Discrete Painlevé equations for maps and partitions

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Web-seminar on Painlevé Equations and related topics

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1 Discrete Painlevé equations

2 dPII and discrete probabilities in random partitions

3 dPI and counting planar maps

Outline

1. Discrete Painlevé equations
2. dPII and discrete probabilities in random partitions
3. dPI and counting planar maps

The first two discrete Painlevé equations

The first and second discrete Painlevé equations are the second order nonlinear discrete equations

$$\text{dPI:} \quad y_n(y_{n+1} + y_n + y_{n-1}) = c_1 + c_2 n + c_3 y_n$$

$$\text{dPII:} \quad (1 - y_n^2)(y_{n+1} + y_{n-1}) = (c_1 + c_2 n) y_n + c_3, \quad c_i \in \mathbb{C}$$

admitting as continuous limit (for different scaling limits) the classical first and second Painlevé equations

$$\text{PI:} \quad u''(t) = 6u^2(t) + t$$

$$\text{PII:} \quad u''(t) = 2u^3(t) + tu(t) + \alpha, \quad \alpha \in \mathbb{C}.$$

Some history

- [Picard, 1889 - Painlevé, 1900 - Fuchs, 1905 - Gambier, 1910] Painlevé equations are classified as 6 nonlinear second order ODEs which have the Painlevé property and generically do not admit solutions in terms of classical special functions.
- [Grammaticos - Ramani..., 1990] Classification of their discrete analogues using the singularity confinement property.
- [Okamoto, 1980 - Sakai, 2000] Algebraic geometry setting : classification in terms of affine Weyl group related to the study of certain rational surfaces.

Their main properties (for this talk)

[Cresswell - Joshi, 1998] Both the first and second discrete Painlevé equations can be extended to a hierarchy i.e. a sequence of higher order equations, as their continuous versions.

The k-th equation of each hierarchy is encoded by discrete Lax pairs i.e. linear systems of type

$$\Phi^{(k)}_{n+1}(\lambda) = L_n(\lambda; y_n)\,\Phi^{(k)}_n(\lambda)$$
$$\frac{\partial}{\partial\lambda}\Phi^{(k)}_n(\lambda) = M^{(k)}_n\left(\lambda; \{y_\ell\}_{\ell=n-k}^{n+k}\right)\Phi^{(k)}_n(\lambda)$$

where L_n, M_n are rational in λ (eventually matrix-valued) with coefficients depending on y_ℓ. The compatibility condition of this system

$$\frac{\partial}{\partial\lambda}L_n + L_n M^{(k)}_n - M^{(k)}_{n+1} L_n = 0$$

corresponds to the k-th equation of the hierarchy.

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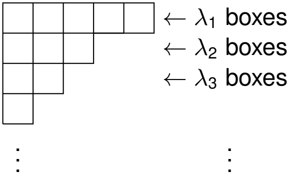
Random partitions models

For a given $M \in \mathbb{N}$ a partition of M is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_r > 0)$ with $\lambda_i \in \mathbb{N}^*$

$$\sum_{i=1}^{r} \lambda_i = M \; (= |\lambda|).$$

We can represent a given partition λ via the Young diagram of shape λ



And a standard Young tableau (SYT) of shape λ is obtained by filling in the boxes of the Young diagram of shape λ with numbers $1, \ldots, |\lambda|$ with increasing sequences in both directions $\rightarrow$ and $\downarrow$.

A random partition model is then given by the definition of a probability measure on the set of partitions.

