

Eventual and Strong Eventual Notions in Iterated Announcements

Eiji Yamada

Department of Mathematical and Computing Science
Institute of Science Tokyo
MLG 2026

October 24, 2026

Contents

1 Introduction

2 Basic definitions

3 Eventual and strong eventual notions

4 Conclusion and future work

Contents

1 Introduction

2 Basic definitions

3 Eventual and strong eventual notions

4 Conclusion and future work

My research

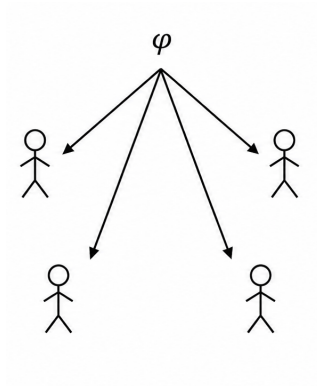
My research is the mathematical study of knowledge and belief in a group of agents.

Related fields are mathematical logic, linguistics, game theory, multi-agent systems, and artificial intelligence.

My current research fields are in particular **epistemic logic**, **dynamic epistemic logic**, and **epistemic game theory**.

Today's talk

Public announcements are one of the main topics in dynamic epistemic logic. A public announcement of φ is an announcement to a group of agents that φ holds.



Today's talk

The four notions of **success**, **self-refutation**, **true lies**, and **impossible lies** have been discussed in the context of public announcements ¹.

Successful	$\models \varphi \rightarrow [\uparrow \varphi] \varphi$
Self-refuting	$\models \varphi \rightarrow [\uparrow \varphi] \neg \varphi$
True lie	$\models \neg \varphi \rightarrow [\uparrow \varphi] \varphi$
Impossible lie	$\models \neg \varphi \rightarrow [\uparrow \varphi] \neg \varphi$

Here, $[\uparrow \varphi] \psi$ means “after announcing φ , ψ holds.”

¹Holliday and Icard [2010], Ågotnes et al. [2018]

Today's talk

In this talk, we introduce **eventual** and **strong eventual** versions of the four notions, as well as their **transifnite** versions. We also introduce the notions of **always informativeness when true/false**².

We have two main results on the logical relationship among eventual and strong eventual notions.

²Eventual self-refutation and always informativeness (when true) were introduced by Holliday and Icard [2010].

Contents

1 Introduction

2 Basic definitions

3 Eventual and strong eventual notions

4 Conclusion and future work

Believed public announcement logic

Let G be a nonempty finite set of agents and $\mathbf{Prop}$ be a countably infinite set of proposition letters.

Definition 1

Define **formulas** in the multi-agent epistemic logic $\mathcal{L}$ by

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i\varphi \quad (i \in G, p \in \mathbf{Prop})$$

Definition 2 ([Ågotnes et al., 2018])

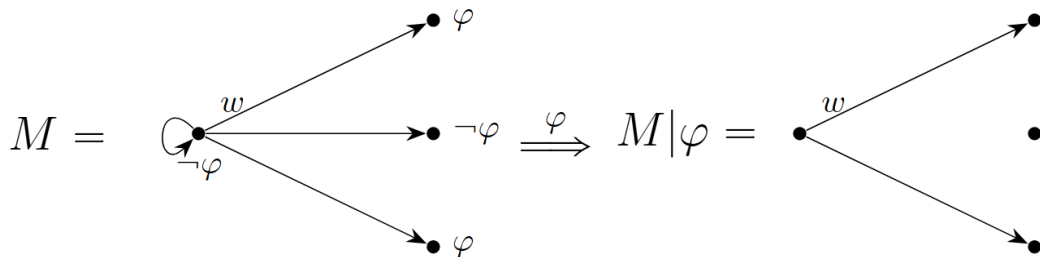
Define **formulas** in believed public announcement logic $\mathcal{L}_{\text{BPAL}}$ by

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i\varphi \mid [\uparrow\varphi]\psi \quad (i \in G, p \in \mathbf{Prop})$$

Believed public announcement logic

For a model $M = (W, \{R_i\}_{i \in G}, V)$, let $\llbracket \varphi \rrbracket_M := \{w \in W : M, w \models \varphi\}$ be the set of states in which φ holds.

Define the updated model under the believed public announcement of φ as the model $M|\varphi = (W, \{R_i|\varphi\}_{i \in G}, V)$ where $R_i|\varphi = R_i \cap (W \times \llbracket \varphi \rrbracket_M)$.



