if and only if*

$$\varepsilon_1\sqrt{2\alpha} + \varepsilon_2\sqrt{-2\beta} + \varepsilon_3\gamma = 2n+1, \qquad \textit{or} \qquad \left(\sqrt{2\alpha}-n\right)\left(\sqrt{-2\beta}-n\right) = 0$$

where $n \in \mathbb{Z}$, *with* $\varepsilon_j = \pm 1$, $j = 1,2,3$, *independently.*

- The **Kummer functions** $M(a,b,z)$, $U(a,b,z)$ satisfy

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0$$

- The **Whittaker functions** $M_{\kappa,\mu}(z)$, $W_{\kappa,\mu}(z)$ satisfy

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right)w = 0$$

Special Function Solutions of $\mathrm{P_V}$

Theorem (Okamoto [1987], Masuda [2004])

Let

$$\tau_n^{(i,j)}(z) = \det\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{k+\ell} \varphi_{i,j}(z)\right]_{k,\ell=0}^{n-1}$$

where

$$\varphi_{i,j}(z) = c_1\frac{\Gamma(a+i)\Gamma(b-a-i+j)}{\Gamma(b+j)}M(a+i;b+j;z) + \frac{c_2}{\sin\{\pi(b-a-i+j)\}\Gamma(2-b-j)}U(a+i;b+j;z)$$

with $M(a,b,z)$*,* $U(a,b,z)$ ***Kummer functions*** *and* c_1*,* c_2 *arbitrary constants. Then*

$$w(z;\alpha,\beta,\gamma) = -\left(\frac{b-a}{b-a-1}\right)^n \frac{\tau_n^{(0,0)}(z)\,\tau_{n+1}^{(1,1)}(z)}{\tau_n^{(1,0)}(z)\,\tau_{n+1}^{(0,1)}(z)}$$

is a solution of $\mathrm{P_V}$ *for*

$$(\alpha,\beta,\gamma) = \left(\tfrac{1}{2}(b-a)^2, -\tfrac{1}{2}(a+n)^2, n+1-b\right)$$

Theorem (PAC, Dunning & Mitchell [2026])

Let $\mathcal{K}_n^{(a,b)}(z)$ be the bi-directional Wronskian determinant given by

$$\mathcal{K}_n^{(a,b)}(z;c_1,c_2) = \det\left|\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}}\left\{c_1 M(a,b,z) + c_2 U(a,b,z)\right\}\right|_{j,k=0}^{n-1}.$$

Then

$$w_n(z;a,b) = \frac{1}{a}\frac{\mathcal{K}_{n+1}^{(a,b)}(z;c_1(a+1-b),c_2)\,\mathcal{K}_n^{(a+1,b)}(z;c_1,c_2)}{\mathcal{K}_n^{(a,b)}(z;c_1(a+1-b),c_2)\,\mathcal{K}_{n+1}^{(a+1,b)}(z;c_1,c_2)},$$

is a solution of $\mathrm{P_V}$ *for the parameters*

$$(\alpha,\beta,\gamma) = \left(\tfrac{1}{2}(a+n)^2, -\tfrac{1}{2}(b+n-a+1)^2, b-2\right)$$

Example. When $n=0$

$$w_0(z;a,b) = \frac{(a+1-b)c_1 M(a,b,z) + c_2 U(a,b,z)}{a\left[c_1 M(a+1,b,z) + c_2 U(a+1,b,z)\right]}$$

which is a solution of $\mathrm{P_V}$ for the parameters

$$(\alpha,\beta,\gamma) = \left(\tfrac{1}{2}a^2, -\tfrac{1}{2}(a-b+1)^2, b-2\right)$$

Relationship between Kummer and Bessel Functions

If $b = 2a = 2\nu + 1$, then it is well known that

$$M\left(\nu+\tfrac{1}{2}, 2\nu+1, 2z\right) = \Gamma(\nu+1)\, e^{z} \left(\tfrac{1}{2}z\right)^{-\nu} I_\nu(z)$$
$$U\left(\nu+\tfrac{1}{2}, 2\nu+1, 2z\right) = \pi^{-1/2} \mathrm{e}^{z} \left(2z\right)^{-\nu} K_\nu(z)$$

with $I_\nu(z)$ and $K_\nu(z)$ the **modified Bessel functions**.

More generally, if $b = 2a + n$, $n \in \mathbb{Z}$, and $a = \nu + \frac{1}{2}$, then for $n \geq 0$

$$M\left(\nu+\tfrac{1}{2}, 2\nu+1+n, 2z\right) = \Gamma\left(\nu\right) \mathrm{e}^{z} \left(\tfrac{1}{2}z\right)^{-\nu} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2\nu)_k(\nu+k)}{(2\nu+1+n)_k} I_{\nu+k}(z)$$

$$M\left(\nu+\tfrac{1}{2}, 2\nu+1-n, 2z\right) = \Gamma\left(\nu-n\right) \mathrm{e}^{z} \left(\tfrac{1}{2}z\right)^{n-\nu} \times \sum_{k=0}^{n} \binom{n}{k} \frac{(2\nu-2n)_k(\nu-n+k)}{(2\nu+1-n)_k} I_{\nu+k-n}(z)$$

$$U\left(\nu+\tfrac{1}{2}, 2\nu+1+n, 2z\right) = \frac{2\mathrm{e}^{z}\left(2z\right)^{-\nu}\left(\nu+\frac{1}{2}\right)_n}{\sqrt{\pi}} \sum_{k=0}^{n} \binom{n}{k} \frac{\nu+k}{(2\nu+k)_{n+1}} K_{\nu+k}(z)$$

$$U\left(\nu+\tfrac{1}{2}, 2\nu+1-n, 2z\right) = \frac{(-1)^n\, 2\mathrm{e}^{z}\left(2z\right)^{n-\nu}}{\sqrt{\pi}} \times \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\nu+k-n}{(2\nu+k-2n)_{n+1}} K_{\nu+k-n}(z)$$

University of Kent

Bessel Function Solutions of $\mathrm{P_V}$

Theorem (PAC, Dunning & Mitchell [2026])

For $n\in\mathbb{N}$ *and* $m\in\mathbb{Z}$ *define Wronskian*

$$\mathcal{B}_{m,n}^{(\nu)}(z)=\mathrm{Wr}\left(\left\{f_{m-j,\nu+j}(z)\right\}_{j=0}^{n-1}\right)$$

where

$$f_{m,\nu}(z)=z^{-\nu}\sum_{j=0}^{m}\binom{m}{j}\frac{\nu+j}{(2\nu+j)_{m+1}}(-1)^j\,\mathcal{Z}_{\nu+j}\left(\tfrac{1}{2}z\right)$$

$$f_{-m,\nu}(z)=z^{m-\nu}\sum_{j=0}^{m}\binom{m}{j}\frac{j-\nu}{(j-2\nu)_{m+1}}\mathcal{Z}_{\nu-j}\left(\tfrac{1}{2}z\right)$$

with

$$\mathcal{Z}_\nu(z)=\begin{cases}c_1I_\nu(z)+c_2I_{-\nu}(z), & \textit{if}\quad \nu\notin\mathbb{N}\\ c_1I_n(z)+c_2(-1)^nK_n(z), & \textit{if}\quad n\in\mathbb{N}\end{cases}$$

Then

$$w_{m,n}^{(\nu)}(z)=c_{m,n}^{(\nu)}\frac{\mathcal{B}_{m,n+1}^{(\nu)}(z)\mathcal{B}_{m-2,n}^{(\nu+1)}(z)}{\mathcal{B}_{m,n}^{(\nu)}(z)\mathcal{B}_{m-2,n+1}^{(\nu+1)}(z)}$$

with $c_{m,n}^{(\nu)}$ *a (known) constant, is a solution of* $\mathrm{P_V}$ *for the parameters*

$$(\alpha,\beta,\gamma)=\left(\tfrac{1}{8}\left(2\nu+2n+1\right)^2,-\tfrac{1}{8}\left(2\nu+2n+2m-1\right)^2,2\nu+m-1\right)$$

Rational Solutions of $\mathrm{P_V}$

$$\frac{d^2w}{dz^2} = \left(\frac{1}{2w}+\frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2(\alpha w^2+\beta)}{z^2\,w} + \frac{\gamma w}{z} - \frac{w(w+1)}{2(w-1)}$$

Theorem (Kitaev, Law & McLeod [1994])

$\mathrm{P_V}$ with $\delta=-\frac{1}{2}$ has a rational solution if and only if one of the following holds:

(i) $(\alpha,\beta,\gamma)=\left(\frac{1}{2}m^2,-\frac{1}{2}(m+2n+1+\mu)^2,\mu\right)$, *for* $m\geq 1$*;*

(ii) $(\alpha,\beta,\gamma)=\left(\frac{1}{2}(m+\mu)^2,-\frac{1}{2}(n+\varepsilon\mu)^2,m+\varepsilon n\right)$, *with* $\varepsilon=\pm1$, *provided that* $m\neq 0$ *or* $n\neq 0$*;*

(iii) $(\alpha,\beta,\gamma)=\left(\frac{1}{2}(m+\frac{1}{2})^2,-\frac{1}{2}(n+\frac{1}{2})^2,\mu\right)$, *provided that* $m\neq 0$ *or* $n\neq 0$,

where $m,n\in\mathbb{Z}^+$ *and* μ *is an arbitrary constant, together with the solutions obtained through the symmetries*

$$\mathcal{S}_1:\qquad w_1(z;\alpha_1,\beta_1,\gamma_1)=w(-z;\alpha,\beta,\gamma),\qquad (\alpha_1,\beta_1,\gamma_1)=(\alpha,\beta,-\gamma)$$

$$\mathcal{S}_2:\qquad w_2(z;\alpha_2,\beta_2,\gamma_3)=\frac{1}{w(z;\alpha,\beta,\gamma)},\qquad (\alpha_2,\beta_2,\gamma_3)=(-\beta,-\alpha,-\gamma)$$

where $w(z;\alpha,\beta,\gamma)$ *is a solution of* $\mathrm{P_V}$.

Case (i): Generalised Laguerre Polynomials

The rational solutions in case (i) are special cases of the solutions of $\mathrm{P_V}$ expressible in terms of **Kummer functions** $M(a,b,z)$ and $U(a,b,z)$, with $a=-n\in\mathbb{Z}^-$, since

$$U(-n,\alpha+1,z)=(-1)^n(\alpha+1)_nM(-n,\alpha+1,z)=(-1)^nn!L_n^{(\kappa)}(z)$$

where $L_n^{(\alpha)}(z)$ is the **Laguerre polynomial**.

Definition

The ***generalised Laguerre polynomial*** $T_{m,n}^{(\mu)}(z)$ is given by

$$T_{m,n}^{(\mu)}(z)=\det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}}L_{m+n}^{(\mu+1)}(z)\right]_{j,k=0}^{n-1},\qquad m\geq 0,\qquad n\geq 1$$

with μ a parameter, $L_n^{(\alpha)}(z)$ the **Laguerre polynomial** and $T_{m,0}^{(\mu)}(z)=1$.

Remark. The generalised Laguerre polynomial $T_{m,n}^{(\mu)}(z)$ can also be written as the Wronskian

$$T_{m,n}^{(\mu)}(z)=\mathrm{Wr}\left(L_{m+1}^{(n+\mu)}(z),L_{m+2}^{(n+\mu)}(z),\ldots,L_{m+n}^{(n+\mu)}(z)\right)$$

University of Kent

Theorem (PAC & Dunning [2024])

Define the polynomial $\tau_{m,n}^{(\mu)}(z)$

$$\tau_{m,n}^{(\mu)}(z) = \det\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j+k} L_{m+n}^{(n+\mu)}(z)\right]_{j,k=0}^{n-1}$$

with $L_n^{(\alpha)}(z)$ *the* ***Laguerre polynomial****, then*

$$w_{m,n}^{(\mu)}(z) = \left(\frac{m+\mu+2n}{m+\mu+2n+1}\right)^n \frac{\tau_{m-1,n}^{(\mu)}(z)\,\tau_{m-1,n+1}^{(\mu)}(z)}{\tau_{m,n}^{(\mu)}(z)\,\tau_{m-2,n+1}^{(\mu)}(z)}, \qquad m,n \geq 1,$$

is a rational solution of $\mathrm{P_V}$ *for the parameters*

$$(\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n}) = \left(\tfrac{1}{2}m^2, -\tfrac{1}{2}(m+2n+1+\mu)^2, \mu\right)$$

Lemma (PAC & Dunning [2024])

The polynomials $\tau_{m,n}^{(\mu)}(z)$ *and* $T_{m,n}^{(\mu)}(z)$ *are related as follows*

$$\tau_{m,n}^{(\mu)}(z) = a_{m,n} z^{n(n-1)/2} T_{m,n}^{(\mu)}(z), \qquad a_{m,n} = \prod_{j=1}^{n}(m+n+j+\mu)^{j-1}.$$

In Case (i), rational solutions of $\mathrm{P_V}$ arise when

$$\alpha = \tfrac{1}{2}\ell^2, \qquad \beta = -\tfrac{1}{2}(k+\gamma)^2$$

where $\ell > 0$ and $\ell + k$ is odd, with $\ell, k \in \mathbb{Z}$, and $\beta \neq 0$ when $|k| < \ell$. There are three cases (a), $k > \ell$, (b), $|k| \leq \ell$ and (c), $-k > \ell$.

Theorem (PAC & Dunning [2024])

(a) *Let* $\ell = m$ *and* $k = m + 2n + 1$*, with* $m \geq 1$*, so* $\ell + k = 2m + 2n + 1$

$$w_{m,n}^{(\mu)}(z) = \frac{T_{m-1,n}^{(\mu)}(z)\, T_{m-1,n+1}^{(\mu)}(z)}{T_{m,n}^{(\mu)}(z)\, T_{m-2,n+1}^{(\mu)}(z)}, \qquad n \geq 0$$

$$(\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n}) = \left(\tfrac{1}{2}m^2, -\tfrac{1}{2}(m+2n+1+\mu)^2, \mu\right)$$

(b) *Let* $\ell = m + n + 1$ *and* $k = m - n$*, with* $m \geq 0$*, so* $\ell + k = 2m + 1$

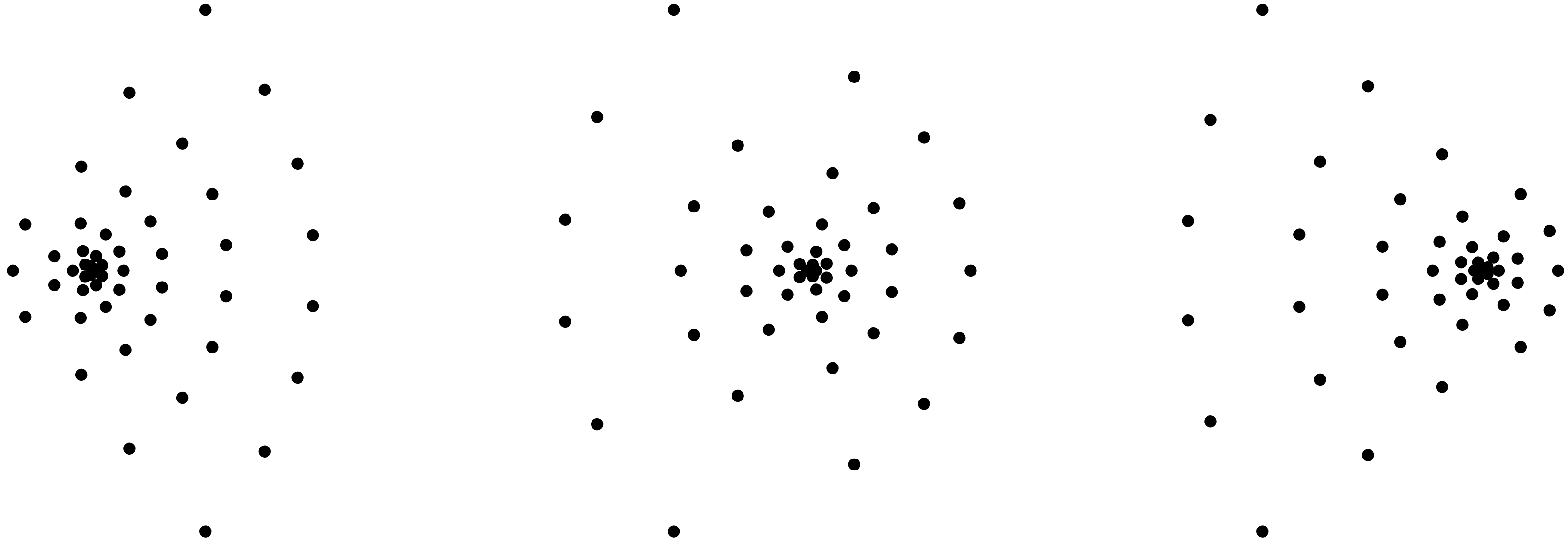
$$v_{m,n}^{(\mu)}(z) = \frac{m-n+\mu}{m+n+1}\, \frac{T_{m,n}^{(\mu-2n)}(z)\, T_{m-1,n+1}^{(\mu-2n-2)}(z)}{T_{m-1,n}^{(\mu-2n)}(z)\, T_{m,n+1}^{(\mu-2n-2)}(z)}, \qquad n \geq 1$$

$$(\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n}) = \left(\tfrac{1}{2}(m+n+1)^2, -\tfrac{1}{2}(m-n+\mu)^2, \mu\right)$$