The Poissonized Plancherel measure

The Plancherel measure is the measure on the set of partitions of M defined by

$$\mathbb{P}_{\mathrm{Pl.}}(\lambda) = \frac{F_\lambda^2}{M!}, \text{ with } F_\lambda = \#\{P \in \mathrm{SYT}_M,\ \mathrm{sh}(P) = \lambda\}.$$

Remark It is induced by the uniform measure on the symmetric group of M via the Robinson–Schensted correspondence

$$RS : \pi_M \ni S_M \to RS(\pi_M) \in \{(P, Q) \in \mathrm{SYT}_M \times \mathrm{SYT}_M,\ \mathrm{sh}(P) = \mathrm{sh}(Q)\}.$$

$$\downarrow$$

The Poissonized Plancherel measure instead consists of taking on the set of all partitions the measure

$$\mathbb{P}_{\mathrm{P.Pl.}}(\lambda) = \mathrm{e}^{-\theta^2} \left(\frac{\theta^{|\lambda|} F_\lambda}{|\lambda|!} \right)^2, \text{ where } |\lambda| = \mathrm{weight}(\lambda).$$

Distribution of first parts and Toeplitz determinants

In the Poissonized Plancherel model, the distributions of first parts are given by the Gessel's formula

$$\mathbb{P}_{\text{P.Pl.}}(\lambda_1 \leq k) = \mathrm{e}^{-\theta^2} D_{k-1}(\varphi)$$

where $D_k = D_k(\varphi)$ are Toeplitz determinants associated to the symbol $\varphi = \varphi[\theta](z) = \mathrm{e}^{w(z)}$ for $w(z) = v(z) + v(z^{-1})$ and $v(z) = \theta z, z \in S^1$. In particular

$$\boxed{D_k := \det(T_k(\varphi))}$$

with $T_k(\varphi)$ being the k-th Toeplitz matrix associated to the symbol $\varphi(z)$

$$T_k(\varphi)_{i,j} := \varphi_{i-j}, \quad i,j = 0, \dots, k$$

where for every $\ell \in \mathbb{Z}$, φ_ℓ is the ℓ-th Fourier coefficient of $\varphi(z)$.

Well-known results

- [Borodin, Adler - Van Moerbeke, Baik, 2000] For every $k \geq 1$ we have

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2$$

where x_k solves the so called discrete Painlevé II equation

$$\theta(x_{k+1} + x_{k-1})(1 - x_k^2) + kx_k = 0$$

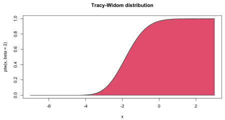
with initial conditions $x_0 = -1, x_1 = \varphi_1/\varphi_0$.

- [Baik - Deift - Johansson, 2000]

$$\lim_{\theta\to\infty} \mathbb{P}_{\text{P.Pl.}} \left(\frac{\lambda_1 - 2\theta}{\theta^{1/3}} \leq s \right) = F(s),$$

where $F(s)$ is the (GUE) Tracy-Widom distribution

$$\begin{cases} F(s) = \exp\left(-\int_s^{+\infty} (r-s) u^2(r) dr \right), \text{with} \\ u''(s) = su(s) + 2u^3(s), \quad u(s) \sim_{s\to\infty} \mathrm{Ai}(s). \end{cases}$$



Generalizations to the (n-minimal multicritical) Schur case

Consider now Toeplitz determinants $D_k(\varphi) = \det_{i,j=0}^{k} \varphi_{i-j}$ but for different symbol $\varphi[\theta_1, \dots, \theta_n](z) = \mathrm{e}^{w(z)}$ for $w(z) = v(z) + v(z^{-1})$ and $v(z) = \theta_1 z + \theta_2 z^2 + \cdots + \theta_n z^n$.

[Chouteau - T., 2023]

For any $n \geq 1$, the Toeplitz determinants $D_k = D_k(\varphi)$, $k \geq 1$, satisfy

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2$$

where x_k is the solution of the n-th equation of the discrete Painlevé II hierarchy.

Remark Relation with Schur measures studied in [Betea–Bouttier–Walsh, 2021]

1. Random partitions with n-minimal multicritical Schur measures have distributions of first part given by these $D_k(\varphi)$ where

$$\theta_r = \theta\gamma_r, r = 1, 2, \dots, n, \quad \gamma_r = \frac{1}{r}\binom{2n}{n+r} \Big/ \binom{2n}{n-1}.$$

2. The continuous limit of this recurrence formula, in this multicritical regime, corresponds to their results for the limiting behavior of the distribution of first part in terms of the higher order (GUE) Tracy-Widom distributions.

