No Small forcing adds extendible cardinals

1. Background:

A notion LC of large cardinals (measurable, super-compact, $C^{(n)}$-supercompact, super $C^{(n)}$-supercompact ... etc) is regular if (1) all LC-cardinals are inaccessible (2) if λ is not LC and $|\mathbb{P}| < \lambda$ then $\Vdash_{\mathbb{P}} \lambda$ is not LC.

For $n \in \mathbb{N}$

$$C^{(n)} := \{ \kappa \in On : V_\kappa \prec_{\Sigma_n} V \}$$

$C^{(0)} = On$, $C^{(1)} = \{\kappa \in Cn : \kappa > \omega, V_\kappa = H(\kappa)\}$

$= \{\kappa \in Cn : \kappa > \omega$, κ is a strong limit $\}$

(Bagaria 2012, "$C^{(n)}$-cardinals")

Ex 1 For each $n \in \mathbb{N}$, "$C^{(n)}$ and inaccessible" is a regular notion of large cardinals.

Proposition 3 Suppose that $(\mathcal{P}, \phi)$-R_cA^* holds for an attenuating iterable class $\mathcal{P}$ of posets. If LC is a regular notion of large cardinals and LC-cardinal exists, then there are class many LC cardinals.

A class $\mathcal{P}$ of posets is called *iterable* if $\{\mathbb{1}\} \in \mathcal{P}$; $\mathcal{P}$ is closed wrt forcing equivalence; taking restrictions (if $\mathbb{P} \in \mathcal{P}$ then $\mathbb{P} \restriction p \in \mathcal{P}$); and for any $\mathbb{P} \in \mathcal{P}$ and $\dot{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \mathcal{P}$ then $\mathbb{P} * \dot{\mathbb{Q}} \in \mathcal{P}$

$\mathcal{P}$ is *attenuating* if for any $\lambda_0 \in Cn$ there is $\lambda \geq \lambda_0$, with $\lambda \geq |\mathbb{P}|$ (and $\mathbb{P} \in \mathcal{P}$) s.t. $\Vdash_{\mathbb{P}}$ "no card $\mu \leq \lambda$ is inaccessible"

<u>Ex.2</u> (1) $\mathcal{P}$ = all c.c.c. posets: is attenuating.

(2) $\mathcal{P}$ = all σ-closed posets : is attenuating.

RcA^* is a variant of Recurrence Axiom introduced in S.F., and T.Usuba (2025)[1] which is a variant of Maximality Principle of Joel Hamkins.

For an iterable $\mathcal{P}$ and a set S,

$(\mathcal{P}, S)$-RcA^{*+} : For any $\mathcal{L}_\in$-formula $\varphi = \varphi(\bar{x})$ and $\bar{a} \in S$, if there is a poset $\mathbb{P} \in \mathcal{P}$ s.t. $\Vdash_{\mathbb{P}} \varphi(\bar{a})$, there is a ground W of V s.t. $\bar{a} \in W$ and $W \models \varphi(\bar{a})$ and the machine to return to V has the size s.t. there is no inaccessible cardinal below it in V.

$\mathcal{P}$-

If W is a ground of V and $\mathbb{R} \in W$ and $\mathbb{H}$ are s.t. $\mathbb{H}$ is $(W, \mathbb{R})$-generic and $V = W[\mathbb{H}]$, then we call $\mathbb{P}$ (and/or $\mathbb{H}$) the machine to return to V (from W).

Proof We work in ZFC + $(\mathcal{P},h)$-RcA* for attenuating iterable $\mathcal{P}$.

Assume that the proposition does not hold. Then there is some regular notion LC of large cardinals s.t. there are LC cardinals but they are only set many.

Let λ_0 be the supremum of these LC-cardinals.

Let $\mathbb{P} \in \underbrace{\mathcal{P} \text{ be}}_{(\text{and } \lambda)}$ as in the def. of attenuating-ness for this λ_0.

Since LC is a regular notion of large cardinals

$\Vdash_{\mathbb{P}}$ there is no LC-cardinal.

Thus by $(\mathbb{P},\phi)$-R_cA^* there is a ground W of V s.t. $V = W[\mathbb{H}]$ for some $\mathbb{Q} \in W$ and $(W,\mathbb{Q})$-gen. $\mathbb{H}$ with $|W| = \lambda$ s.t.

$V \models$ there is no inaccessible $\leq \lambda$. Thus,

again by regularity of LC,

$V \models$ there is LC cardinal.

and $W \models$ there is no LC cardinal

$\lightning$ $\square$

Proposition 3 is a substitute of the following theorem in the small large cardinals setting (Note that the assumption of Proposition 3 for a transfinitely iterable stationary preserving attenuating $\mathcal{P}$ can be forced starting from L with mahlo cardinals):

Theorem 3A. (1) Suppose that $\mathcal{P}$ is a iterable class of posest and the super-$C^{(\infty)}$-$\mathcal{P}$ -LgLCA for huperhuge holds. Then $(\mathcal{P},\mathcal{H}(\text{K\_refl}))$-RcA^{*+} holds, and for any $n \in \mathbb{N}$, there are stationarily many super-$C^{(n)}$-hyperhuge cardinals.