Believed public announcement logic

Define truth as follows:

- $M, w \models \Box_i \varphi \iff M, v \models \varphi$ for all v with wR_iv .
- $M, w \models [\uparrow \varphi] \psi \iff M| \varphi, w \models \psi$.
- $M, w \models C_G \varphi \iff M, v \models \varphi$ for all v with $w(\bigcup_{i \in G} R_i)^+ v$ where $+$ denotes the transitive closure.

$\Box_i \varphi$ means “agent i **believes** φ ” and $C_G \varphi$ means “ φ is **common belief** in G .”

Hereafter, we work in K45. For each ordinal α and a model $M = (W, \{R_i\}_{i \in G}, V)$, let $M^\alpha = (W, \{R_i|^\alpha \varphi\}, V)$ denote the updated model after α announcements of φ .

Four notions

Example 3

The **Moore sentence**³ $\varphi := p \wedge \neg \Box p$ is self-refuting. In fact, whenever φ is true, after its announcement $\Box p$ holds, which in turn makes φ itself false.

The **self-fulfilling sentence** $\psi := p \vee \Box p$ is a true lie.

³The absurdity of asserting “ p but I do not believe p ” was discussed as **Moore’s paradox** [Moore, 1942, Wittgenstein, 1953, Hintikka, 1962].

Contents

1 Introduction

2 Basic definitions

3 Eventual and strong eventual notions

4 Conclusion and future work

Eventual, strong eventual, always informative

Definition 4

Write $\varphi^1 := \varphi$ and $\varphi^0 := \neg\varphi$. For $i, j \in \{0, 1\}$:

$$E_{ij}(\varphi) :\iff \forall M, w \ (M, w \models \varphi^i \Rightarrow \exists n \geq 1 \ M^n, w \models \varphi^j)$$

$$SE_{ij}(\varphi) :\iff \forall M, w \ (M, w \models \varphi^i \Rightarrow \exists N \geq 1 \ \forall n \geq N \ M^n, w \models \varphi^j)$$

$$E_{ij}^{Ord}(\varphi) :\iff \forall M, w \ (M, w \models \varphi^i \Rightarrow \exists \alpha > 0 \ M^\alpha, w \models \varphi^j)$$

$$SE_{ij}^{Ord}(\varphi) :\iff \forall M, w \ (M, w \models \varphi^i \Rightarrow \exists \alpha > 0 \ \forall \beta \geq \alpha \ M^\beta, w \models \varphi^j)$$

$$AI_i(\varphi) :\iff \forall M, w \ (M, w \models \varphi^i \Rightarrow (M|\varphi)_w \neq M_w)$$

We use for example the terms **eventually self-refuting** for E_{10} , **strongly eventually self-refuting** for SE_{10} , **transfinitely eventually self-refuting** for E_{10}^{Ord} , and **always informative when true** for AI_1 .

Eventual, strong eventual, always informative

Definition 5

$$S_{ij}^{<\omega}(\varphi) :\Longleftrightarrow \forall k < \omega \models \varphi^i \rightarrow [\uparrow \varphi]^k (C_G \varphi \rightarrow \varphi^j),$$
$$S_{ij}^{Ord}(\varphi) :\Longleftrightarrow \forall M, w, \alpha \left(M, w \models \varphi^i \wedge M^\alpha, w \models C_G \varphi \Rightarrow M^\alpha, w \models \varphi^j \right)$$

For a pointed model M, w and $n \geq 0$, put

$$\sigma_n^{M,w}(\varphi) := \begin{cases} 1 & \text{if } M|_n \varphi, w \models \varphi, \\ 0 & \text{if } M|_n \varphi, w \models \neg \varphi. \end{cases}$$

Main result I

Theorem 6

Let $\varphi \in \mathcal{L}$. In multi-agent K45, we have the following. In single-agent K45, all the conditions in each item are equivalent. Every displayed right arrow is strict when $|G| \geq 2$.

1

φ is strongly eventually successful

$$\iff \forall M, w [\sigma_0^{M,w} = 1 \Rightarrow \lim_{n \rightarrow \infty} \sigma_n^{M,w} = 1]$$

$\implies \varphi$ is eventually successful

$$\iff \forall M, w [\sigma_0^{M,w} = 1 \Rightarrow \limsup_{n \rightarrow \infty} \sigma_n^{M,w} = 1]$$

$$\iff \exists N \geq 1, \models \varphi \rightarrow \bigvee_{n=1}^N [\uparrow \varphi]^n \varphi$$

$$\implies \forall k \geq 0, \models \varphi \rightarrow [\uparrow \varphi]^k (\varphi \leftrightarrow (\varphi \vee C_G \varphi)).$$