The first equations

$$n = 1: \quad kx_k + \theta_1(x_{k+1} + x_{k-1})(1 - x_k^2) = 0, \leftarrow$$ discrete Painlevé II equation

$$n = 2: \quad kx_k + \theta_1(1 - x_k^2)\,(x_{k+1} + x_{k-1})$$
$$+ \theta_2(1 - x_k^2)\left(x_{k+2}(1 - x_{k+1}^2) + x_{k-2}(1 - x_{k-1}^2) - x_k(x_{k+1} + x_{k-1})^2\right) = 0,$$

$$n = 3: \quad kx_k + \theta_1(1 - x_k^2)\,(x_{k+1} + x_{k-1})$$
$$+ \theta_2(1 - x_k^2)\left(x_{k+2}(1 - x_{k+1}^2) + x_{k-2}(1 - x_{k-1}^2) - x_k(x_{k+1} + x_{k-1})^2\right)$$
$$+ \theta_3(1 - x_k^2)\left(x_k^2(x_{k+1} + x_{k-1})^3 + x_{k+3}(1 - x_{k+2}^2)(1 - x_{k+1}^2) + x_{k-3}(1 - x_{k-2}^2)(1 -$$
$$+ \theta_3(1 - x_k^2)\left(-2x_k(x_{k+1} + x_{k-1})(x_{k+2}(1 - x_{k+1}^2) + x_{k-2}(1 - x_{k-1}^2))\right)$$
$$+ \theta_3(1 - x_k^2)\left(-x_{k-1}x_{k-2}^2(1 - x_{k-1}^2) - x_{k+1}x_{k+2}^2(1 - x_{k+1}^2)\right)$$
$$+ \theta_3(1 - x_k^2)\,(-x_{k+1}x_{k-1}(x_{k+1} + x_{k-1})) = 0.$$

Remark Previously appeared in [Periwal-Schewitz, 1990] on unitary matrix integrals.

Idea of the proof: OPUC

1. We consider the family $\{p_k(z) = \kappa_k z^k + \dots\}_{k\in\mathbb{N}}$, $\kappa_k > 0$ of orthonormal polynomials on the unit circle (OPUC)

$$\int_{-\pi}^{\pi} p_n(\mathrm{e}^{i\alpha})\overline{p_m(\mathrm{e}^{i\alpha})}\varphi(\mathrm{e}^{i\alpha})\frac{\mathrm{d}\alpha}{2\pi} = \delta_{n,m}.$$

2. The Toeplitz determinants $D_k = D_k(\varphi)$ are related to these OPUC by

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2$$

   where $x_k = \frac{1}{\kappa_k} p_k(0)$.

3. From the solution of the Riemann–Hilbert problem associated to this OPUC family, we construct a Lax pair of the discrete Painlevé II hierarchy for x_k.

4. This Lax pair is mapped into the original Lax pair obtained by Cresswell and Joshi in 1998 which first introduced the discrete Painlevé II hierarchy.

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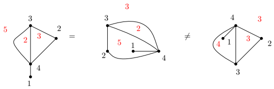
Planar maps



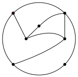
A planar map is a connected graph embedded in the sphere, so that

$$\#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\} = 2,$$

considered up to continuous deformation.

For each face its degree is the number of edges adjacent to it and for each vertex it is the number of edges touching it.

A quadrangulation is a map in which all faces have degree 4 and a 4-valent map is a map in which all vertices have valency 4.



A bipartite map is a map in which all faces have even degree.

A pointed map has a distinguished vertex while a rooted map has a distiguished oriented edge.