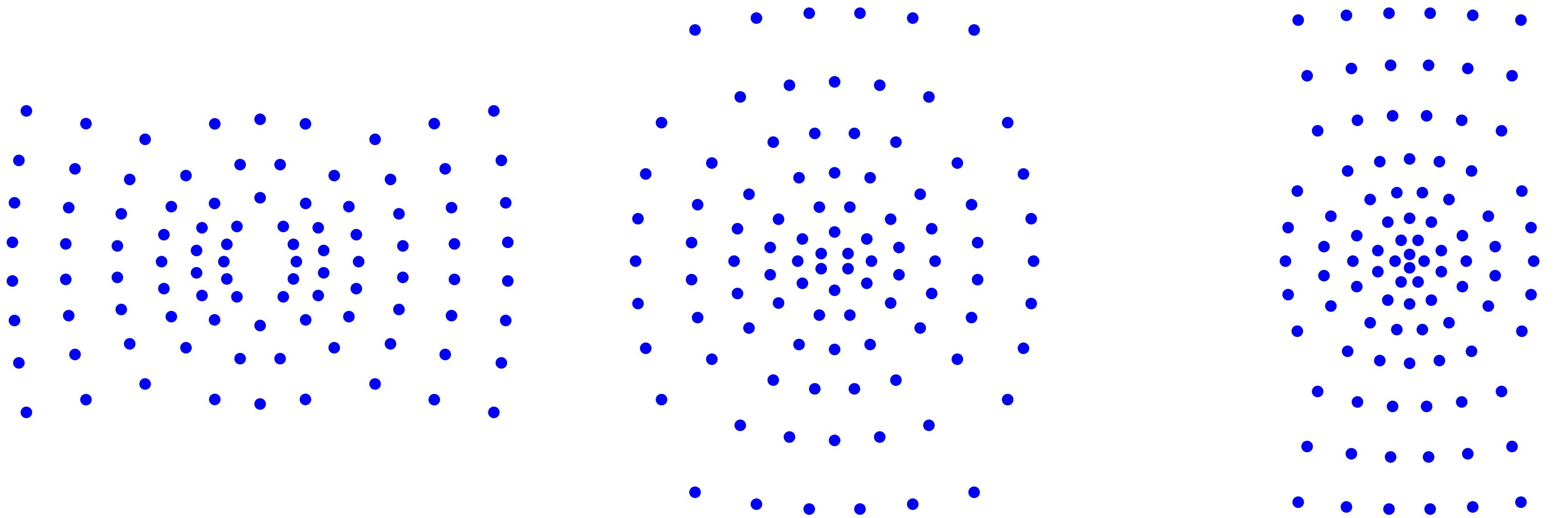
(c) *Let* $\ell = m$ *and* $k = -m - 2n - 1$*, with* $m \geq 1$ *and* $n \geq 0$*, so* $\ell + k = -2m - 1$

$$u_{m,n}^{(\mu)}(z) = w_{m,n}^{(-\mu)}(-z)$$

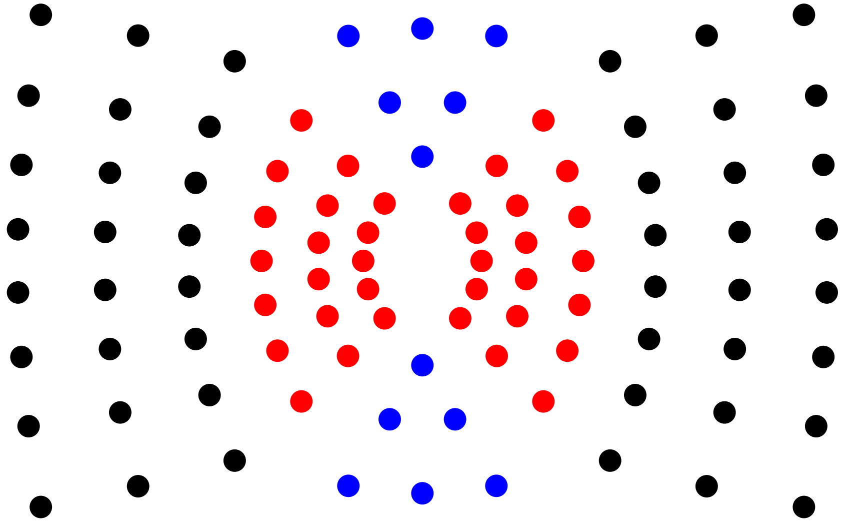
$$(\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n}) = \left(\tfrac{1}{2}m^2, -\tfrac{1}{2}(m+2n+1-\mu)^2, \mu\right)$$



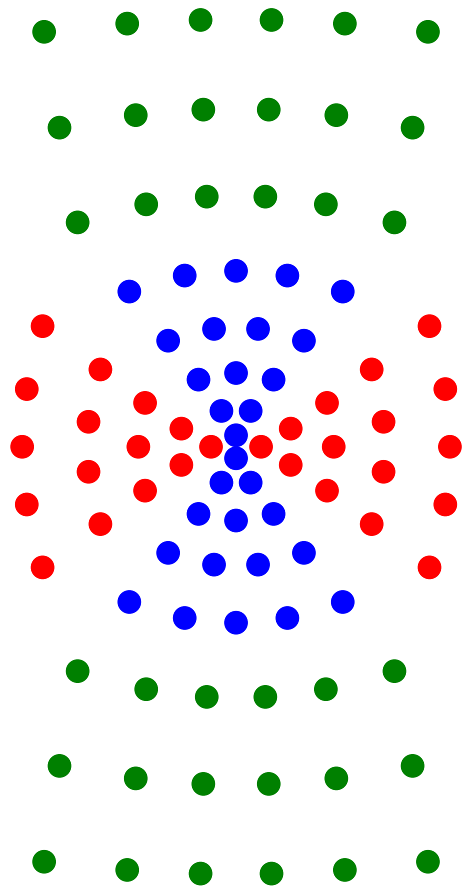
$T_{5,8}^{(-23/2)}(z)$ $T_{5,8}^{(-23/2)}(z^2)$ $T_{5,8}^{(-27/2)}(z)$ $T_{5,8}^{(-27/2)}(z^2)$ $T_{5,8}^{(-31/2)}(z)$ $T_{5,8}^{(-31/2)}(z^2)$



Generalised Laguerre Polynomial $T_{m,n}^{(\mu)}(z^2)$



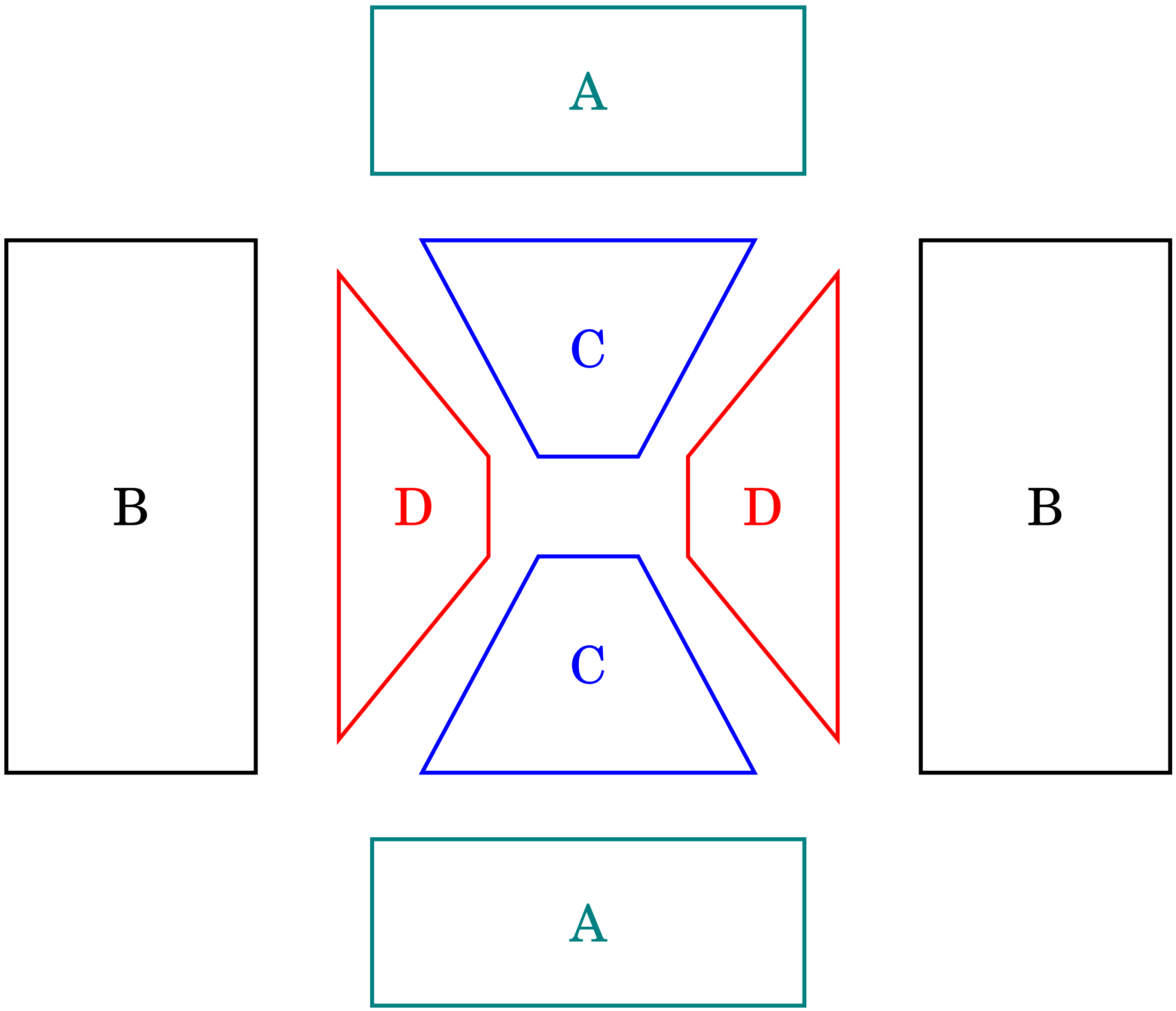
$T_{5,8}^{(-23/2)}(z^2)$



$T_{5,8}^{(-31/2)}(z^2)$

University of Kent

Structure of Roots of $T_{m,n}^{(\mu)}(z^2)$



A, B rectangle   C, D triangle/rhombus

Cases (ii) and (iii): Generalised Umemura Polynomials

Masuda, Ohta & Kajiwara [2001] generalised earlier work of Umemura [1996] and expressed rational solutions of $\mathrm{P_V}$ in terms of the polynomial

$$R_{m,n}^{(\ell)}(z) = \begin{vmatrix} p_1 & \cdots & p_{-m+2} & p_{-m+1} & \cdots & p_{-m-n+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ p_{2m-1} & \cdots & p_m & p_{m-1} & \cdots & p_{m-n} \\ q_{n-m} & \cdots & q_{n+1} & q_n & \cdots & q_{2n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q_{-n-m+2} & \cdots & q_{-n+1} & q_{-n+2} & \cdots & q_1 \end{vmatrix}$$

where $q_k(z;\ell) = L_k^{(\ell)}(\tfrac12 z)$ and $p_k(z;\ell) = L_k^{(\ell)}(-\tfrac12 z)$, with $q_k(z;\ell) = p_k(z;\ell) = 0$ for $k<0$, and $L_k^{(\alpha)}(x)$ the **Laguerre polynomial**.

Definition

The ***generalised Umemura polynomial*** $U_{m,n}^{(\kappa)}(z)$ is given by

$$U_{m,n}^{(\kappa)}(z) = \exp(\tfrac12 mz)\ \mathrm{Wr}\left(\varphi_1^{(\kappa)}, \varphi_3^{(\kappa)}, \ldots, \varphi_{2m-1}^{(\kappa)}; \psi_1^{(\kappa)}, \psi_3^{(\kappa)}, \ldots, \psi_{2n-1}^{(\kappa)}\right), \qquad m,n \geq 0$$

where

$$\varphi_k^{(\kappa)}(z) = \mathrm{e}^{-z/2}\, L_k^{(\kappa)}(\tfrac12 z), \qquad \psi_k^{(\kappa)}(z) = L_k^{(\kappa)}(-\tfrac12 z)$$

with κ a parameter, $L_k^{(\alpha)}(\zeta)$ the **Laguerre polynomial** and $U_{0,0}^{(\kappa)}(z) = 1$.

Theorem

(Masuda, Ohta & Kajiwara [2001])

Suppose $U^{(\kappa)}_{m,n}(z)$ is the generalised Umemura polynomial. In case (ii)

$$\widehat{w}^{(\kappa)}_{m,n}(z) = -\frac{U^{(2\kappa)}_{m,n-1}(z)\,U^{(2\kappa+2)}_{m-1,n}(z)}{U^{(2\kappa+2)}_{m,n-1}(z)\,U^{(2\kappa)}_{m-1,n}(z)}, \qquad m,n \ge 1$$

is a rational solution of $\mathrm{P_V}$ *with*

$$(\alpha_{m,n},\beta_{m,n},\gamma_{m,n}) = \left(\tfrac{1}{2}(m+\kappa)^2, -\tfrac{1}{2}(n+\kappa)^2, m+n\right)$$

and

$$\widehat{v}^{(\kappa)}_{m,n}(z) = -\frac{U^{(2\kappa-2n+3)}_{m-1,n-1}(z)\,U^{(2\kappa-2n-1)}_{m,n}(z)}{U^{(2\kappa-2n+1)}_{m-1,n-1}(z)\,U^{(2\kappa-2n+1)}_{m,n}(z)}, \qquad m,n \ge 1$$

is a rational solution of $\mathrm{P_V}$ *with*

$$(\alpha_{m,n},\beta_{m,n},\gamma_{m,n}) = \left(\tfrac{1}{2}(m+\kappa)^2, -\tfrac{1}{2}(n-\kappa)^2, m-n\right)$$

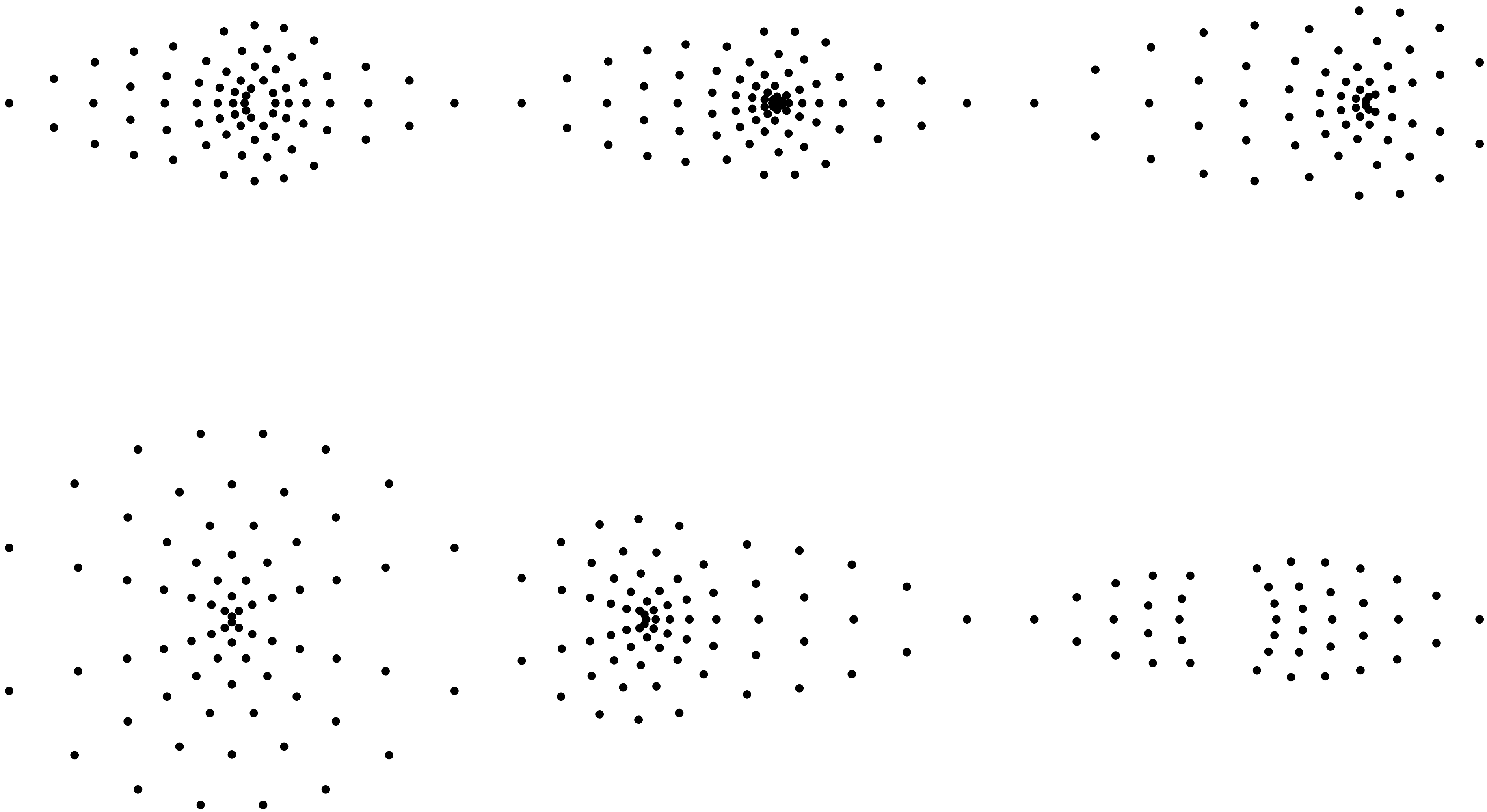
In case (iii)