(2) The super-$C^{(\infty)}$-LgLCAA implies $(\text{all},\mathcal{H}(\aleph_1))$-RcA^{*+}, and that, for any $n \in \mathbb{N}$, there are stationarily many super-$C^{(n)}$-hyperhuge cardinals.

2. Hamkins' Theorem on "adding no large cardinal in an extension"

"MAIN THEOREM 3" in: Joel Hamkims,

Extensions with the approximation and cover properties have no new large cardinals,

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By the theorem, we see immediately that measurable, supercompact, $C^{(n)}$-supercompact, superhuge etc. are regular.

This was known earlier and called Lévy-Solovay theorem

A cardinal κ is extendible if (3) for any $\alpha > \kappa$
There are $\beta \in On$ and $j : V_\alpha \xrightarrow[\kappa]{\prec} V_\beta$ s.t.
(4) $j(\kappa) > \alpha$ (see Kanamori)

κ is Jech extendible if (3)
(but not necessarily (4)) holds

Jech extenibility is proved to be equivalent to extendibility via Kunen's Inconsistency Theorem.

Hamkins' "Main Theorem 3" cannot be applied to these characterizations of extendible cardinals.

The following Theorem gives a characterization to which "Main Theorm 3" can be applied to conclude that extendibility is a regular notion of large cardinals.

[redacted]

3. A (possibly new, but probably folklore) characterization of extendible cardinals

Theorem 4 TFAE: (a) κ is extendible.

(b) κ is Jech-extendible.

(a') For all $\lambda > \kappa$ there are $j, M \subseteq V$ s.t. $j : V \xrightarrow{\prec}_{\kappa} M$

$$j(\kappa) > \lambda, \quad \underline{{}^{\kappa}M \subseteq M}, \quad V_{j(\lambda)} \in M.$$

(b') Like (a') but without "$j(\kappa) > \lambda$".

Hankin's MAIN THEOREM 3 is applicable to (a$'$) of Theorem 4 to show that extendibility is a regular notion of large cardinals.

Sketch of (a) $\Rightarrow$ (a$'$)

For this implication, it is enough to prove the following two lemmas:

Lemma 5 For a cardinal θ and $j_0 : \mathcal{H}(\theta) \xrightarrow{\prec} N$ for a transitive N, let $N_0 := \bigcup j_0{}'' \mathcal{H}(\theta)$. Then

$$j_0 : \mathcal{H}(\theta) \xrightarrow{\prec} N_0 \text{ and } j_0 \text{ is cofinal in } N_0$$

Lemma 8 (A modification of Lemma 7 in Fuchino, Sakai preprint) For any inaccessible θ and any cofinal $j_0 : \mathcal{H}(\theta) \xrightarrow{\prec}_{\kappa} N$ with (5) ${}^{\kappa>}N \subseteq N$, there are $j, M \subseteq V$ s.t. $j : V \xrightarrow{\prec}_{\kappa} M$, (6) ${}^{\kappa>}M \subseteq M$ and $j_0 \subseteq j$.

It is known that if κ is extendible then there are class many measurable cardinals (hence there are class many inaccessible cardinals!)

A sketch of the proof of Lemma 6

Let $\left|\begin{array}{l} \text{We have: } j_0 : H(\theta) \xrightarrow{\prec} N \\ (*) \quad {}^{\kappa >}N \subseteq N \end{array}\right.$

$\mathcal{F} := \{ f \in V : f : \mathrm{dom}(f) \to V,\ \mathrm{dom}(f) \in H(\theta)^{Y} \}$

$\Pi := \{ \langle f, a \rangle : f \in \mathcal{F},\ a \in j_0(\mathrm{dom}(f)) \}$

Note that, for $\langle f, a \rangle \in \Pi$ then $a \in N$

For $\langle f, a\rangle, \langle g, b\rangle \in \Pi$ let

$$\langle f, a\rangle \sim \langle g, b\rangle :\Longleftrightarrow \langle a, b\rangle \in j_0\left(S_{f(x)=g(y)}\right)$$

where $S_{f(x)=g(y)} := \{\langle x, y\rangle \in \mathrm{dom}(f) \times \mathrm{dom}(g) : f(x) = g(y)\}$

$$\langle f, a\rangle \, E \, \langle g, b\rangle :\Longleftrightarrow \langle a, b\rangle \in j_0\left(S_{f(x)\in g(y)}\right)$$

where $S_{f(x)=g(y)} := \{\langle x, y\rangle \in \mathrm{dom}(f) \times \mathrm{dom}(g) : f(x) \in g(y)\}$

<u>Claim 1</u> (1) $\sim$ is an equivalence relation of Π

(2) $\sim$ is congruent to E.