Counting problems [Tutte (1963)] The number of rooted planar maps with n edges is

$$\frac{2}{n+2} 3^n \mathrm{cat}_n, \quad \text{with} \quad \mathrm{cat}_n = \frac{(2n)!}{n!(n+1)!}.$$

The matrix integrals approach

[Brézin - Itzykson - Parisi - Zuber, (1978)] , [Bessis - Itzykson -Zuber (1980)] . . .
[Bleher - Garlakhoo - McLaughlin (2022)] Over the space of $N \times N$ Hermitian matrices study the large N asymptotic expansion of deformed Gaussian integrals

$$Z_N(g) = \int \exp\left(-\frac{\operatorname{tr} M^2}{2} - \frac{g}{N}\operatorname{tr} M^4\right) dM, \quad dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d(ReM_{ij})d(ImM_{ij})$$

with $Z_N(0)$ being the GUE partition function.

They realized that the perturbative expansion (in powers of g) of the free energy

$$E_N(g) = -\frac{1}{N^2}\ln\frac{Z_N(g)}{Z_N(0)}$$

will have the asymptotic expansion

$$E_N(g) = \sum_{H\geq 0} e_H(g) N^{-2H},$$

$e_H(g)$ the generating series of $\#\{$4-valent maps embedded in a genus H surface$\}$.

Connection with orthogonal polynomials on the real line and dPI

[Bessis - Itzykson - Zuber (1980)] The planar case $e_0(g) = \lim_{N\to\infty} E_N(g)$ was studied by using the expression

$$E_N(g) = -\frac{1}{N}\sum_{k=1}^{N}\left(1-\frac{k}{N}\right)\ln\frac{r_k(g)}{r_k(0)} - \frac{1}{N}\ln\frac{h_0(g)}{h_0(0)}$$

in terms recurrence coefficients of orthogonal polynomials on the real line w.r.t. the measure $d\mu(\lambda) = \mathrm{e}^{-\frac{\lambda^2}{2}-\frac{g}{N}\lambda^4}d\lambda$. They are defined by the orthonormality relation

$$\int_{\mathbb{R}} p_n(\lambda)p_m(\lambda)d\mu(\lambda) = \delta_{n,m},$$

where $p_n(x) = \gamma_n x^n + \gamma_{n,n-1}x^{n-1} + \cdots + \gamma_{n,0}$ and $\gamma_n > 0$. The corresponding monic orthogonal polynomials are denoted by $\pi_n(x) = \gamma_n^{-1}p_n(x)$.

- the orthogonal polynomials $\pi_\ell(x) = \pi_\ell(x;g)$ satisfy a 3-terms recurrence relation with coefficients the $r_\ell = r_\ell(g)$, s. t. $x\pi_\ell(x) = \pi_{\ell+1}(x) + r_\ell\pi_{\ell-1}(x)$;
- The $r_k(g)$ satisfied the discrete Painlevé I equation

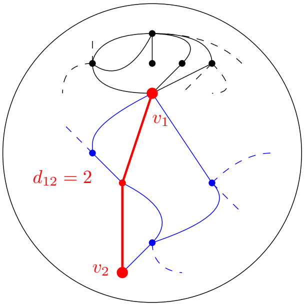
$$k = r_k(g) + \frac{4g}{N}r_k(g)(r_{k+1}(g) + r_k(g) + r_{k-1}(g)).$$

The two-point function

The graph distance between two vertices v_1, v_2 in a planar map is the minimal number of edges which separates v_1 from v_2.

The question is

Counting the number of planar maps with given faces and having two marked vertices at a given distance $d_{12} \in \mathbb{N}$.



The 2-point function is the generating function for maps with two marked points at given distance (ℓ) $\rightsquigarrow$ the integrated two-point function is the generating function R_ℓ for maps with two marked points at distance at most the given one.

Remark It describes the probability distribution of distances between two uniformly chosen random points of uniform random planar maps.

Autonomous dPI and the two-point function for planar quadrangulations

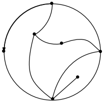
[Bouttier - Di Francesco - Guitter (2003)] By using bijective combinatorics methods, it is shown that R_ℓ, in the case of quadrangulations, satisfies the recursion

$$R_\ell = 1 + gR_\ell(R_{\ell+1} + R_\ell + R_{\ell-1}), \ \ell \geq 1, \quad R_0 = 0.$$

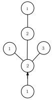
with $\lim_{\ell \to \infty} R_\ell = R$ and $3gR^2 + 1 = R$ and $g = g_4$ being the counting parameter for the faces of the quadrangulations.