$$\widehat{u}^{(\kappa)}_{m,n}(z) = -\frac{U^{(\kappa+1)}_{m,n-1}(z)\,U^{(\kappa-1)}_{m,n+1}(z)}{U^{(\kappa+1)}_{m-1,n}(z)\,U^{(\kappa-1)}_{m+1,n}(z)}, \qquad m,n \ge 1$$

is a rational solution of $\mathrm{P_V}$ *with*

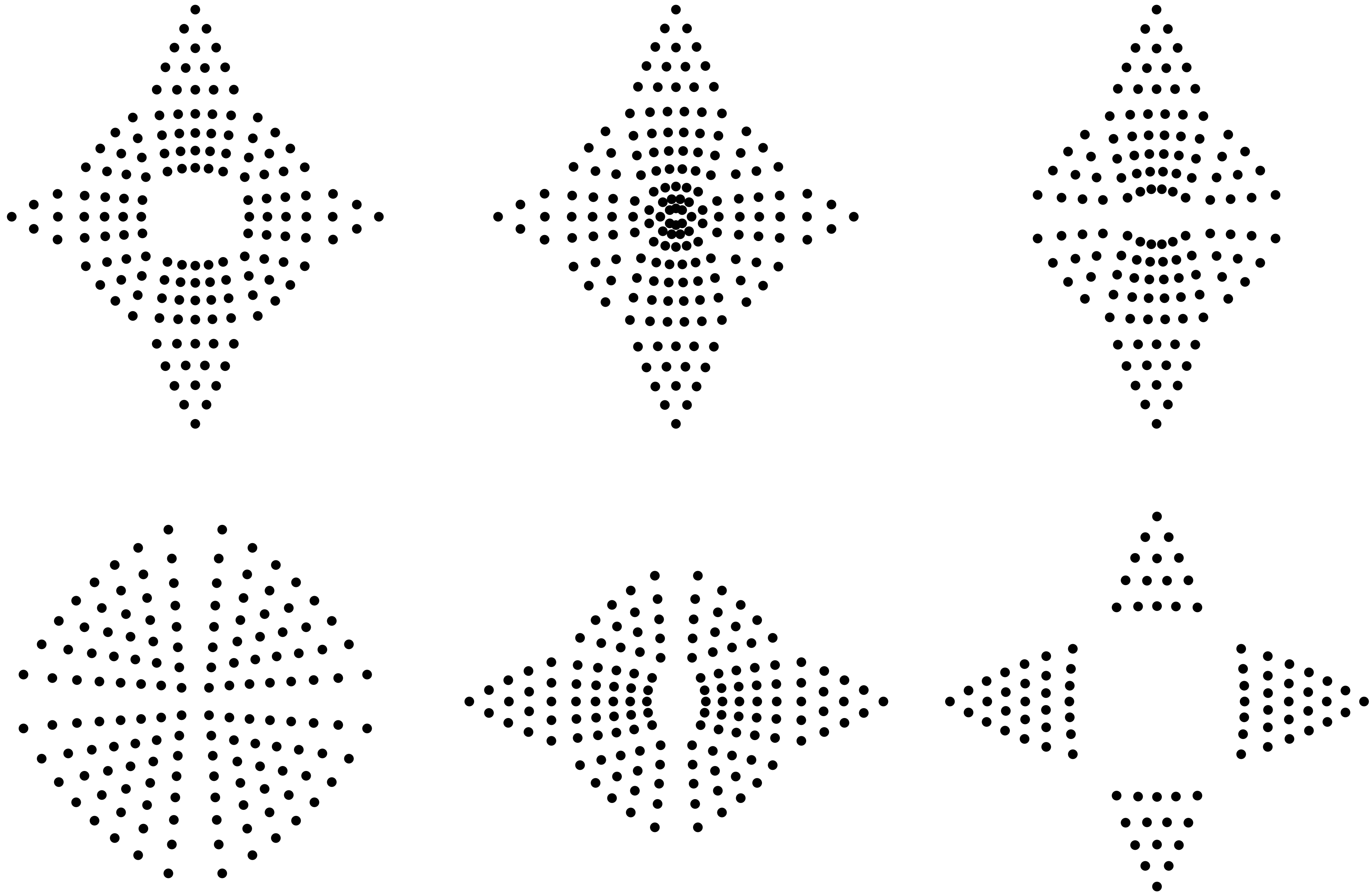
$$(\alpha_{m,n},\beta_{m,n},\gamma_{m,n}) = \left(\tfrac{1}{2}(m+\tfrac{1}{2})^2, -\tfrac{1}{2}(n+\tfrac{1}{2})^2, m+n+\kappa\right)$$

Generalised Umemura Polynomial $U^{(\kappa)}_{m,n}(z)$

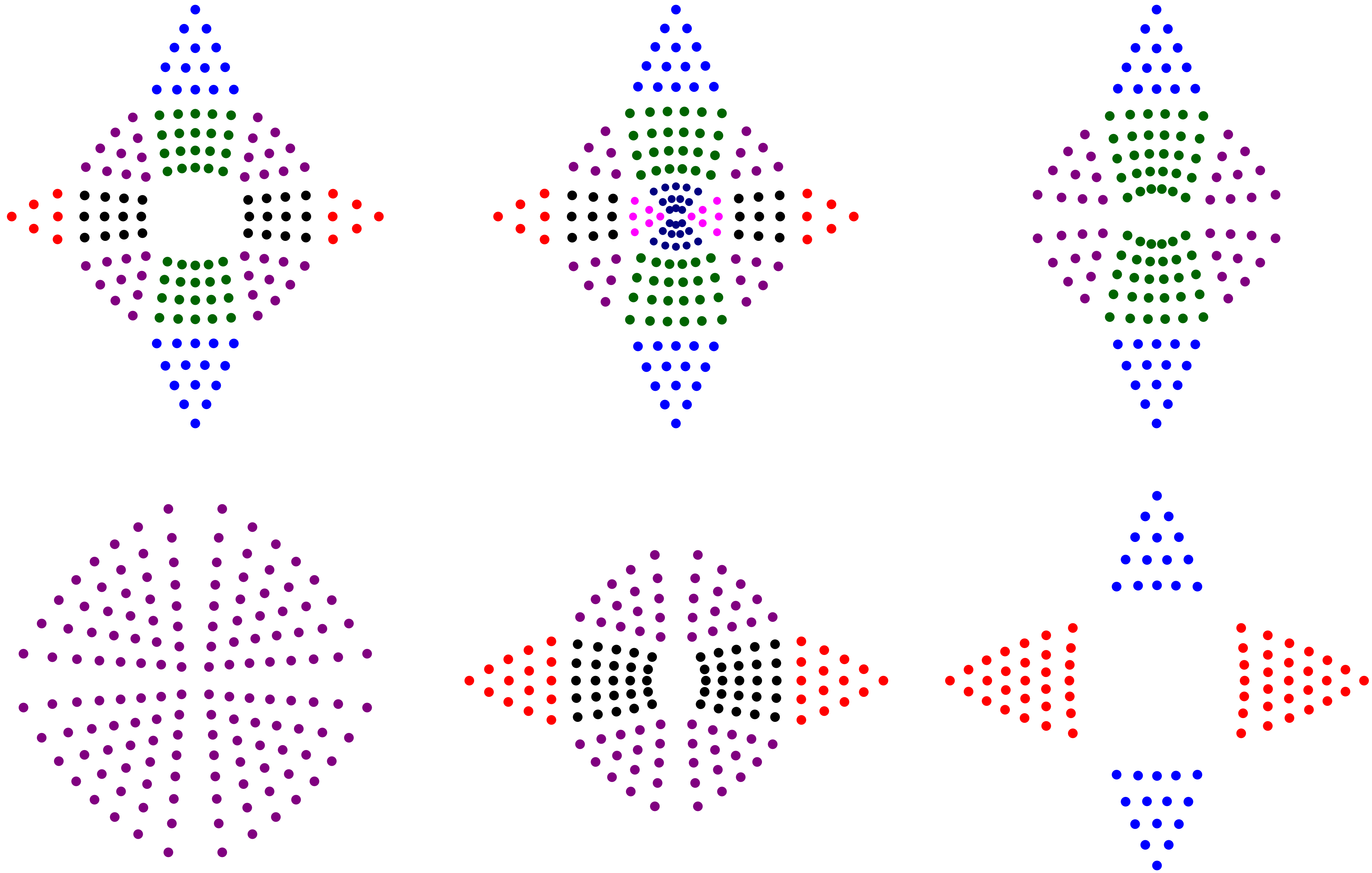


University of Kent

Generalised Umemura Polynomial $U_{m,n}^{(\kappa)}(z^2)$

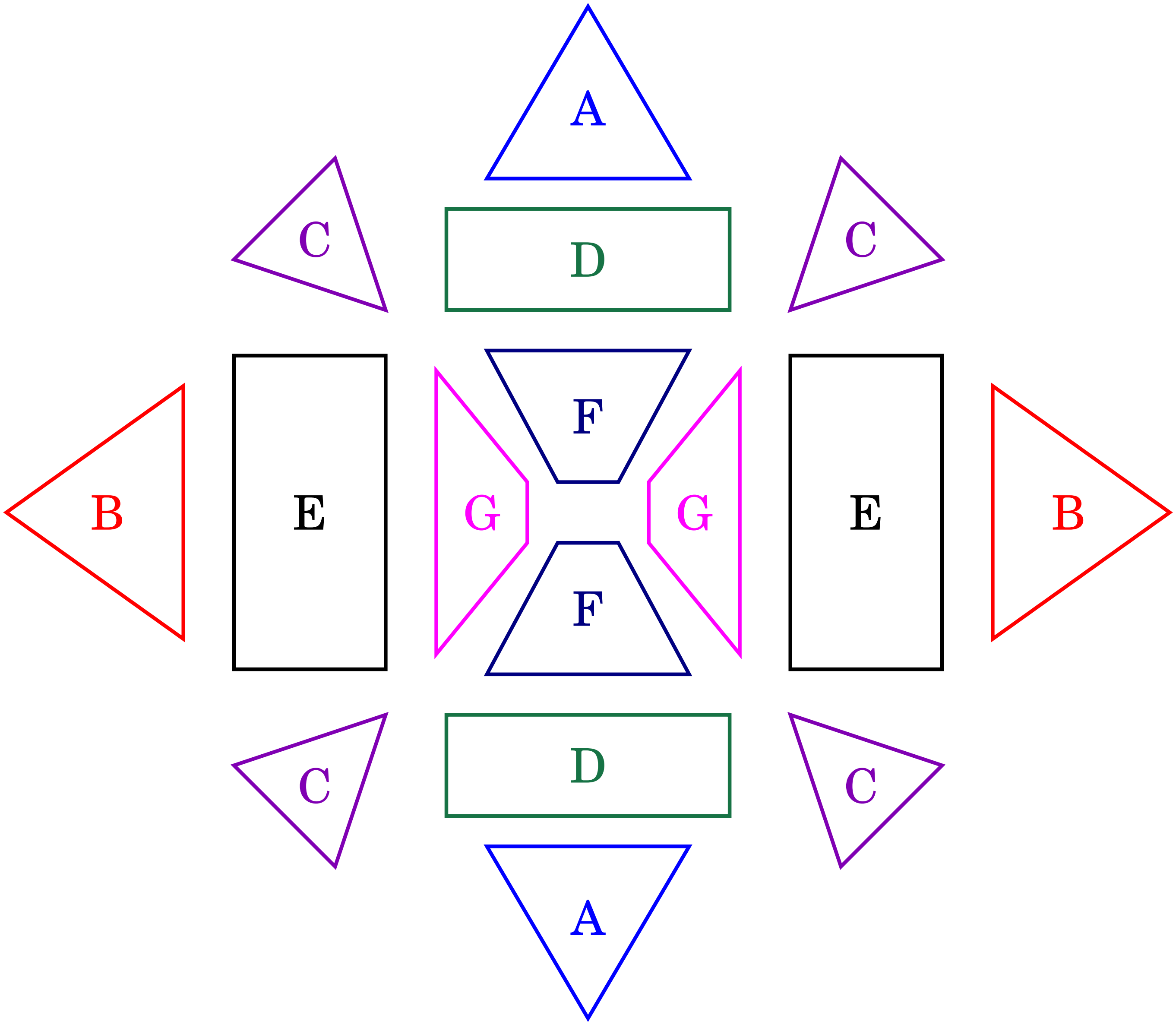


Generalised Umemura Polynomial $U^{(\kappa)}_{m,n}(z^2)$



University of Kent

Structure of Roots of $U_{m,n}^{(\kappa)}(z^2)$



A, B, C triangle   D, E rectangle   F, G triangle/rhombus

University of Kent

Non-uniqueness of Rational Solutions of $\mathrm{P_V}$

There exist distinct rational solutions in terms of generalised Laguerre polynomials which satisfy $\mathrm{P_V}$ for the **same** parameters.

- Kitaev, Law & McLeod [1994] state that if $\gamma \in \mathbb{Z}$ and $\alpha\beta \neq 0$ then there are two rational solutions.
- The rational solutions $v_{m,n}^{(\mu)}$ and $v_{n-\mu,m+\mu}^{(\mu)}$ both satisfy $\mathrm{P_V}$ for the parameters

$$\alpha = \tfrac{1}{2}(m+n+1)^2, \qquad \beta = -\tfrac{1}{2}(m-n+\mu)^2, \qquad \gamma = \mu$$

  so we need $\mu \in \mathbb{Z}$ and $|\mu| \leq \min(m,n)$.
- For example, if $m=4$, $n=1$ and $\mu=-1$ then

$$v_{4,1}^{(-1)}(z) = -\frac{2(z^2-8z+20)(z^3-15z^2+60z-60)}{(z^2-8z+12)(z^4-20z^3+150z^2-480z+600)}$$

$$v_{2,3}^{(-1)}(z) = \frac{4(z^5+21z^4+180z^3+780z^2+1800z+1800)}{z^6+24z^5+252z^4+1440z^3+4680z^2+8640z+7200} - \frac{2(z^3+9z^2+24z+24)}{z^4+12z^3+54z^2+96z+72}$$

  both satisfy $\mathrm{P_V}$ for the parameters

$$\alpha = 18, \qquad \beta = -2, \qquad \gamma = -1$$

University of Kent

Non-uniqueness of Rational Solutions of $\mathrm{P_V}$

Example. The rational functions

$$w_1(z) = \frac{1}{v_{3,0}^{(-6)}(z)} = \frac{z^4 + 12z^3 + 72z^2 + 240z + 360}{3(z^3 + 12z^2 + 60z + 120)}$$

and

$$w_2(z) = v_{0,2}^{(6)}(z) = -\frac{4(z^2 - 10z + 30)}{z^3 - 12z^2 + 60z - 120}$$

are both solutions of

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\frac{9w}{2} - \frac{8}{w}\right) + \frac{6w}{z} - \frac{w(w+1)}{2(w-1)}$$

i.e. $\mathrm{P_V}$ with the parameters

$$\alpha = \tfrac{9}{2}, \qquad \beta = -8, \qquad \gamma = 6$$

In fact $w_1(z)$ and $w_2(z)$ are special cases of the more general solution

$$w(z) = \frac{(z^4 + 12z^3 + 72z^2 + 240z + 360)C_1 - 4(z^2 - 10z + 30)C_2\,\mathrm{e}^z}{3(z^3 + 12z^2 + 60z + 120)C_1 + (z^3 - 12z^2 + 60z - 120)C_2\,\mathrm{e}^z}$$

with C_1 and C_2 arbitrary constants, which also satisfies $\mathrm{P_V}$ for the same parameters.