By Claim 1 we may consider $\prod/_\sim$ (with Scott's trick)

we denote $\langle f, a\rangle/_\sim$ for the class of $\langle f, a\rangle$ and we simply write $\langle f, a\rangle/_\sim \; E \; \langle g, b\rangle$ if $\langle f, a\rangle E \langle g, b\rangle$

Claim 2 (Łoś's Lemma) For any $\mathcal{L}_\in$-formula $\varphi = \varphi(x_0, \ldots, x_{n-1})$ and $\langle f_0, a_0\rangle, \ldots \langle f_{n-1}, a_{n-1}\rangle \in \prod$, we have

$\langle \prod/_\sim, E\rangle \models \varphi(\langle f_0, a_0\rangle/_\sim, \ldots, \langle f_{n-1}, a_{n-1}\rangle/_\sim)$

$\iff (a_0, \ldots, a_n) \in j_0(S_\varphi)$

where

$S_\varphi = \{\langle u_0, \ldots u_{n-1}\rangle \in \mathrm{dom}(f_0) \times \cdots : V \models \varphi(f_0(u_0), \ldots)\}$

For $u \in V$ let $f_u : \underset{\{0\}}{\overset{\|}{1}} \to V$ be s.t.

$f_u(\phi) = u$.

$i : V \longrightarrow \Pi/\sim \; ; \; u \longmapsto \langle f_u \, \phi \rangle/\sim$

<u>Claim 3</u>

i is an elementary embedding of $\langle V, \in \rangle$ into $\langle \Pi/\sim, E \rangle$

<u>Claim 4</u> (1) E well-founded

(2) E is set like

By Claims 3, 4 there is a Mostowski collapse

$$m : \langle \Pi/\mathcal{U}, E\rangle \longrightarrow \langle V, \in\rangle$$

Let $M := m''\Pi/\mathcal{U}$ and $j = m \circ i$

$$\begin{tikzcd}
(V, \in) \arrow[r, "i", "\prec"'] \arrow[rr, bend left, "j"] & (\Pi/\mathcal{U}, E) \arrow[r, "m", "\cong"'] & (M, \in)
\end{tikzcd}$$

<u>Claim 5</u> $j \restriction H(\theta)^V = j_0$ in particular,

$$j : V \xrightarrow[\kappa]{\prec} M .$$

M satisfies (6):

Let $\langle f_\xi, a_\xi \rangle \in \Pi$, $\xi < \delta$ for some $\delta < \kappa$

We want to prove is that there is $\langle f, a \rangle \in \Pi$

which represents the sequence $\langle f_\xi\ a_\xi \rangle$, $\xi < \delta$.

Let

$$f : \prod_{\xi<\delta} \mathrm{dom}(f_\xi) \longrightarrow V \quad \langle u_\xi : \xi < \delta \rangle \longmapsto \langle f_\xi(u_\xi) : \xi < \delta \rangle$$

and $a = \langle a_\xi : \xi < \delta \rangle$

Since θ is inaccessible and (5) $\prod_{\xi<\delta} \mathrm{dom}(f_\xi) \in H(\theta)$ $a \in N$ and hence $\langle f, a \rangle \in \Pi$

By Łoś's Lemma this $\langle f, a \rangle$ is as desired. $\boxtimes$

To Claim 4, (1)

Suppose not. Let $\langle f_n, b_n \rangle \in \Pi$ $n \in \omega$ be

s.t.

$\langle f_0, b_0 \rangle \ni \langle f_1, b_1 \rangle \ni \langle f_2, b_2 \rangle \ni \dots$

By def. E this means

$$\langle b_{n+1}, b_n \rangle \in j_0\left(\mathbb{S}_{f_{n+1}(x) \in f_n(y)} \right)$$

Let $Q = \{ \langle n, u \rangle : n \in \omega ;\ u \in \mathrm{dom}(f_n) \}$

$\langle n, u \rangle \sqsubset \langle n', u' \rangle : \Longleftrightarrow n' = n+1 \quad f_{n'}(u') \in f_n(u)$

Then $\langle b_i : i < \omega \rangle$ is a descending sequence

in $\langle j_0(Q), j_0(\sqsubset) \rangle$

Since well-foundedness is Δ_1

$N \models j_0(\langle Q, \sqsubset \rangle)$ is non well founded

$\left(j_0 : \mathcal{H}(\theta) \xrightarrow[\kappa]{\prec} N \right)$ So

$\mathcal{H}(\theta) \models \langle Q, \sqsubset \rangle$ is non-well founded.

But a witness of non well foundedness generates an

infinite descending $\in$ chain.