In the case of bipartite planar maps with bounded face degrees, other (even) higher order equations are satisfied by R_ℓ, corresponding to (autonomous versions of the) discrete Painlevé I hierarchy.

Remark Based on the Bouttier-Di Francesco-Guitter bijection generalizing the bijection between (rooted) planar quadrangulations with n faces and (rooted) well-labeled trees with n edges by [Cori-Vauquelin (1981), Schaeffer (1998)].



$\leftrightarrow$



Two-point function characterization in terms of Hankel determinants

[Bouttier - Guitter (2010)] R_ℓ is written as

$$R_\ell = \frac{D_\ell D_{\ell-2}}{D_{\ell-1}^2}$$

where D_ℓ are the Hankel determinants $D_\ell = \det_{i,j=0}^{\ell} F_{i+j}$. Their entries, in the case of quadrangulations, are computed as

$$F_n = \begin{cases} R(1-2Rg)\mathrm{cat}_k R^k - Rg\mathrm{cat}_{k+1} R^{k+1}, & n = 2k, \\ 0, & \text{otherwise.} \end{cases}$$

In the case of bipartite planar maps, the result is the same but the formula for F_ℓ is more convoluted. An analogue result is also proved for general planar maps.

Remark [Bergère - Eynard - Guitter - Oukassi (2023)] recently generalized this type of result in the case of hypermaps.

F_n, in the bounded bipartite case, are moments w.r.t. the measure

$$\boxed{\mu^{(R,2N)}(x)dx = \frac{1}{2\pi}\sqrt{4R - x^2}P_{2N}(x) \quad -2\sqrt{R} \le x \le 2\sqrt{R}}$$

where $P_{2N}(x)$ is an even polynomial of degree $2N$ with coefficients depending on R and the counting parameters $g = g_4, g_6, \ldots, g_{2N+2}$.

Connection with the recurrence coefficients of orthogonal polynomials

Consider the normalized measure on $[-1,1]$ given by

$$\rho^{(R,2N)}(x)dx = \underbrace{\frac{2}{\pi}\sqrt{1-x^2}}_{\text{Wigner semicircle law}} \widetilde{P}_{2N}(x)dx,$$

$\widetilde{P}_{2N}(x) = \prod_{i=1}^{N}(t_i - x)(t_i + x)$, t_i depending on R and the counting parameters $g = g_4, g_6, \dots g_{2N+2}$.

The family of orthogonal polynomials associated to $\rho^{(R,2N)}(x)dx$, satisfies a 3-terms recurrence relation of type $x\pi_\ell(x) = \pi_{\ell+1}(x) + r_\ell \pi_{\ell-1}(x)$ where

$$r_\ell = \frac{H_{\ell-2}H_\ell}{H_{\ell-1}^2}, \quad H_\ell = \det_{i,j=0}^{\ell} m_{i+j}^{(\rho^{(R,2N)})}.$$

[Bouttier - T., 2026+]

The two-point function for planar bipartite maps with bounded face degrees (order $2N+2$) R_ℓ corresponds to the recurrence coefficient r_ℓ via

$$R_\ell = 4Rr_\ell.$$

Remark An analogue statement holds in the non-bipartite case.

The case of quadrangulations

In the case of quadrangulations, one see that F_ℓ are the moments for

$$\mu^{(R,4)}(x) = \frac{1}{2\pi}\sqrt{4R - x^2}\left(1 - 2gR - gx^2\right), -2\sqrt{R} \le x \le 2\sqrt{R}.$$

Thus the two-point function R_ℓ can be identified to the coefficient r_ℓ of the 3-terms recurrence relation for the OP w.r.t. the measure

$$\rho^{(R,4)}(x) = \frac{2}{\pi}(1 - x^2)^{1/2}(t^2 - x^2),\ t^2 = \frac{R+2}{4(R-1)}$$

via $R_\ell = 4Rr_\ell$.