- Recall that $\mathrm{P_V}$ has the special function solution

$$w\left(z; \tfrac{1}{2}a^2, -\tfrac{1}{2}\left(a-b+1\right)^2, b-2\right) = \frac{(a+1-b)c_1 M(a,b,z) + c_2 U(a,b,z)}{a\left[c_1 M(a+1,b,z) + c_2 U(a+1,b,z)\right]}$$

with $M(a,b,z)$ and $U(a,b,z)$ the **Kummer functions**.

- Setting $a = \nu + \frac{1}{2}$ and $b = 2\nu + 3$ gives

$$w\left(z; \tfrac{1}{2}(\nu+\tfrac{1}{2})^2, -\tfrac{1}{2}(\nu+\tfrac{3}{2})^2, 2\nu+1\right) = \frac{2\nu+1+z}{2\nu+1} - \frac{z}{2\nu+1}\,\frac{c_1 I_\nu(\frac{1}{2}z) + c_2 I_{-\nu}(\frac{1}{2}z)}{c_1 I_{\nu+1}(\frac{1}{2}z) + c_2 I_{-\nu-1}(\frac{1}{2}z)}$$

with $I_\nu(z)$ the **modified Bessel function**.

- Then setting $\nu = \frac{5}{2}$ gives

$$w(z) = \frac{(z^4+12z^3+72z^2+240z+360)(c_1-c_2) - 12(z^2-10z+30)(c_1+c_2)\mathrm{e}^z}{3(z^3+12z^2+60z+120)(c_1-c_2) + 3(z^3-12z^2+60z-120)(c_1+c_2)\mathrm{e}^z}$$

so if $c_1 = c_2$ or $c_1 = -c_2$ we obtain the two rational solutions.

Remark. The **spherical Bessel functions** $I_{\pm(n+1/2)}(z)$, with $n \in \mathbb{N}$, have the form

$$I_{\pm(n+1/2)}(z) = \frac{1}{\sqrt{2\pi z}} \sum_{k=0}^{n} \frac{(n+k)!}{(2z)^k k!(n-k)!}\left\{(-1)^k \mathrm{e}^z \pm (-1)^{n+1}\mathrm{e}^{-z}\right\}$$

Joint Moments of the Characteristic Polynomial of CUE Random Matrices

In their study of joint moments of the characteristic polynomial of CUE random matrices, Basor, Bleher, Buckingham, Grava, A. Its, E. Its & Keating [2019] and recently Alvarez, Conrey, Rubenstein & Snaith [2025] were interested in solutions of the $\mathrm{P_V}$ σ-equation

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 = \left\{2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - (2N+z)\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right\}^2 - 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+k\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}-N\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}-k-N\right) \tag{1a}$$

with $k, N \in \mathbb{N}$, satisfying

$$\sigma(z) = -Nk + \tfrac{1}{2}Nz + \mathcal{O}(z^2), \qquad \text{as} \quad z \to 0 \tag{1b}$$

Basor *et al.* [2019] obtain the solution

$$\widehat{S}_{N,k}(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\left\{\det\left[L^{(2k-1)}_{N+k-1-i-j}(-z)\right]_{i,j=0}^{k-1}\right\} - Nk$$

whilst Alvarez *et al.* [2025] obtain the solution

$$S_{N,k}(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\left\{\det\left[L^{(2k-1)}_{N+i-j}(z)\right]_{i,j=0}^{k-1}\right\} + Nz - Nk$$

with $L_n^{(\alpha)}(z)$ the **Laguerre polynomial**.

University of Kent

In terms of the **generalised Laguerre polynomial**

$$T^{(\mu)}_{m,n}(z) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}} L^{(\mu+1)}_{m+n}(z)\right]_{j,k=0}^{n-1}$$

the rational solution of (1) is given by

$$S_{N,k}(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\left\{T^{(0)}_{N-1,k}(z)\right\} - Nk + Nz, \qquad N \geq 1, \quad k \geq 1$$

which is an equivalent representation of the solution derived by Alvarez *et al.* [2025]. Alternatively using

$$\widehat{T}^{(\mu)}_{m,n}(z) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}} L^{(\mu+1)}_{m+n}(-z)\right]_{j,k=0}^{n-1} = T^{(\mu)}_{m,n}(-z)$$

then the solution is given by

$$\widehat{S}_{N,k}(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\left\{\widehat{T}^{(0)}_{N-1,k}(z)\right\} - Nk, \qquad N \geq 1, \quad k \geq 1$$

which is an equivalent representation of the solution derived by Basor *et al.* [2019]. These solutions are related by

$$\widehat{S}_{N,k}(z) = S_{N,k}(-z) + Nz$$

since (1) is invariant under the transformation

$$\sigma(z) \to \sigma(z) - Nz, \qquad z \to -z$$

Example. Suppose that $N = 2$ and $k = 2$, then

$$S_{2,2}(z) = \frac{2(z^5 - 16z^4 + 104z^3 - 336z^2 + 600z - 480)}{z^4 - 16z^3 + 96z^2 - 240z + 240}$$

$$= 2z + \frac{16(z^3 - 12z^2 + 45z - 60)}{z^4 - 16z^3 + 96z^2 - 240z + 240}$$

$$\widehat{S}_{2,2}(z) = -\frac{16(z^3 + 12z^2 + 45z + 60)}{z^4 + 16z^3 + 96z^2 + 240z + 240}$$

and so as $z \to 0$

$$S_{2,2}(z) = -4 + z - \frac{z^2}{5} + \frac{3z^4}{100} - \frac{z^5}{48} + \mathcal{O}(z^6)$$

$$\widehat{S}_{2,2}(z) = -4 + z - \frac{z^2}{5} + \frac{3z^4}{100} + \frac{z^5}{48} + \mathcal{O}(z^6)$$

However as $z \to \infty$

$$S_{2,2}(z) = \quad - \frac{16}{z} + \frac{64}{z^2} - \frac{208}{z^3} + \frac{64}{z^4} + \frac{7424}{z^5} + \mathcal{O}(z^{-6})$$

$$\widehat{S}_{2,2}(z) = 2z + \frac{16}{z} + \frac{64}{z^2} + \frac{208}{z^3} + \frac{64}{z^4} - \frac{7424}{z^5} + \mathcal{O}(z^{-6})$$

Note that

$$S_{2,2}(z) = 2z + \widehat{S}_{2,2}(-z)$$

A more general solution when $N = 2$ and $k = 2$ is

$$\sigma(z; c_1, c_2) = \frac{f_{2,2}(z; c_1, c_2)}{g_{2,2}(z; c_1, c_2)}$$

where

$$\begin{aligned} f_{2,2}(z; c_1, c_2) = 6c_2^2(&z^5 - 16z^4 + 104z^3 - 336z^2 + 600z - 480)\,\mathrm{e}^{2z} \\ &- c_1c_2(z^7 + 2z^6 + 18z^5 - 144z^3 + 288z^2 + 1440z - 5760)\,\mathrm{e}^{z} \\ &- 48c_1^2(z^3 + 12z^2 + 45z + 60) \\ g_{2,2}(z; c_1, c_2) = 3c_2^2(&z^4 - 16z^3 + 96z^2 - 240z + 240)\,\mathrm{e}^{2z} \\ &- c_1c_2(z^6 + 18z^4 - 144z^2 + 1440)\,\mathrm{e}^{z} \\ &+ 3c_1^2(z^4 + 16z^3 + 96z^2 + 240z + 240) \end{aligned}$$

with c_1 and c_2 arbitrary constants. Setting either $c_1 = 0$ or $c_2 = 0$ gives the two rational solutions

$$\sigma(z; 0, c_2) = S_{2,2}(z) = \frac{2(z^5 - 16z^4 + 104z^3 - 336z^2 + 600z - 480)}{z^4 - 16z^3 + 96z^2 - 240z + 240}$$

$$\sigma(z; c_1, 0) = \widehat{S}_{2,2}(z) = -\frac{16(z^3 + 12z^2 + 45z + 60)}{z^4 + 16z^3 + 96z^2 + 240z + 240}$$

Also, as $z \to 0$

$$\sigma(z; c_1, c_2) = -4 + z - \frac{z^2}{5} + \frac{3z^4}{100} - \frac{c_1 + c_2}{c_1 - c_2}\frac{z^5}{48} + \mathcal{O}(z^6)$$

A more general solution of the initial value problem

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 = \left\{2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - (2N+z)\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right\}^2 - 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+k\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}-N\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}-k-N\right) \tag{1a}$$

with $k, N \in \mathbb{N}$, satisfying

$$\sigma(z) = -Nk + \tfrac{1}{2}Nz + \mathcal{O}(z^2), \qquad \text{as} \quad z \to 0 \tag{1b}$$

is

$$\sigma(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln \mathrm{Wr}\left(\left\{(c_2+c_1)I_{k+j+1/2}(\tfrac{1}{2}z) + (c_2-c_1)I_{-k-j-1/2}(\tfrac{1}{2}z)\right\}_{j=0}^{N-1}\right) + \tfrac{1}{2}N(z+2N-1)$$

where $I_{\pm(n+1/2)}(z)$, with $n \in \mathbb{Z}$, is the **spherical Bessel function** and c_1 and c_2 are arbitrary constants. Setting $c_2 = 0$ or $c_1 = 0$ gives the rational solution obtained by Basor *et al.* [2019] or Alvarez *et al.* [2025], respectively.

Recall that the **spherical Bessel functions** $I_{\pm(n+1/2)}(z)$ have the form

$$I_{\pm(n+1/2)}(z) = \frac{1}{\sqrt{2\pi z}}\left\{\sum_{k=0}^{n}\frac{(n+k)!}{(2z)^k k!(n-k)!}\left[(-1)^k \mathrm{e}^{z} \pm (-1)^{n+1}\mathrm{e}^{-z}\right]\right\}$$

Bäcklund Transformations

Definition

A ***Bäcklund transformation*** maps solutions of a given Painlevé equation to solutions of the same Painlevé equation, with different values of the parameters, or another equation.

Bäcklund Transformations of $\mathrm{P_V}$

Theorem (Gromak [1976], Gromak & Filipuk [2001])

Let $w = w(z; \alpha, \beta, \gamma)$ *be a solution of* $\mathrm{P_V}$ *such that*

$$\Phi(w; z) = z\frac{\mathrm{d}w}{\mathrm{d}z} - \varepsilon_2 a w^2 + (\varepsilon_2 a - \varepsilon_3 b + \varepsilon_1 z)w + \varepsilon_3 b \neq 0$$

and

$$\Phi(w; z) - 2\varepsilon_1 z w \neq 0$$

where $a = \sqrt{2\alpha}$, $b = \sqrt{-2\beta}$ *and* $\gamma = c$. *Then*

$$\mathcal{T}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}: \quad w_1(z; \alpha_1, \beta_1, \gamma_1) = 1 - \frac{2\varepsilon_1 z w}{\Phi(w; z)} = 1 - \frac{2\varepsilon_1 z w}{z w' - \varepsilon_2 a w^2 + (\varepsilon_2 a - \varepsilon_3 b + \varepsilon_1 z)w + \varepsilon_3 b}$$

with $\varepsilon_j = \pm 1$, $j = 1, 2, 3$, *independently, is a solution of* $\mathrm{P_V}$ *for the parameters*

$$(\alpha_1, \beta_1, \gamma_1) = \left(\tfrac{1}{8}\big(c + \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)\big)^2, -\tfrac{1}{8}\big(c - \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)\big)^2, \varepsilon_1(\varepsilon_3 b - \varepsilon_2 a)\right)$$

University of Kent

To apply a sequence of Bäcklund transformations there is an issue

$$\alpha_1 = \tfrac{1}{2}a_1^2 = \tfrac{1}{8}\big(c + \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)\big)^2 \qquad \Rightarrow \qquad a_1 = \pm\tfrac{1}{2}\big(c + \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)\big)$$
$$\beta_1 = -\tfrac{1}{2}b_1^2 = -\tfrac{1}{8}\big(c - \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)\big)^2 \qquad \Rightarrow \qquad b_1 = \pm\tfrac{1}{2}\big(c - \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)\big)$$

Question: How do you decide the sign for a_1 and b_1?

Answer: Define the Bäcklund transformation in terms of a, b and c rather than $\alpha = \frac{1}{2}a^2$, $\beta = -\frac{1}{2}b^2$ and $\gamma = c$, i.e.

$$\mathcal{T}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}: \qquad w_1(z; a_1, b_1, c_1) = 1 - \frac{2\varepsilon_1 z w}{z w' - \varepsilon_2 a w^2 + (\varepsilon_2 a - \varepsilon_3 b + \varepsilon_1 z) w + \varepsilon_3 b}$$

with

$$(a_1, b_1, c_1) = \big(\tfrac{1}{2}c + \tfrac{1}{2}\varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a), \tfrac{1}{2}c - \tfrac{1}{2}\varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a), \varepsilon_1(\varepsilon_3 b - \varepsilon_2 a)\big)$$

Objective: To find a combination of Bäcklund transformations which change a, b and c by an integer. This will give a procedure for generating hierarchies of solutions.

- To generate a hierarchy of solutions, then we need to change a, b and c by an integer.

Example. Suppose $w = w(z; a, b, c)$ satisfies $\mathrm{P_V}$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2\big(a^2w^2 - b^2\big)}{2z^2w} + \frac{cw}{z} - \frac{w(w+1)}{2(w-1)}$$

Then

$$\begin{aligned}\mathcal{T}_{1,-1,-1}(a,b,c) &= \left(\tfrac{1}{2}(a+b+c+1), \tfrac{1}{2}(-a-b+c-1), a-b\right)\\ \mathcal{T}_{1,-1,1}\circ\mathcal{T}_{1,1,1}(a,b,c) &= \big(a+1, -b-1, c\big)\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_{1,-1,-1}(w) &= \frac{zw' + aw^2 - (a-b+z)w - b}{zw' + aw^2 - (a-b-z)w - b}\\ \mathcal{T}_{1,-1,1}\circ\mathcal{T}_{1,1,1}(w) &= \frac{z^2(w')^2 + 2z(w-1)(bw-w-b)w' + f(w,z)}{w\left\{z^2(w')^2 - 2z(w-1)(aw-a-1)w' + g(w,z)\right\}}\end{aligned}$$

with

$$\begin{aligned}f(w,z) &= -z^2w^2 + 2czw^2(w-1) - (w-1)^2(aw+b)\big[(a+2b+2)w - b\big]\\ g(w,z) &= -z^2w^2 + 2czw(w-1) + (w-1)^2(aw+b)(aw-2a-b-2)\end{aligned}$$

Deriving Discrete Painlevé Equations from Bäcklund Transformations for the Painlevé Equations

General Principle: Suppose that there are two Bäcklund transformation for a Painlevé equation in the form

$$w_\pm(z; a_\pm, b_\pm, c_\pm) = F_\pm(w(z; a, b, c), w'(z; a, b, c), z) \tag{1}$$

where $w(z; a, b, c)$ is a solution with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$, and $w_\pm(z; a_\pm, b_\pm, c_\pm)$ solutions with $(\alpha_\pm, \beta_\pm, \gamma_\pm) = \left(\frac{1}{2}a_\pm^2, -\frac{1}{2}b_\pm^2, c_\pm\right)$.