Main consequences

I. Analytical (re)proof of a determinantal formula for R_ℓ in terms of the Tchebychev polynomials of second kind $U_n(x)$ (associated to the measure $\frac{2}{\pi}(1-x^2)^{1/2}dx$) and thus in terms of the solutions of the so called *characteristic equation*.

II. Analytical (re)proof of the discrete equation satisfied by R_ℓ, in the case of quadrangulations

$$R_\ell = 1 + gR_\ell(R_{\ell+1} + R_\ell + R_{\ell-1}),$$

seeing it as orthogonality conditions for the family of orthogonal polynomials $\pi_n(x)$ w.r.t. $\rho^{(R,2N)}(x)dx$.

A (useful) formula from Christoffel

Let $\pi_n, n \geq 0$ be the monic OP w.r.t. a measure $d\mu(x) = \mu(x)dx$ with compact support on $\mathbb{R}$.

Now consider the following deformation of the measure

$$d\mu(x)dx \to d\mu^{(\ell)}(x) = \underbrace{c(t_1 - x)(t_2 - x)\dots(t_\ell - x)}_{T(x)}\mu(x)dx$$

for given $t_i, c \neq 0$.

Then $\pi_n^{(\ell)}, n \geq 0$ the monic OP w.r.t. $d\mu^{(\ell)}(x)$ are expressed as

$$\pi_n^{(\ell)}(x) = \frac{1}{T(x)} \frac{\begin{vmatrix} \pi_n(t_1) & \dots & \pi_{n+\ell}(t_1) \\ \vdots & & \vdots \\ \pi_n(t_\ell) & \dots & \pi_{n+\ell}(t_\ell) \\ \pi_n(x) & \dots & \pi_{n+\ell}(x) \end{vmatrix}}{\begin{vmatrix} \pi_n(t_1) & \dots & \pi_{n+\ell-1}(t_1) \\ \vdots & & \vdots \\ \pi_n(t_\ell) & \dots & \pi_{n+\ell-1}(t_\ell) \end{vmatrix}}.$$

I. Determinantal formula involving Tchebychev polynomials

$$R_n = R\frac{\det\limits_{i,j=1\dots N}\left(U_{n+2j}(t_i)\right)\det\limits_{i,j=1\dots N}\left(U_{n+2j-3}(t_i)\right)}{\det\limits_{i,j=1\dots N}\left(U_{n+2j-2}(t_i)\right)\det\limits_{i,j=1\dots N}\left(U_{n+2j-1}(t_i)\right)},$$

where $U_k(\cdot)$ are the Tchebychev polynomials of the second kind and the t_i are the roots of the polynomial factor in the measure $\mu^{(R,2N)}(x)dx$.

In particular we can recover the result of [Bouttier - Guitter (2010)]

$$R_n = R\,\frac{\det\limits_{i,j=1\dots N}\left(y_i^{\frac{n}{2}+j+\frac{1}{2}} - y_i^{-(\frac{n}{2}+j+\frac{1}{2})}\right)\det\limits_{i,j=1\dots N}\left(y_i^{\frac{n}{2}+j-1} - y_i^{-(\frac{n}{2}+j-1)}\right)}{\det\limits_{i,j=1\dots N}\left(y_i^{\frac{n}{2}+j-\frac{1}{2}} - y_i^{-(\frac{n}{2}+j-\frac{1}{2})}\right)\det\limits_{i,j=1\dots N}\left(y_i^{\frac{n}{2}+j} - y_i^{-(\frac{n}{2}+j)}\right)}$$

where

$$t_i = \frac{x_i + x_i^{-1}}{2}, i = 1, \dots, N, y_i = x_i^2$$

for x_i solutions of the characteristic equation.