- Eliminating w' in (1) yields an algebraic relation, between w_+, w and w_-.
- Given two Bäcklund transformations $\mathcal{T}_\pm$ where $\mathcal{T}_+ \circ \mathcal{T}_- = \mathcal{T}_- \circ \mathcal{T}_+ = \mathcal{I}$, the identity transformation, then

$$w_-(z; a_-, b_-, c_-) \ \underset{\mathcal{T}_-}{\overset{\mathcal{T}_+}{\rightleftarrows}} \ w(z; a, b, c) \ \underset{\mathcal{T}_-}{\overset{\mathcal{T}_+}{\rightleftarrows}} \ w_+(z; a_+, b_+, c_+)$$

- If $w(z; a, b, c) = w_n$, $w_\pm(z; a_\pm, b_\pm, c_\pm) = w_{n\pm1}$ then

$$w_{n-1} \ \underset{\mathcal{T}_-}{\overset{\mathcal{T}_+}{\rightleftarrows}} \ w_n \ \underset{\mathcal{T}_-}{\overset{\mathcal{T}_+}{\rightleftarrows}} \ w_{n+1}$$

First Discrete Equation

Consider the Bäcklund transformation $\mathcal{R}_1 = \mathcal{T}_{1,-1,1}$ which has the inverse $\mathcal{R}_1^{-1} = \mathcal{T}_{-1,1,-1}$ and $w_n = w_n(z; \alpha_n, \beta_n, \gamma_n)$, with $\alpha_n = \frac{1}{2}a_n^2$, $\beta_n = -\frac{1}{2}b_n^2$ and $\gamma_n = c_n$, is a solution of $\mathrm{P_V}$, then

$$w_{n+1} = \mathcal{T}_{1,-1,1}(w_n) = 1 - \frac{2zw_n}{zw_n' + a_n w_n^2 + (z - a_n - b_n)w_n + b_n}$$

$$w_{n-1} = \mathcal{T}_{-1,1,-1}(w_n) = 1 + \frac{2zw_n}{zw_n' - a_n w_n^2 - (z - a_n - b_n)w_n - b_n}$$

Eliminating w_n' yields

$$\frac{1}{w_{n+1} - 1} + \frac{1}{w_{n-1} - 1} + \frac{a_n w_n^2 + (z - a_n - b_n)w_n + b_n}{zw_n} = 0$$

and setting

$$q_n = \frac{w_n + 1}{w_n - 1}, \qquad q_{n+1} = \frac{w_{n+1} + 1}{w_{n+1} - 1}, \qquad q_{n-1} = \frac{w_{n-1} + 1}{w_{n-1} - 1}$$

gives

$$q_{n+1} + q_{n-1} = \frac{4}{z}\,\frac{(a_n - b_n)q_n + a_n + b_n}{1 - q_n^2}$$

The parameters evolve as

$$\mathcal{T}_{1,-1,1}(a_n, b_n, c_n) = \left(\tfrac{1}{2}(a_n - b_n + c_n + 1), \tfrac{1}{2}(-a_n + b_n + c_n - 1), a_n + b_n\right)$$

and so

$$\left.\begin{aligned} a_{n+1} &= \tfrac{1}{2}(a_n - b_n + c_n + 1) \\ b_{n+1} &= \tfrac{1}{2}(-a_n + b_n + c_n - 1) \\ c_{n+1} &= a_n + b_n \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} a_n &= \tfrac{1}{2}n + \tfrac{1}{2}\lambda + \tfrac{1}{2}\rho + \tfrac{1}{2}(-1)^n\varphi \\ b_n &= -\tfrac{1}{2}n + \tfrac{1}{2}\lambda - \tfrac{1}{2}\rho + \tfrac{1}{2}(-1)^n\varphi \\ c_n &= \lambda - (-1)^n\varphi \end{aligned}$$

with λ, ρ and φ arbitrary constants. Hence we obtain the **asymmetric** $\mathrm{dP_{II}}$ equation

$$q_{n+1} + q_{n-1} = \frac{4}{z}\,\frac{(n+\lambda)q_n + \rho + (-1)^n\varphi}{1 - q_n^2}$$

studied by Satsuma, Kajiwara, Grammaticos, Hietarinta & Ramani [1994].

Setting $x_n = q_{2n}$ and $y_n = q_{2n+1}$ gives

$$x_{n+1} + x_n = \frac{4}{z}\,\frac{(n+\lambda)y_n + \rho - \varphi}{1 - y_n^2}$$

$$y_n + y_{n-1} = \frac{4}{z}\,\frac{(n+\lambda)x_n + \rho + \varphi}{1 - x_n^2}$$

Second Discrete Equation

Consider the Bäcklund transformation $\mathcal{R}_2 = \mathcal{T}_{-1,-1,-1} \circ \mathcal{T}_{-1,1,-1}$, which has the inverse $\mathcal{R}_2^{-1} = \mathcal{T}_{1,1,1} \circ \mathcal{T}_{1,-1,1}$, and $w_n = w_n(z;\alpha_n,\beta_n,\gamma_n)$, with $\alpha_n = \frac{1}{2}a_n^2$, $\beta_n = -\frac{1}{2}b_n^2$ and $\gamma_n = c_n$, is a solution of $\mathrm{P_V}$, then

$$w_{n+1} = \mathcal{R}_2(w_n) = \frac{1}{w_n}\left\{1 + \frac{(a_n - b_n - c_n - 1)(w_n - 1)^2}{zw_n' - a_n w_n^2 + (2a_n - c_n - 1 + z)w_n - (a_n - c_n - 1)}\right\}$$

$$w_{n-1} = \mathcal{R}_2^{-1}(w_n) = \frac{1}{w_n}\left\{1 - \frac{(a_n - b_n - c_n + 1)(w_n - 1)^2}{zw_n' + a_n w_n^2 - (2a_n - c_n + 1 + z)w_n + (a_n - c_n + 1)}\right\}$$

Eliminating w_n' gives the difference equation

$$\frac{a_n - b_n - c_n - 1}{w_n w_{n+1} - 1} + \frac{a_n - b_n - c_n + 1}{w_n w_{n-1} - 1} + a_n - \frac{z - c_n - 1}{w_n - 1} - \frac{z}{(w_n - 1)^2} = 0$$

and setting

$$q_n = \frac{w_n + 1}{w_n - 1}, \qquad q_{n+1} = \frac{w_{n+1} + 1}{w_{n+1} - 1}, \qquad q_{n-1} = \frac{w_{n-1} + 1}{w_{n-1} - 1}$$

gives

$$\frac{a_n - b_n - c_n - 1}{q_{n+1} + q_n} + \frac{a_n - b_n - c_n + 1}{q_n + q_{n-1}} + z + \frac{2\big[(a_n - b_n)q_n + a_n + b_n\big]}{1 - q_n^2} = 0$$

The parameters evolve as

$$\mathcal{R}_2(a_n, b_n, c_n) = \left(\tfrac{1}{2}\big(a_n + b_n - c_n - 1\big), \tfrac{1}{2}\big(a_n + b_n + c_n + 1\big), -a_n + b_n + 1\right)$$

and so

$$\left.\begin{aligned} a_{n+1} &= \tfrac{1}{2}\big(a_n + b_n - c_n + 1\big) \\ b_{n+1} &= \tfrac{1}{2}\big(a_n + b_n - c_n + 1\big) \\ c_{n+1} &= -a_n + b_n - 1 \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} a_n &= -\tfrac{1}{2}\big[n + \lambda + \rho + (-1)^n \varphi\big] \\ b_n &= \tfrac{1}{2}\big[n + \lambda - \rho + (-1)^n \varphi\big] \\ c_n &= n + \lambda - (-1)^n \varphi \end{aligned}$$

with λ, ρ and φ arbitrary constants. Hence we obtain

$$\frac{\xi_{n+1} + \xi_n}{q_{n+1} + q_n} + \frac{\xi_n + \xi_{n-1}}{q_n + q_{n-1}} = z - \frac{2\big[\xi_n + \varphi(-1)^n\big] q_n + 2\rho}{1 - q_n^2}$$

with $\xi_n = n + \lambda$, which was studied by Ramani, Ohta & Grammaticos [2000]. Setting $q_{2n} = x_n$ and $q_{2n+1} = y_n$ yields the discrete system

$$\frac{\xi_{2n+2} + \xi_{2n+1}}{x_{n+1} + y_n} + \frac{\xi_{2n+1} + \xi_{2n}}{x_n + y_n} = z - \frac{2[(\xi_{2n+1} - \varphi) y_n + \rho]}{1 - y_n^2}$$

$$\frac{\xi_{2n+1} + \xi_{2n}}{x_n + y_n} + \frac{\xi_{2n} + \xi_{2n-1}}{x_n + y_{n-1}} = z - \frac{2[(\xi_{2n} + \varphi) x_n + \rho]}{1 - x_n^2}$$

Third Discrete Equation

Consider the Bäcklund transformation $\mathcal{R}_3 = \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1}$, which has an inverse $\mathcal{R}_3^{-1} = \mathcal{T}_{-1,1,1}$, and $w_n = w_n(z; \alpha_n, \beta_n, \gamma_n)$, with $\alpha_n = \frac{1}{2}a_n^2$, $\beta_n = -\frac{1}{2}b_n^2$ and $\gamma_n = c_n$, is a solution of $\mathrm{P_V}$, then

$$w_{n+1} = \mathcal{R}_3(w_n) = 1 + \frac{2zw_n}{a_n w_n' - a_n w_n^2 + (a_n - b_n - z)w_n + b_n}$$
$$w_{n-1} = \mathcal{R}_3^{-1}(w_n) = 1 + \frac{2zw_n}{a_n w_n' - a_n w_n^2 + (a_n - b_n - z)w_n + b_n}$$

Eliminating w_n' gives the difference equation

$$\frac{1}{w_{n+1} - 1} + \frac{1}{w_{n-1} - 1} = -\frac{a_n(w_n - 1)}{z} - 1$$

then setting $w_n = 1 + 1/x_n$ and $w_{n\pm1} = 1 + 1/x_{n\pm1}$ gives the discrete equation

$$x_{n+1} + x_{n-1} + 1 + \frac{a_n}{zx_n} = 0$$

The parameters (a_n, b_n, c_n) evolve as follows

$$\left.\begin{aligned} a_{n+1} &= \tfrac{1}{2}\big(a_n + b_n + c_n - 1\big) \\ b_{n+1} &= -\tfrac{1}{2}\big(a_n + b_n - c_n - 1\big) \\ c_{n+1} &= a_n - b_n \end{aligned}\right\} \Rightarrow \begin{aligned} a_n &= \tfrac{1}{3}n + \rho + \lambda\cos\big(\tfrac{2}{3}\pi n\big) + \tfrac{1}{3}\sqrt{3}\,\varphi\sin\big(\tfrac{2}{3}\pi n\big) \\ b_n &= \tfrac{1}{3} + \sqrt{3}\,\lambda\sin\big(\tfrac{2}{3}\pi n\big) - \varphi\cos\big(\tfrac{2}{3}\pi n\big) \\ c_n &= \tfrac{1}{3}n + \rho - 2\lambda\cos\big(\tfrac{2}{3}\pi n\big) - \tfrac{2}{3}\sqrt{3}\,\varphi\sin\big(\tfrac{2}{3}\pi n\big) \end{aligned}$$

with λ, ρ and φ arbitrary constants (the factor of $\sqrt{3}$ is for convenience).

University of Kent

Hence we obtain the **ternary** $\mathrm{dP_I}$ equation (Grammaticos, Ramani & Papageorgiou [1997]; Tokihiro, Grammaticos & Ramani [2002])

$$x_{n+1} + x_{n-1} + 1 + \frac{a_n}{z x_n} = 0 \tag{1}$$

with

$$a_n = \tfrac{1}{3}n + \rho + \lambda\cos\left(\tfrac{2}{3}\pi n\right) + \tfrac{1}{3}\sqrt{3}\,\varphi\sin\left(\tfrac{2}{3}\pi n\right)$$

Setting $x_{3n} = X_n$, $x_{3n+1} = Y_n$ and $x_{3n+2} = Z_n$ gives

$$\begin{aligned} X_n(Y_n + Z_{n-1} + 1) + \frac{n+\rho+\lambda}{z} &= 0 \\ Y_n(Z_n + X_n + 1) + \frac{n+\rho-\frac{1}{2}\lambda+\frac{1}{2}\varphi+\frac{1}{3}}{z} &= 0 \\ Z_n(X_{n+1} + Y_n + 1) + \frac{n+\rho-\frac{1}{2}\lambda-\frac{1}{2}\varphi+\frac{2}{3}}{z} &= 0 \end{aligned}$$

Remark: The "standard" version of $\mathrm{dP_I}$ is the equation (Grammaticos, Ramani & Papageorgiou [1991])

$$x_{n+1} + x_n + x_{n-1} = 1 + \frac{\lambda n + \rho + (-1)^n\varphi}{x_n}$$

with λ, ρ and φ arbitrary constants, which arises from the Bäcklund transformations of $\mathrm{P_{IV}}$ (Fokas, Grammaticos & Ramani [1993]; PAC, Mansfield & Webster [2000]).

Fourth Discrete Equation

Lemma (PAC, Dunning & Mitchell [2026])

Suppose that x_n, x_{n+2} and x_{n-2} are solutions of ternary $\mathrm{dP_I}$

$$x_n(x_{n+1}+x_{n-1}+1)+\frac{a_n}{z}=0 \tag{1}$$

where

$$a_n=\tfrac{1}{3}n+\rho+\lambda\cos\left(\tfrac{2}{3}\pi n\right)+\tfrac{1}{3}\sqrt{3}\,\varphi\sin\left(\tfrac{2}{3}\pi n\right)$$

with λ, ρ and φ arbitrary constants, then x_n, x_{n+2} and x_{n-2} satisfy the discrete equation

$$\frac{a_{n+1}}{x_{n+2}+x_n+1}+\frac{a_{n-1}}{x_n+x_{n-2}+1}=z+\frac{a_n}{x_n}$$

Proof. Letting $n\to n-1$ in ternary $\mathrm{dP_I}$ (1) gives

$$x_{n-1}(x_n+x_{n-2}+1)+\frac{a_{n-1}}{z}=0$$

Solving this equation and ternary $\mathrm{dP_I}$ for x_{n+1} and x_{n-1} gives

$$x_{n+1}=\frac{a_{n-1}}{z(x_n+x_{n-2}+1)}-1-\frac{a_n}{zx_n},\qquad x_{n-1}=-\frac{a_{n-1}}{z(x_n+x_{n-2}+1)}$$

Letting $n\to n+2$ in the expression for x_{n-1} gives

$$x_{n+1}=-\frac{a_{n+1}}{z(x_{n+2}+x_n+1)}$$

Equating the two expressions for x_{n+1} gives the result.

University of Kent

Alternative Proof

Consider the Bäcklund transformation $\mathcal{R}_4 = \mathcal{R}_3^2 = \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1}^2 \circ \mathcal{T}_{1,-1,1}$

$$w_{n+2} = \mathcal{R}_4(w_n) = \frac{1}{w_n}\left\{1 + \frac{(a_n - b_n + c_n + 1)(w_n - 1)^2}{zw_n' + a_n w_n^2 - (2a_n + c_n + 1 - z)w_n + a_n + c_n + 1}\right\}$$

$$w_{n-2} = \mathcal{R}_4^{-1}(w_n) = \frac{1}{w_n}\left\{1 + \frac{(a_n + b_n + c_n - 1)(w_n - 1)^2}{zw_n' - a_n w_n^2 + (2a_n + c_n - 1 - z)w_n - a_n - c_n + 1}\right\}$$

where a_n, b_n and c_n are given by

$$\begin{aligned}
a_n &= \tfrac{1}{3}n + \rho + \lambda\cos\left(\tfrac{2}{3}\pi n\right) + \tfrac{1}{3}\sqrt{3}\,\varphi\sin\left(\tfrac{2}{3}\pi n\right)\\
b_n &= \tfrac{1}{3} + \sqrt{3}\,\lambda\sin\left(\tfrac{2}{3}\pi n\right) - \varphi\cos\left(\tfrac{2}{3}\pi n\right)\\
c_n &= \tfrac{1}{3}n + \rho - 2\lambda\cos\left(\tfrac{2}{3}\pi n\right) - \tfrac{2}{3}\sqrt{3}\,\varphi\sin\left(\tfrac{2}{3}\pi n\right)
\end{aligned}$$

Then eliminating w_n' and letting $w_n = 1 + 1/x_n$ and $w_{n\pm2} = 1 + 1/x_{n\pm2}$ gives

$$\frac{a_{n+1}}{x_{n+2} + x_n + 1} + \frac{a_{n-1}}{x_n + x_{n-2} + 1} = z + \frac{a_n}{x_n}$$

since $a_{n+1} = \frac{1}{2}(a_n - b_n + c_n + 1)$ and $a_{n-1} = \frac{1}{2}(a_n + b_n + c_n - 1)$.

Suppose that x_n satisfies

$$\frac{a_{n+1}}{x_{n+2}+x_n+1}+\frac{a_{n-1}}{x_n+x_{n-2}+1}=z+\frac{a_n}{x_n}$$

where

$$a_n=\tfrac{1}{3}n+\rho+\lambda\cos\left(\tfrac{2}{3}\pi n\right)+\tfrac{1}{3}\sqrt{3}\,\varphi\sin\left(\tfrac{2}{3}\pi n\right)$$

with λ, ρ and φ arbitrary constants, then $q_n=x_{2n}$ and $p_n=x_{2n+1}$ respectively satisfy

$$\frac{a_{2n+1}}{q_{n+1}+q_n+1}+\frac{a_{2n-1}}{q_n+q_{n-1}+1}=z+\frac{a_{2n}}{q_n}$$

$$\frac{a_{2n+2}}{p_{n+1}+p_n+1}+\frac{a_{2n}}{p_n+p_{n-1}+1}=z+\frac{a_{2n+1}}{p_n}$$

Further $X_n=x_{3n}$, $Y_n=x_{3n+1}$ and $Z_n=x_{3n+2}$ satisfy

$$\frac{a_{3n+1}}{X_n+Z_n+1}+\frac{a_{3n-1}}{X_n+Y_{n-1}+1}=z+\frac{a_{3n}}{X_n}$$

$$\frac{a_{3n+2}}{Y_n+X_{n+1}+1}+\frac{a_{3n}}{Y_n+Z_{n-1}+1}=z+\frac{a_{3n+1}}{Y_n}$$

$$\frac{a_{3n+3}}{Y_{n+1}+Z_n+1}+\frac{a_{3n+1}}{X_n+Z_n+1}=z+\frac{a_{3n+2}}{Z_n}$$

Differential-difference equations

We obtained asymmetric $\mathrm{dP_{II}}$

$$q_{n+1}+q_{n-1}=\frac{4}{z}\,\frac{(n+\lambda)q_n+\rho+(-1)^n\varphi}{1-q_n^2}$$

by adding the equations

$$q_{n+1}+\frac{2}{1-q_n^2}\frac{\mathrm{d}q_n}{\mathrm{d}z}-\frac{2}{z}\,\frac{(n+\lambda)q_n+\rho+(-1)^n\varphi}{1-q_n^2}=0$$

$$q_{n-1}-\frac{2}{1-q_n^2}\frac{\mathrm{d}q_n}{\mathrm{d}z}-\frac{2}{z}\,\frac{(n+\lambda)q_n+\rho+(-1)^n\varphi}{1-q_n^2}=0$$

Subtracting these equations gives the **modified Volterra lattice**

$$\frac{\mathrm{d}q_n}{\mathrm{d}z}=\tfrac{1}{4}(q_n^2-1)(q_{n+1}-q_{n-1})$$

The associated differential-difference equation for

$$\frac{\xi_{n+1}+\xi_n}{q_{n+1}+q_n}+\frac{\xi_n+\xi_{n-1}}{q_n+q_{n-1}}=z-\frac{2\big[\xi_n+(-1)^n\varphi\big]q_n+2\rho}{1-q_n^2}$$

with $\xi_n=n+\lambda$, is

$$z\frac{\mathrm{d}q_n}{\mathrm{d}z}=\tfrac{1}{2}(1-q_n^2)\left\{\frac{\xi_{n+1}+\xi_n}{q_{n+1}+q_n}-\frac{\xi_n+\xi_{n-1}}{q_n+q_{n-1}}\right\}$$

Application to Quantum Minimal Surfaces
(PAC, Dzhamay, Hone & Mitchell [2025])

The discrete equation

$$v_n(v_{n+1} + v_{n-1} + 1) = \varepsilon(n+1), \qquad \varepsilon > 0 \tag{1}$$

with $v_{-1} = 0$, which is the special case of **ternary** $\mathrm{dP_I}$ that was discussed by Arnlind, Hoppe & Kontsevich [2019] and Hoppe [2025] in a study of quantum minimal surfaces.

In this case $a_n = -\frac{1}{3}(n+1)$, $b_n = \frac{1}{3}$ and $c_n = -\frac{1}{3}(n+1)$, so need a solution of $\mathrm{P_V}$ with

$$(\alpha, \beta, \gamma) = \left(\tfrac{1}{2}a_n^2, -\tfrac{1}{2}b_n^2, c_n\right) = \left(\tfrac{1}{18}(n+1)^2, -\tfrac{1}{18}, -\tfrac{1}{3}(n+1)\right)$$

Special function solutions exist for these parameters since

$$\begin{aligned} a_{3n} - b_{3n} + c_{3n} &= -2n - 1 \\ a_{3n+1} + b_{3n+1} + c_{3n+1} &= -2n - 1 \\ a_{3n+2} &= -n - 1 \end{aligned}$$

Recall that special function solutions exist for $\mathrm{P_V}$ if and only if

$$\varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = 2n + 1, \qquad \text{or} \qquad (a-n)(b-n) = 0$$

where $a = \sqrt{2\alpha}$, $b = \sqrt{-2\beta}$ and $\gamma = c$ and $\varepsilon_j^2 = 1$, $j = 1, 2, 3$ independently.

When $n=0$ then $(\alpha,\beta,\gamma)=\left(\frac{1}{18},-\frac{1}{18},-\frac{1}{3}\right)$ and the Riccati equation is

$$z\frac{\mathrm{d}w_0}{\mathrm{d}z}=\tfrac{1}{3}w_0^2-zw_0-\tfrac{1}{3}$$

which has solution

$$w_0(z)=-\frac{C_1\left\{I_{1/6}(\frac{1}{2}z)-I_{-5/6}(\frac{1}{2}z)\right\}+C_2\left\{K_{1/6}(\frac{1}{2}z)+K_{5/6}(\frac{1}{2}z)\right\}}{C_1\left\{I_{1/6}(\frac{1}{2}z)+I_{-5/6}(\frac{1}{2}z)\right\}+C_2\left\{K_{1/6}(\frac{1}{2}z)-K_{5/6}(\frac{1}{2}z)\right\}}$$

with $I_\nu(z)$, $K_\nu(z)$ **modified Bessel functions** and C_1, C_2 arbitrary constants. Hence

$$w_1(z)=\mathcal{R}_3(w_0;-\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3})=\frac{2\mathcal{Z}_{1/6}(\frac{1}{2}z)}{(3z+2)\mathcal{Z}_{1/6}(\frac{1}{2}z)+3z\mathcal{Z}_{-5/6}(\frac{1}{2}z)}$$

where

$$\mathcal{Z}_{1/6}(\tfrac{1}{2}z)=C_1I_{1/6}(\tfrac{1}{2}z)+C_2K_{1/6}(\tfrac{1}{2}z),\qquad \mathcal{Z}_{-5/6}(\tfrac{1}{2}z)=C_1I_{-5/6}(\tfrac{1}{2}z)-C_2K_{5/6}(\tfrac{1}{2}z)$$

and so

$$v_0=\frac{1}{w_0-1}=-\frac{1}{2}-\frac{\mathcal{Z}_{-5/6}(\frac{1}{2}z)}{2\mathcal{Z}_{1/6}(\frac{1}{2}z)},\qquad v_1=\frac{1}{w_1-1}=-\frac{(3z+2)\mathcal{Z}_{1/6}(\frac{1}{2}z)+3z\mathcal{Z}_{-5/6}(\frac{1}{2}z)}{3z\mathcal{Z}_{1/6}(\frac{1}{2}z)+\mathcal{Z}_{-5/6}(\frac{1}{2}z)}$$

Note that

$$v_{-1}=-v_1-1+\frac{1}{3zv_0}=0$$

Theorem (PAC, Dzhamay, Hone & Mitchell [2025])

For each $\varepsilon > 0$, the unique positive solution of the ternary dP_{I} equation

$$v_n(v_{n+1} + v_{n-1} + 1) = \varepsilon(n+1) \tag{1}$$

subject to the initial condition

$$v_{-1} = 0, \qquad v_0 > 0$$

is determined by the value of $v_0 = v_0(\varepsilon)$, which is given by a ratio of modified Bessel functions, that is

$$v_0(\varepsilon) = \frac{1}{2}\left\{\frac{K_{5/6}(z)}{K_{1/6}(z)} - 1\right\}, \qquad z = \frac{1}{6\varepsilon}$$

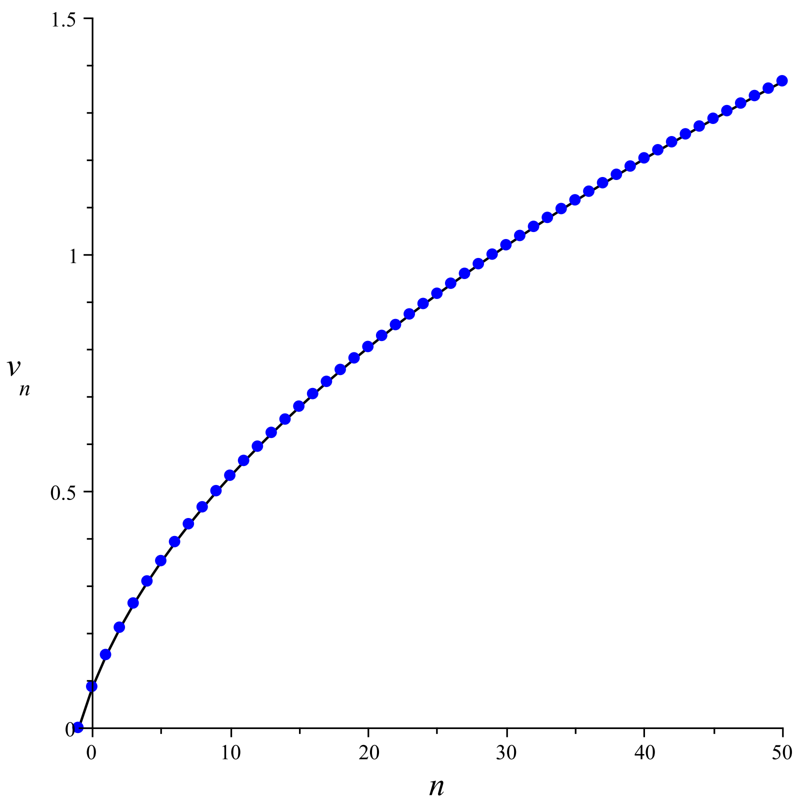
For each $n > 0$, the corresponding quantities $v_n > 0$ are written explicitly as ratios of Wronskian determinants whose entries are specified in terms of modified Bessel functions.

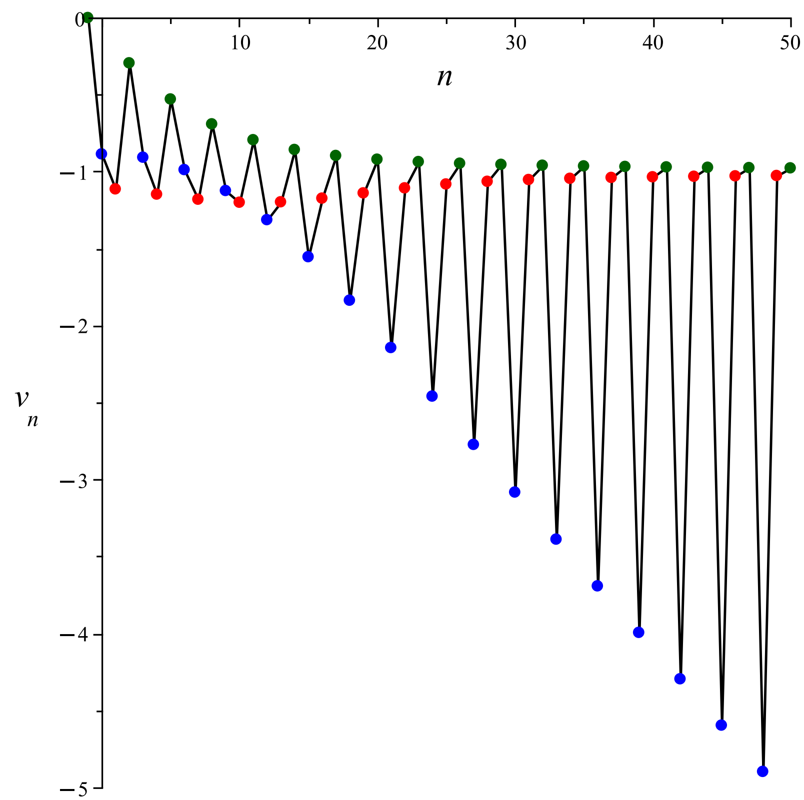
Remark: The expressions for v_{3n}, v_{3n+1} and v_{3n+2} have different structures illustrating the three-fold nature of the solutions, despite the lack of ternary symmetry in equation (1).

University of Kent

Plots when $\varepsilon = 0.1$

$$v_n(v_{n+1} + v_{n-1} + 1) = \varepsilon(n+1), \quad v_{-1} = 0, \quad \varepsilon > 0$$





$$v_0 = \frac{1}{2}\left\{\frac{K_{5/6}(\frac{3}{5})}{K_{1/6}(\frac{3}{5})} - 1\right\}$$

$$v_0 = -\frac{1}{2}\left\{\frac{I_{-5/6}(\frac{3}{5})}{I_{1/6}(\frac{3}{5})} + 1\right\}$$

v_{3n} v_{3n+1} v_{3n+2}

University of Kent

Remarks

- Felder & Hoppe [2025] show that equation

$$v_n(v_{n+1} + v_{n-1} + 1) = \varepsilon(n + 1), \qquad \varepsilon > 0 \tag{1}$$

  arises as the recurrence relation for orthogonal polynomials in the complex plane with respect to the weight

$$w(z;t) = \exp\left\{-t|z|^2 + \tfrac{1}{3}\mathrm{i}(z^3 + \overline{z}^3)\right\}, \qquad t > 0$$

  with $z = x + \mathrm{i}y \in \mathbb{C}$. Further the problem has the **same** unique solution as the problem solved by Clarkson *et al.* [2025].

- Equation (1) arises in the theory of random normal matrices Teodorescu, Bettelheim, Agam, Zabrodin & Wiegmann [2005]

Rational solutions of discrete equations

Each of the Bäcklund transformations is actually four transformations since there are two independent choices of sign for $a = \pm\sqrt{2\alpha}$ and $b = \pm\sqrt{2\beta}$. This can lead to four different solutions from applying the Bäcklund transformation to a particular solution.

The effect on the parameters of applying the transformations $\mathcal{R}_1$ and $\mathcal{R}_1^2$ on a solution w of $\mathrm{P_V}$ with $(\alpha, \beta, \gamma) = \left(\frac{1}{2}a^2, -\frac{1}{2}b^2, c\right)$

	α_1	β_1	γ_1
$\mathcal{R}_1(a, b, c)$	$\frac{1}{8}(a-b+c+1)^2$	$-\frac{1}{8}(a-b-c+1)^2$	$a+b$
$\mathcal{R}_1(-a, b, c)$	$\frac{1}{8}(a+b-c-1)^2$	$-\frac{1}{8}(a+b+c-1)^2$	$-a+b$
$\mathcal{R}_1(a, -b, c)$	$\frac{1}{8}(a+b+c+1)^2$	$-\frac{1}{8}(a+b-c+1)^2$	$a-b$
$\mathcal{R}_1(-a, -b, c)$	$\frac{1}{8}(a-b-c-1)^2$	$-\frac{1}{8}(a-b+c-1)^2$	$-a-b$
$\mathcal{R}_1^2(a, b, c)$	$\frac{1}{2}(a+1)^2$	$-\frac{1}{2}(b-1)^2$	c
$\mathcal{R}_1^2(-a, b, c)$	$\frac{1}{2}(a-1)^2$	$-\frac{1}{2}(b-1)^2$	c
$\mathcal{R}_1^2(a, -b, c)$	$\frac{1}{2}(a+1)^2$	$-\frac{1}{2}(b+1)^2$	c
$\mathcal{R}_1^2(-a, b, c)$	$\frac{1}{2}(a-1)^2$	$-\frac{1}{2}(b-1)^2$	c

Rational solutions of discrete equations: Generalised Laguerre polynomials

Example. Consider the rational solution

$$w_{1,1}^{(\mu-3)}(z) = -\frac{(z-\mu+1)\left[z^2-2\mu z+\mu(\mu+1)\right]}{z^2-2\mu z+\mu(\mu-1)}$$

which satisfies $\mathrm{P_V}$ with $(\alpha,\beta,\gamma) = \left(\frac{1}{2}, -\frac{1}{2}(\mu+1)^2, \mu-3\right)$. There is only one choice of parameters for which the Bäcklund transformation $\mathcal{R}_1$ gives an infinite sequence of rational functions ($a=1$, $b=-\mu-1$, $c=\mu-3$); in the other three cases the sequence terminates after a few iterations.