The case of planar quadrangulations

$d\mu^{(2)}(x)x = \frac{2}{\pi}(1-x^2)^{1/2}(t^2-x^2)$, so the Christoffel formula gives

$$\pi_n^{(2)}(x) = \frac{1}{t^2-x^2}\frac{\pi_n(x)\pi_{n+2}(t)-\pi_{n+2}(x)\pi_n(t)}{\pi_n(t)}, \quad \text{where} \quad \pi_n(x) = 2^{-n}U_n(x).$$

- Plugging itin the 3 terms recurrence relation for the $\pi_n^{(2)}$, and looking in the basis π_n

$$R_n = 4Rr_n = 4R\underbrace{r_n^{(\text{Tch.})}}_{=\frac{1}{4}}\frac{U_{n+2}(t)U_{n-1}(t)}{U_n(t)U_{n+1}(t)} = R\frac{U_{n+2}(t)U_{n-1}(t)}{U_n(t)U_{n+1}(t)}.$$

- Property of Tchevychev polynomials of the second kind

$$U_k\left(\frac{\lambda+\lambda^{-1}}{2}\right) = \frac{\lambda^{k+1}-\lambda^{-k-1}}{\lambda-\lambda^{-1}}, \ \lambda \neq 0, k \geq 0.$$

- Identification of t with $(x+x^{-1})/2$, for x solving the characteristic equation $(y+y^{-1})+1 = (gR^2)^{-1}$ and $y = x^2$.
- The expression for R_n recovers exactly

$$R_n = R\frac{U_{n+2}(t)U_{n-1}(t)}{U_n(t)U_{n+1}(t)} = R\frac{(1-y^n)(1-y^{n+3})}{(1-y^{n+2})(1-y^{n+1})}.$$

II. The discrete equations as orthogonality conditions

Let $\widetilde{T}(x)$ be the polynomial part of the expansion of $T(x)x\sqrt{1-x^{-2}}$ for $x \to \infty$. Then, for any n, the following identities hold

$$\left(\widetilde{T}(x)\pi_n(x), \pi_n(x)\right)_{d\mu^{(\ell)}} = 0,$$
$$\left(\widetilde{T}(x)\pi_n(x), \pi_{n-1}(x)\right)_{d\mu^{(\ell)}} = \frac{h_0 h_{n-1}}{4}$$

where $(\cdot,\cdot)_{d\mu^{(\ell)}}$ denotes the scalar product induced by the measure and h_k the normalization constants.
They are equivalent to the discrete equations satisfied by R_n.

Remark Their proof relies on the fact that the integrals can be rewritten as contour integrals and by the properties of the secondary orthogonal polynomials.

The case of quadrangulations

For $T(x) = t^2 - x^2$ even, we find $\widetilde{T}(x) = (t^2 + \frac{1}{2})x - x^3$ and $h_0 = t^2 - \frac{1}{4}$, then the system

$$\int_{-1}^{1} ((t^2 + \frac{1}{2})x - x^3)\pi_n(x)\pi_{n-1}(x)d\mu^{(2)}(x) = 0$$
$$\int_{-1}^{1} ((t^2 + \frac{1}{2})x - x^3)\pi_n^2(x)\mathrm{d}\mu^{(2)}(x) = \frac{h_n}{4}(t^2 - \frac{1}{4})$$

after applying three times the 3-terms recurrence relation gives

$$(t^2 + \frac{1}{2})r_n - r_n(r_{n-1} + r_n + r_{n+1}) = \frac{4t^2 - 1}{16}$$

that after taking $R_n = 4Rr_n$ and using the relation $R = 1 + 3gR^2$, this equation coincides with

$$R_\ell = 1 + gR_\ell(R_{\ell+1} + R_\ell + R_{\ell-1}).$$

Conclusion

1. We can compute the distributions of first parts of multicritical random partitions via solutions of the discrete Painlevé II hierarchy.
2. We can count bipartite planar maps with fixed graph distance between two vertices with an autonomous version of the discrete Painlevé I hierarchy.
3. Orthogonal polynomials are always behind the scene.

Thank you!