Hence we obtain a sequence of rational solutions of $\mathrm{P_V}$

$$w_{1,1}^{(\mu-3)}(z) \xrightarrow{\mathcal{R}_1} \frac{1}{v_{1,1}^{(\mu)}(z)} \xrightarrow{\mathcal{R}_1} w_{2,1}^{(\mu-3)}(z) \xrightarrow{\mathcal{R}_1} \frac{1}{v_{2,1}^{(\mu)}(z)} \xrightarrow{\mathcal{R}_1} w_{3,1}^{(\mu-3)}(z)$$

which gives a sequence of rational solutions of **asymmetric** $\mathrm{dP_{II}}$. The first two are

$$q_0(z) = \frac{w_{1,1}^{(\mu-3)}(z)+1}{w_{1,1}^{(\mu-3)}(z)-1} = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z-\mu}{z^2-2(\mu+2)z+\mu(\mu-1)}$$

$$q_1(z) = \frac{1+v_{1,1}^{(\mu)}(z)}{1-v_{1,1}^{(\mu)}(z)} = 1 - \frac{2\mu}{z} + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^2-2\mu z+\mu(\mu-1)}{z^2-2\mu z+\mu(\mu+1)}$$

University of Kent

If we define the rational functions

$$q_{2n}(z) = x_n(z) = \frac{w_{n+1,1}^{(\mu-3)}(z)+1}{w_{n+1,1}^{(\mu-3)}(z)-1} = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T_{n,1}^{(\mu-2)}(z)}{T_{n,2}^{(\mu-4)}(z)}$$

$$q_{2n+1}(z) = y_n(z) = \frac{1+v_{n+1,1}^{(\mu)}(z)}{1-v_{n+1,1}^{(\mu)}(z)} = 1 - \frac{2\mu}{z} + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T_{n+1,1}^{(\mu-3)}(z)}{T_{n,2}^{(\mu-3)}(z)}$$

so the structure of the even and odd terms is different, then $q_n(z)$ satisfies

$$q_{n+1} + q_{n-1} = \frac{4}{z}\,\frac{(n+\mu+2)q_n - \frac{3}{2} - (\mu-\frac{3}{2})(-1)^n}{1-q_n^2}$$

which is **asymmetric** $\mathrm{dP_{II}}$ with $\lambda = \mu+2$, $\rho = -\frac{3}{2}$ and $\varphi = -\mu+\frac{3}{2}$, whilst $x_n(z)$ and $y_n(z)$ satisfy the discrete system

$$x_{n+1} + x_n = \frac{4}{z}\,\frac{(2n+\mu+3)y_n + \mu - 3}{1-y_n^2}$$

$$y_n + y_{n-1} = \frac{4}{z}\,\frac{(2n+\mu+2)x_n - \mu}{1-x_n^2}$$

In general, for fixed m and n

$$w^{(\mu)}_{m,n} \xrightarrow{\mathcal{R}_1} \frac{1}{v^{(\mu+2n+1)}_{m,n}} \xrightarrow{\mathcal{R}_1} w^{(\mu)}_{m+1,n} \xrightarrow{\mathcal{R}_1} \frac{1}{v^{(\mu+2n+1)}_{m+1,n}} \xrightarrow{\mathcal{R}_1} w^{(\mu)}_{m+2,n}$$

Lemma

(PAC, Dunning & Mitchell [2026])

Consider the rational functions

$$Q_{2N}(z) = X_N(z) = \frac{w^{(\mu)}_{N+m,n}(z)+1}{w^{(\mu)}_{N+m,n}(z)-1}, \qquad Q_{2N+1}(z) = Y_N(z) = \frac{1+v^{(\mu+2n+1)}_{N+m,n}(z)}{1-v^{(\mu+2n+1)}_{N+m,n}(z)}$$

then $Q_N(z)$ *satisfies*

$$Q_{N+1}+Q_{N-1} = \frac{4}{z}\,\frac{(N+2m+2n+\mu+1)Q_N - n - \frac{1}{2} - (n+\mu+\frac{1}{2})(-1)^N}{1-Q_N^2}$$

which is **asymmetric** dP$_{\rm II}$ *with* $\lambda = 2m+2n+\mu+1$*,* $\rho = -n-\frac{1}{2}$ *and* $\varphi = -n-\mu-\frac{1}{2}$*, whilst* $X_N(z)$ *and* $Y_N(z)$ *satisfy the discrete system*

$$X_{N+1}+X_N = \frac{4}{z}\,\frac{(2N+2m+2n+\mu+2)Y_N+\mu}{1-Y_N^2}$$

$$Y_N+Y_{N-1} = \frac{4}{z}\,\frac{(2N+2m+2n+\mu+1)X_N-2n-\mu-1}{1-X_N^2}$$

Second Bäcklund transformation

Example. Applying the Bäcklund transformation $\mathcal{R}_2$ to

$$v_{1,1}^{(\mu)}(z) = 1 - \frac{z\left[z^2 - 2\mu z + \mu(\mu+1)\right]\left[z^2 - 2\mu z + \mu(\mu-1)\right]}{(z-\mu)\left[z^4 - 4\mu z^3 + 6\mu^2 z^2 - 4\mu(\mu^2-1)z + \mu^2(\mu^2-1)\right]}$$

gives the sequence of rational solutions of P_{V}

$$v_{1,1}^{(\mu)}(z) \xrightarrow{\mathcal{R}_2} u_{2,1}^{(\mu+4)}(z) \xrightarrow{\mathcal{R}_2} v_{1,2}^{(\mu+2)}(z) \xrightarrow{\mathcal{R}_2} u_{3,1}^{(\mu+6)}(z) \xrightarrow{\mathcal{R}_2} v_{1,3}^{(\mu+4)}(z)$$

where

$$u_{2,1}^{(\mu+4)}(z) = \tfrac{1}{2}(z-\mu+1) + \tfrac{1}{2}z\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^2 - 2(\mu+1) + (\mu+1)(\mu+2)}{z^3 - 3\mu z^2 + 3\mu(\mu+1)z - \mu(\mu+1)(\mu+2)}$$

Hence the first two rational solutions of the discrete equation are

$$q_0(z) = \frac{v_{1,1}^{(\mu)}(z)+1}{v_{1,1}^{(\mu)}(z)-1} = 1 + \frac{2\mu}{z} + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^2 - 2\mu + \mu(\mu+1)}{z^2 - 2\mu + \mu(\mu-1)}$$

$$q_1(z) = \frac{u_{2,1}^{(\mu+4)}(z)+1}{u_{2,1}^{(\mu+4)}(z)-1}$$

$$= 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^4 - 4(\mu+1)z^3 + 6(\mu+1)^2z^2 - 4\mu(\mu+1)(\mu+2)z + \mu(\mu+1)^2(\mu+2)}{z^2 - 2\mu + \mu(\mu+1)}$$

University of Kent

If we define the rational functions

$$q_{2n}(z) = x_n(z) = \frac{v_{1,n+1}^{(\mu+2n)}(z)+1}{v_{1,n+1}^{(\mu+2n)}(z)-1} = -1 + \frac{2(\mu+2n)}{z} + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T_{0,n+2}^{(\mu-3)}(z)}{T_{1,n+1}^{(\mu-3)}(z)}$$

$$q_{2n+1}(z) = y_n(z) = \frac{u_{n+2,1}^{(\mu+2n+4)}(z)+1}{u_{n+2,1}^{(\mu+2n+4)}(z)-1} = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T_{1,n+2}^{(\mu-3)}(z)}{T_{0,n+2}^{(\mu-3)}(z)}$$

then $q_n(z)$ satisfies

$$\frac{n+\mu+2}{q_{n+1}+q_n} + \frac{n+\mu+1}{q_n+q_{n-1}} = \tfrac{1}{2}z - \frac{\left\{n+\mu+\frac{3}{2}[1+(-1)^n]\right\}q_n + 3 - \mu}{1-q_n^2}$$

whilst $x_n(z)$ and $y_n(z)$ satisfy the discrete system

$$\frac{2n+\mu+3}{x_{n+1}+y_n} + \frac{2n+\mu+2}{x_n+y_n} = \tfrac{1}{2}z - \frac{(2n+\mu+4)y_n + 3 - \mu}{1-y_n^2}$$

$$\frac{2n+\mu+2}{x_n+y_n} + \frac{2n+\mu+1}{x_n+y_{n-1}} = \tfrac{1}{2}z - \frac{(2n+\mu+1)x_n + 3 - \mu}{1-x_n^2}$$

Third Bäcklund transformation: Ternary $\mathrm{dP_I}$

Example. Applying the Bäcklund transformation $\mathcal{R}_3$ to

$$\frac{1}{w_{1,1}^{(\mu-3)}(z)} = -\frac{z^2-2\mu z+\mu(\mu-1)}{(z-\mu+1)\left[z^2-2\mu z+\mu(\mu+1)\right]}$$

gives the sequence of rational solutions of $\mathrm{P_V}$

$$\frac{1}{w_{1,1}^{(\mu-3)}} \xrightarrow{\mathcal{R}_3} v_{1,1}^{(\mu)} \xrightarrow{\mathcal{R}_3} u_{2,1}^{(\mu+3)} \xrightarrow{\mathcal{R}_3} \frac{1}{w_{1,2}^{(\mu-4)}} \xrightarrow{\mathcal{R}_3} v_{1,2}^{(\mu+1)} \xrightarrow{\mathcal{R}_3} u_{3,1}^{(\mu+4)}$$

so that

$$\mathcal{R}_3^3\left(1/w_{1,1}^{(\mu)}\right) = 1/w_{1,2}^{(\mu-1)}, \qquad \mathcal{R}_3^3\left(v_{1,1}^{(\mu)}\right) = v_{1,2}^{(\mu+1)}, \qquad \mathcal{R}_3^3\left(u_{2,1}^{(\mu)}\right) = u_{3,1}^{(\mu+1)}$$

Hence we obtain rational solutions of **ternary** $\mathrm{dP_I}$

$$x_0(z) = \frac{w_{1,1}^{(\mu-3)}(z)}{1-w_{1,1}^{(\mu-3)}(z)} = -1+\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^2-2(\mu-1)z+\mu(\mu-1)}{z-\mu}$$

$$x_1(z) = \frac{1}{v_{1,1}^{(\mu)}(z)-1} = -1+\frac{\mu}{z}+\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^2-2\mu z+\mu(\mu-1)}{z^2-2\mu z+\mu(\mu+1)}$$

$$x_2(z) = \frac{1}{u_{2,1}^{(\mu+3)}(z)-1} = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^4-4\mu z^3+6\mu^2z^2-4\mu(\mu^2-1)z+\mu^2(\mu^2-1)z}{z^2-2(\mu+1)z+\mu(\mu-1)}$$

University of Kent

If we define the rational functions

$$x_{3n}(z) = X_n(z) = \frac{w_{1,n+1}^{(\mu-n-3)}(z)}{1 - w_{1,n+1}^{(\mu-n-3)}(z)} = -1 + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T_{0,n+2}^{(\mu-n-4)}(z)}{T_{0,n+1}^{(\mu-n-2)}(z)}$$

$$x_{3n+1}(z) = Y_n(z) = \frac{1}{v_{1,n+1}^{(\mu+n)}(z) - 1} = -1 + \frac{\mu+n}{z} + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T_{0,n+2}^{(\mu-n-3)}(z)}{T_{1,n+1}^{(\mu-n-3)}(z)}$$

$$x_{3n+2}(z) = Z_n(z) = \frac{1}{u_{n+2,1}^{(\mu+n+3)}(z) - 1} = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T_{1,n+2}^{(\mu-n-4)}(z)}{T_{0,n+2}^{(\mu-n-4)}(z)}$$

then $x_n(z)$ satisfies

$$x_n(x_{n+1} + x_{n-1} + 1) + \frac{n + \mu + 5 + 2(\mu - 1)\cos\left(\frac{2}{3}\pi n\right) + \frac{4}{3}\sqrt{3}\,\sin\left(\frac{2}{3}\pi n\right)}{3z} = 0$$

whilst $X_n(z)$, $Y_n(z)$ and $Z_n(z)$ satisfy the discrete system

$$X_n(Y_n + Z_{n-1} + 1) + \frac{n+\mu+1}{z} = 0$$

$$Y_n(Z_n + X_n + 1) + \frac{n+3}{z} = 0$$

$$Z_n(X_{n+1} + Y_n + 1) + \frac{n+2}{z} = 0$$

Rational solutions of discrete equations: Generalised Umemura polynomials

The generalised Umemura polynomial $U^{(\kappa)}_{m,n}(z)$ is given by

$$U^{(\kappa)}_{m,n}(z) = \exp(\tfrac{1}{2}mz)\ \mathrm{Wr}\left(\varphi^{(\kappa)}_1, \varphi^{(\kappa)}_3, \ldots, \varphi^{(\kappa)}_{2m-1}; \psi^{(\kappa)}_1, \psi^{(\kappa)}_3, \ldots, \psi^{(\kappa)}_{2n-1}\right), \qquad m, n \geq 0$$

where

$$\varphi^{(\kappa)}_k(z) = \mathrm{e}^{-z/2}\, L^{(\kappa)}_k(\tfrac{1}{2}z), \qquad \psi^{(\kappa)}_k(z) = L^{(\kappa)}_k(-\tfrac{1}{2}z)$$

with κ a parameter, $L^{(\alpha)}_k(\zeta)$ the Laguerre polynomial and $U^{(\kappa)}_{0,0}(z) = 1$.

The associated rational solutions of $\mathrm{P_V}$ are

$$\widehat{w}^{(\kappa)}_{m,n} = -\frac{U^{(2\kappa)}_{m,n-1}\, U^{(2\kappa+2)}_{m-1,n}}{U^{(2\kappa+2)}_{m,n-1}\, U^{(2\kappa)}_{m-1,n}} \qquad (\alpha, \beta, \gamma) = \left(\tfrac{1}{2}(m+\kappa)^2, -\tfrac{1}{2}(n+\kappa)^2, m+n\right)$$

$$\widehat{v}^{(\kappa)}_{m,n} = -\frac{U^{(2\kappa-2n+3)}_{m-1,n-1}\, U^{(2\kappa-2n-1)}_{m,n}}{U^{(2\kappa-2n+1)}_{m-1,n-1}\, U^{(2\kappa-2n+1)}_{m,n}} \qquad (\alpha, \beta, \gamma) = \left(\tfrac{1}{2}(m+\kappa)^2, -\tfrac{1}{2}(n-\kappa)^2, m-n\right)$$

$$\widehat{u}^{(\kappa)}_{m,n} = -\frac{U^{(\kappa+1)}_{m,n-1}\, U^{(\kappa-1)}_{m,n+1}}{U^{(\kappa+1)}_{m-1,n}\, U^{(\kappa-1)}_{m+1,n}} \qquad (\alpha, \beta, \gamma) = \left(\tfrac{1}{2}(m+\tfrac{1}{2})^2, -\tfrac{1}{2}(n+\tfrac{1}{2})^2, m+n+\kappa\right)$$

First Bäcklund transformation: Asymmetric $\mathrm{dP_I}$

Example. Applying the Bäcklund transformation $\mathcal{R}_1$ to

$$\widehat{w}_{2,1}^{(\kappa-3/2)} = -\frac{\{z^2-16\kappa(\kappa+1)\}\{z^3-6(2\kappa-1)z^2+48\kappa(\kappa-1)z-32\kappa(\kappa-1)(2\kappa-1)\}}{\{z^2-16\kappa(\kappa-1)\}\{z^3-6(2\kappa+1)z^2+48\kappa(\kappa+1)z-32\kappa(\kappa+1)(2\kappa+1)\}}$$

gives the sequence of rational solutions of $\mathrm{P_V}$

$$\widehat{w}_{2,1}^{(\kappa-3/2)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{2,1}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{2,1}^{(\kappa-1/2)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{2,1}^{(\kappa+1)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{2,1}^{(\kappa+1/2)}$$

so only the parameter changes. Hence the first few rational solutions of **asymmetric** $\mathrm{dP_I}$ are

$$q_0 = \frac{\widehat{w}_{2,1}^{(\kappa-3/2)}+1}{\widehat{w}_{2,1}^{(\kappa-3/2)}-1} = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^4-8\kappa z^3+128\kappa(\kappa^2-1)z-256\kappa^2(\kappa^2-1)}{z-4\kappa}$$

$$q_1 = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^2-16\kappa(\kappa+1)}{z^3-6(2\kappa+1)z^2+48\kappa(\kappa+1)z-32\kappa(\kappa+1)(2\kappa+1)}$$

$$q_2 = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^4-8(\kappa+1)+128\kappa(\kappa+1)(\kappa+2)-256\kappa(\kappa+1)^2(\kappa+2)}{z-4(\kappa+1)}$$

$$q_3 = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^2-16(\kappa+1)(\kappa+2)}{z^3-6(2\kappa+3)z^2+48(\kappa+1)(\kappa+2)z-32(\kappa+1)(\kappa+2)(2\kappa+3)}$$

University of Kent

If we define the rational functions

$$q_{2n}(z) = x_n(z) = \frac{w_{2,1}^{(n+\kappa-3/2)}(z)+1}{w_{2,1}^{(n+\kappa-3/2)}(z)-1} = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{U_{2,1}^{(2n+2\kappa-3)}(z)}{U_{1,0}^{(2n+2\kappa-1)}(z)}$$

$$q_{2n+1}(z) = y_n(z) = \frac{v_{2,1}^{(n+\kappa)}(z)+1}{v_{2,1}^{(n+\kappa)}(z)-1} = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{U_{2,0}^{(2n+2\kappa-1)}(z)}{U_{1,1}^{(2n+2\kappa-1)}(z)}$$

then $q_n(z)$ satisfies

$$q_{n+1}+q_{n-1} = \frac{4}{z}\,\frac{(n+2\kappa)q_n+2-(-1)^n}{1-q_n^2}$$

which is **asymmetric** $\mathrm{dP_{II}}$ with $\lambda = 2\kappa$, $\rho = 2$ and $\varphi = -1$, whilst $x_n(z)$ and $y_n(z)$ satisfy the discrete system

$$x_{n+1}+x_n = \frac{4}{z}\,\frac{(2n+2\kappa+1)y_n+3}{1-y_n^2}$$

$$y_n+y_{n-1} = \frac{4}{z}\,\frac{2(n+\kappa)x_n+1}{1-x_n^2}$$

For fixed m and n

$$\widehat{w}_{m,n}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{m,n}^{(\kappa+n+1/2)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{m,n}^{(\kappa+1)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{m,n}^{(\kappa+n+3/2)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{m,n}^{(\kappa+2)}$$

Lemma

(PAC, Dunning & Mitchell [2026])

Define the rational functions

$$Q_{2N}(z) = X_N(z) = \frac{w_{m,n}^{(N+\kappa)}(z)+1}{w_{m,n}^{(N+\kappa)}(z)-1} = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{U_{m,n}^{(2N+2\kappa)}(z)}{U_{m-1,n-1}^{(2N+2\kappa+2)}(z)}$$

$$Q_{2N+1}(z) = Y_N(z) = \frac{v_{m,n}^{(N+\kappa+n+1/2)}(z)+1}{v_{m,n}^{(N+\kappa+n+1/2)}(z)-1} = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{U_{m,n-1}^{(2N+2\kappa+2)}(z)}{U_{m-1,n}^{(2N+2\kappa+2)}(z)}$$

then $Q_N(z)$ *satisfies*

$$Q_{N+1}+Q_{N-1} = \frac{4}{z}\,\frac{(N+m+n+2\kappa)Q_N+m-n(-1)^N}{1-Q_N^2}$$

which is **asymmetric** $\mathrm{dP_{II}}$ *with* $\lambda = m+n+2\kappa$, $\rho = m$ *and* $\varphi = -n$, *whilst* $X_N(z)$ *and* $Y_N(z)$ *satisfy the discrete system*

$$X_{N+1}+X_N = \frac{4}{z}\,\frac{(2N+m+n+2\kappa+1)Y_N+m+n}{1-Y_N^2}$$

$$Y_N+Y_{N-1} = \frac{4}{z}\,\frac{(2N+m+n+2\kappa)X_N+m-n}{1-X_N^2}$$

University of Kent

Second Bäcklund transformation

Example. Applying the Bäcklund transformation $\mathcal{R}_2$ gives the sequence

$$\widehat{w}_{1,m}^{(\kappa)}(z) \quad \xrightarrow{\mathcal{R}_2} \quad \widehat{v}_{2,m}^{(\kappa+m-1/2)}(z) \quad \xrightarrow{\mathcal{R}_2} \quad \widehat{w}_{3,m}^{(\kappa-1)}(z) \quad \xrightarrow{\mathcal{R}_2} \quad \widehat{v}_{4,m}^{(\kappa+m-3/2)}(z)$$

If we define the rational functions

$$q_{2n}(z) = x_n(z) = \frac{\widehat{w}_{2n+1,m}^{(\kappa-n)}(z)+1}{\widehat{w}_{2n+1,m}^{(\kappa-n)}(z)-1} = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{U_{2n+1,m}^{(2\kappa-2n)}(z)}{U_{2n,m-1}^{(2\kappa-2n+2)}(z)}$$

$$q_{2n+1}(z) = y_n(z) = \frac{\widehat{v}_{2n+2,m}^{(\kappa+m-n-1/2)}(z)+1}{\widehat{v}_{2n+2,m}^{(\kappa+m-n-1/2)}(z)-1} = 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{U_{2n+2,m-1}^{(2\kappa-2n-2m+2)}(z)}{U_{2n+1,m}^{(2\kappa-2n-2m+2)}(z)}$$

then $q_n(z)$ satisfies

$$\frac{n+\frac{3}{2}}{q_{n+1}+q_n} + \frac{n+\frac{1}{2}}{q_n+q_{n-1}} = \tfrac{1}{2}z - \frac{\left[n+1-m(-1)^n\right]q_n + 2\kappa+m+1}{1-q_n^2}$$

whilst $x_n(z)$ and $y_n(z)$ satisfy the discrete system

$$\frac{2n+\frac{5}{2}}{x_{n+1}+y_n} + \frac{2n+\frac{3}{2}}{x_n+y_n} = \tfrac{1}{2}z - \frac{(2n+m+1)y_n + 2\kappa+m+1}{1-y_n^2}$$

$$\frac{2n+\frac{3}{2}}{x_n+y_n} + \frac{2n+\frac{1}{2}}{x_n+y_{n-1}} = \tfrac{1}{2}z - \frac{(2n-m+1)x_n + 2\kappa+m+1}{1-x_n^2}$$

Third Bäcklund transformation: Ternary $\mathrm{dP_I}$

Example. Applying the Bäcklund transformation $\mathcal{R}_3$ gives the sequence

$$\widehat{w}^{(\kappa)}_{1,m} \xrightarrow{\ \mathcal{R}_3\ } \widehat{v}^{(\kappa+m+1/2)}_{1,m} \xrightarrow{\ \mathcal{R}_3\ } \widehat{u}^{(2\kappa+1)}_{1,m} \xrightarrow{\ \mathcal{R}_3\ } \widehat{w}^{(\kappa)}_{2,m} \xrightarrow{\ \mathcal{R}_3\ } v^{(\kappa+m+1/2)}_{2,m}$$

Define the rational functions

$$X_n(z) = \frac{1}{\widehat{w}^{(\kappa)}_{n+1,m}(z)-1} = -\frac{1}{2} + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{U^{(2\kappa)}_{n+1,m}(z)}{U^{(2\kappa+2)}_{n,m-1}(z)}$$

$$Y_n(z) = \frac{1}{\widehat{v}^{(\kappa+m+1/2)}_{n+1,m}(z)-1} = -\frac{1}{2} + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{U^{(2\kappa+2)}_{n+1,m-1}(z)}{U^{(2\kappa+2)}_{n,m}(z)}$$

$$Z_n(z) = \frac{1}{\widehat{u}^{(2\kappa+1)}_{n+1,m}(z)-1} = -\frac{1}{2} + \frac{n+m+2\kappa+2}{z} + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{U^{(2\kappa+2)}_{n+1,m}(z)}{U^{(2\kappa)}_{n+1,m}(z)}$$

then $X_n(z)$, $Y_n(z)$ and $Z_n(z)$ satisfy

$$X_n\big(Y_n + Z_{n-1} + 1\big) + \frac{n+\kappa+1}{z} = 0$$

$$Y_n\big(Z_n + X_n + 1\big) + \frac{n+m+\kappa+\frac{3}{2}}{z} = 0$$

$$Z_n\big(X_{n+1} + Y_n + 1\big) + \frac{n+\frac{3}{2}}{z} = 0$$

University of Kent

Non-unique rational solutions of discrete equations

Example. The rational solutions

$$w_{1,1}^{(1)}(z) = -\frac{(z-3)(z^2-8z+20)}{(z-2)(z-6)}$$

$$u_{1,2}^{(1)}(z) = \frac{(z^2+4z+6)(z^3+9z^2+36z+60)}{z^4+12z^3+54z^2+96z+72}$$

both satisfy $\mathrm{P_V}$ for the parameters $(\alpha,\beta,\gamma) = \left(\frac{1}{2}, -\frac{25}{2}, 1\right)$. Hence

$$q(z) = \frac{w_{1,1}^{(1)}(z)+1}{w_{1,1}^{(1)}(z)-1} = \frac{z^3-12z^2+52z-72}{(z-4)(z^2-6z+12)}$$

$$p(z) = \frac{u_{1,2}^{(1)}(z)+1}{u_{1,2}^{(1)}(z)-1} = \frac{z^5+14z^4+90z^3+312z^2+552z+432}{(z^2+6z+12)(z^3+6z^2+18z+24)}$$

both satisfy

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{q}{q^2-1}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} - \frac{(q+1)^2-25(q-1)^2}{z^2(q^2-1)} - \frac{q^2-1}{2z} + \frac{q(q^2-1)}{4}$$

We can use such pairs of solutions to derive two different hierarchies of rational solutions which satisfy the same discrete equation.

Asymmetric $\mathrm{dP_{II}}$

Example. Define

$$q_{2n}(z) = 1 - \frac{8}{z} + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T^{(1)}_{n,1}(z)}{T^{(1)}_{n-1,2}(z)} \qquad p_{2n}(z) = -1 - \frac{8}{z} + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T^{(-2n-5)}_{1,n+1}(z)}{T^{(-2n-5)}_{2,n}(z)}$$

$$q_{2n+1}(z) = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T^{(2)}_{n,1}(z)}{T^{(0)}_{n,2}(z)} \qquad p_{2n+1}(z) = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{T^{(-2n-6)}_{2,n+1}(z)}{T^{(-2n-6)}_{1,n+1}(z)}$$

then both q_n and p_n satisfy

$$q_{n+1} + q_{n-1} = \frac{4}{z}\,\frac{(n+5)q_n - \frac{3}{2} + \frac{5}{2}(-1)^n}{1-q_n^2}$$

which is **asymmetric** $\mathrm{dP_{II}}$ with $\lambda = 5$, $\rho = -\frac{3}{2}$ and $\varphi = \frac{5}{2}$.

Further, if either $(x_n, y_n) = (q_{2n}, q_{2n+1})$ or $(x_n, y_n) = (p_{2n}, p_{2n+1})$ then we obtain solutions to the discrete system

$$x_{n+1} + x_n = \frac{4}{z}\,\frac{(2n+6)y_n - 4}{1-y_n^2}$$

$$y_n + y_{n-1} = \frac{4}{z}\,\frac{(2n+5)x_n + 1}{1-x_n^2}$$

The first few solutions in the hierarchies are given by

$$q_0 = 1 - \frac{8}{z} + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln(z-3)$$

$$q_1 = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z-4}{z^2-6z+12}$$

$$q_2 = 1 - \frac{8}{z} + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^2-8z+12}{z^2-8z+20}$$

$$q_3 = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^2-10z+20}{z^4-16z^3+96z^2-240z+240}$$

$$p_0 = -1 - \frac{8}{z} + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln(z^2+4z+6)$$

$$p_1 = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^3+6z^2+18z+24}{z^2+6z+12}$$

$$p_2 = -1 - \frac{8}{z} + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^4+12z^3+54z^2+96z+72}{z^3+9z^2+36z+60}$$

$$p_3 = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{z^6+18z^5+144z^4+624z^3+1512z^2+2160z+1440}{z^4+16z^3+96z^2+240z+240}$$

Second Discrete Equation

Example. Define

$$q_{2n}(z) = 1 + \frac{4n}{z} - 2\frac{\mathrm{d}}{\mathrm{d}z}\ln T_{n-1,1}^{(-2n-1)}(z) \qquad p_{2n}(z) = -1 + \frac{4n}{z} - 2\frac{\mathrm{d}}{\mathrm{d}z}\ln T_{0,n-1}^{(1)}(z)$$

$$q_{2n+1}(z) = -1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln T_{n,1}^{(-2n-3)}(z) \qquad p_{2n+1}(z) = 1 + 2\frac{\mathrm{d}}{\mathrm{d}z}\ln T_{0,n}^{(1)}(z)$$

then both q_n and p_n satisfy

$$\frac{n+1}{q_{n+1}+q_n} + \frac{n}{q_n+q_{n-1}} = \tfrac{1}{2}z - \frac{\left\{n+\tfrac{1}{2}\left[1+(-1)^n\right]\right\}q_n - 1}{1-q_n^2}$$

Further, if either $(x_n, y_n) = (q_{2n}, q_{2n+1})$ or $(x_n, y_n) = (p_{2n}, p_{2n+1})$ then we obtain solutions of the discrete system

$$\frac{2n+2}{x_{n+1}+y_n} + \frac{2n+1}{x_n+y_n} = \tfrac{1}{2}z - \frac{(2n+1)y_n - 1}{1-y_n^2}$$

$$\frac{2n+1}{x_n+y_n} + \frac{2n}{x_n+y_{n-1}} = \tfrac{1}{2}z - \frac{(2n+1)x_n - 1}{1-x_n^2}$$

The first few solutions in the hierarchies are given by

$$q_1 = -\frac{z-1}{z+1} \qquad p_1 = 1$$

$$q_2 = -\frac{z^2+3z+4}{z(z+1)} \qquad p_2 = -\frac{z-4}{z}$$

$$q_3 = -\frac{z^2-2}{z^2+4z+6} \qquad p_3 = \frac{z-1}{z-3}$$

$$q_4 = -\frac{z^3+8z^2+30z+48}{z(z^2+4z+6)} \qquad p_4 = -\frac{z^2-9z+24}{z(z-3)}$$

$$q_5 = -\frac{z^3+3z^2-12}{z^3+9z^2+36z+60} \qquad p_5 = \frac{(z-2)^2}{z^2-8z+20}$$

$$q_6 = -\frac{z^4+15z^3+108z^2+420z+720}{z(z^3+9z^2+36z+60)} \qquad p_6 = -\frac{(z-6)(z^2-10z+40)}{z(z^2-8z+20)}$$

$$q_7 = -\frac{z^4+8z^3+24z^2-120}{z^4+16z^3+120z^2+480z+840} \qquad p_7 = \frac{z^3-9z^2+30z-30}{z^3-15z^2+90z-210}$$

Ternary $\mathrm{dP_I}$

Example. Define

$$x_{3n}(z) = -1 + \frac{n+3}{z} - \frac{\mathrm{d}}{\mathrm{d}z}\ln T_{0,n}^{(2-n)}(z) \qquad y_{3n}(z) = \frac{n+3}{z} - \frac{\mathrm{d}}{\mathrm{d}z}\ln T_{1,1}^{(-n-4)}(z)$$

$$x_{3n+1}(z) = -1 + \frac{\mathrm{d}}{\mathrm{d}z}\ln \frac{T_{0,n+1}^{(1-n)}(z)}{T_{0,n}^{(3-n)}(z)} \qquad y_{3n+1}(z) = -1 + \frac{\mathrm{d}}{\mathrm{d}z}\ln \frac{T_{1,1}^{(-n-5)}(z)}{T_{2,1}^{(-n-5)}(z)}$$

$$x_{3n+2}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln T_{0,n+1}^{(2-n)}(z) \qquad y_{3n+2}(z) = -1 + \frac{\mathrm{d}}{\mathrm{d}z}\ln T_{2,1}^{(-n-6)}(z)$$

then both x_n and y_n satisfy **ternary** $\mathrm{dP_I}$

$$x_n(x_{n+1} + x_{n-1} + 1) + \frac{a_n}{z} = 0$$

with

$$a_n = \tfrac{1}{3}n + \tfrac{5}{3} - \tfrac{2}{3}\cos(\tfrac{2}{3}\pi n) + \tfrac{10}{9}\sqrt{3}\,\sin(\tfrac{2}{3}\pi n)$$

If $(X_n, Y_n, Z_n) = (x_{3n}, x_{3n+1}, x_{3n+2})$ or $(X_n, Y_n, Z_n) = (y_{3n}, y_{3n+1}, y_{3n+2})$, then

$$X_n\big(Y_n + Z_{n-1} + 1\big) + \frac{n+1}{z} = 0$$

$$Y_n\big(Z_n + X_n + 1\big) + \frac{n+4}{z} = 0$$

$$Z_n\big(X_{n+1} + Y_n + 1\big) + \frac{n+1}{z} = 0$$

$$x_0 = -1 + \frac{3}{z}$$

$$x_3 = -1 + \frac{4}{z} - \frac{1}{z-3}$$

$$x_6 = -1 + \frac{5}{z} - \frac{2(z-3)}{z^2-6z+12}$$

$$x_1 = -1 + \frac{\mathrm{d}}{\mathrm{d}z} \ln(z-3)$$

$$x_4 = -1 + \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{z^2-6z+12}{z-4}$$

$$x_7 = -1 + \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{z^3-9z^2+36z-60}{z^2-8z+20}$$

$$x_2 = \frac{1}{z-4}$$

$$x_5 = \frac{2(z-4)}{z^2-8z+20}$$

$$x_8 = \frac{3(z^2-8z+20)}{z^3-12z^2+60z-120}$$

$$y_0 = \frac{3}{z} - \frac{2(z+1)}{z^2+2z+2}$$

$$y_3 = \frac{4}{z} - \frac{2(z+2)}{z^2+4z+6}$$

$$y_6 = \frac{5}{z} - \frac{2(z+3)}{z^2+6z+12}$$

$$y_1 = -1 + \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{z^2+4z+6}{z^3+3z^2+6z+6}$$

$$y_4 = -1 + \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{z^2+6z+12}{z^3+6z^2+18z+24}$$

$$y_7 = -1 + \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{z^2+8z+20}{z^3+9z^2+36z+60}$$

$$y_2 = -1 + \frac{3(z^2+4z+6)}{z^3+6z^2+18z+25}$$

$$y_5 = -1 + \frac{3(z^2+6z+12)}{z^3+9z^2+36z+60}$$

$$y_8 = -1 + \frac{3(z^2+8z+20)}{z^3+12z^2+60z+120}$$

Thank You For Your Attention!

Some References

1. PAC, Special polynomials associated with rational solutions of the fifth Painlevé equation, *J. Comp. Appl. Math.*, **178** (2005) 111–129.
2. PAC and C. Dunning, Rational solutions of the fifth Painlevé equation. Generalised Laguerre polynomials, *Stud. Appl. Math.*, **152** (2024) 453–507.
3. PAC, C. Dunning and B. Mitchell, Discrete equations from Bäcklund transformations of the fifth Painlevé equation, arXiv:2602.09756.
4. PAC, A. Dzhamay, A.N.W. Hone and B. Mitchell, Special solutions of a discrete Painlevé equation for quantum minimal surfaces, *Theo. Math. Phys.*, **224** (2025) 1359–1397