Main result I

Theorem 6 (continued)

2

 φ is strongly eventually self-refuting

$$\iff \forall M, w \lim_{n \rightarrow \infty} \sigma_n^{M,w} = 0$$

 $\iff \varphi$ is eventually self-refuting

$$\iff \forall M, w \liminf_{n \rightarrow \infty} \sigma_n^{M,w} = 0$$

$$\iff \exists N \geq 1, \models [\uparrow \varphi]^N \left(\neg \varphi \wedge \bigwedge_{i \in G} \Box_i \perp \right)$$

 $\implies \varphi$ is always informative when true

$$\iff \forall k \geq 0, \models \varphi \rightarrow [\uparrow \varphi]^k (\varphi \leftrightarrow (\varphi \wedge \neg C_G \varphi))$$

$$\iff \models C_G \varphi \rightarrow \neg \varphi$$

$$\iff \models \varphi \leftrightarrow (\varphi \wedge \neg C_G \varphi).$$

Main result I

Theorem 6 (continued)

3

 φ is a strong eventual true lie

$$\iff \forall M, w \lim_{n \rightarrow \infty} \sigma_n^{M,w} = 1$$

 $\implies \varphi$ is an eventual true lie

$$\iff \forall M, w \limsup_{n \rightarrow \infty} \sigma_n^{M,w} = 1$$

$$\iff \exists N \geq 1 \quad \models \bigvee_{n=1}^N [\uparrow \varphi]^n \varphi$$

 $\implies \varphi$ is always informative when false

$$\iff \forall k \geq 0, \quad \models \neg \varphi \rightarrow [\uparrow \varphi]^k (\varphi \leftrightarrow (\varphi \vee C_G \varphi))$$

$$\iff \models C_G \varphi \rightarrow \varphi$$

$$\iff \models \varphi \leftrightarrow (\varphi \vee C_G \varphi).$$

Main result I

Theorem 6 (continued)

4

 φ is a strong eventual impossible lie

$$\iff \forall M, w [\sigma_0^{M,w} = 0 \Rightarrow \lim_{n \rightarrow \infty} \sigma_n^{M,w} = 0]$$

 $\implies \varphi$ is an eventual impossible lie

$$\iff \forall M, w [\sigma_0^{M,w} = 0 \Rightarrow \liminf_{n \rightarrow \infty} \sigma_n^{M,w} = 0]$$

$$\iff \exists N \geq 1, \quad \models \neg\varphi \rightarrow \bigvee_{n=1}^N [\uparrow \varphi]^n \neg\varphi$$

$$\implies \forall k \geq 0, \quad \models \neg\varphi \rightarrow [\uparrow \varphi]^k (\varphi \leftrightarrow (\varphi \wedge \neg C_G \varphi)).$$

Main result I

In all the four cases, strong eventual notions are characterized by limit and eventual notions are characterized by (i) limsup or liminf, and (ii) uniform bounds. Eventual notions further imply (iii) the fixed-point views of the sentences $\varphi \wedge \neg C_G \varphi$ and $\varphi \vee C_G \varphi$.

Main result II

Next, we analyze the logical relationship among E_{ij} , SE_{ij} , E_{ij}^{Ord} , SE_{ij}^{Ord} , $S_{ij}^{<\omega}$, and S_{ij}^{Ord} .

We take the three perspectives: strong eventual vs eventual, finite vs transfinite, and what condition S_{ij}^{Ord} is equivalent to.

Main result II

Theorem 7

For formulas in $\mathcal{L}$: In multi-agent K45, we have the following. In single-agent K45, all the conditions in each item are equivalent. Every displayed right arrow is strict when $|G| \geq 2$.

1

$$SE_{11} \Rightarrow E_{11} \Rightarrow E_{11}^{Ord} \Rightarrow S_{11}^{<\omega}, \quad S_{11}^{Ord} \Leftrightarrow SE_{11}^{Ord} \Rightarrow E_{11}^{Ord}.$$

Moreover,

$$E_{11} \parallel SE_{11}^{Ord}, \quad SE_{11} \parallel SE_{11}^{Ord}.$$

2

$$SE_{10} \Leftrightarrow E_{10} \Rightarrow E_{10}^{Ord} \Leftrightarrow SE_{10}^{Ord} \Leftrightarrow S_{10}^{Ord} \Leftrightarrow S_{10}^{<\omega}$$

Main result II

Theorem 7 (continued)

3

$$SE_{01} \Rightarrow E_{01} \Rightarrow E_{01}^{Ord} \Leftrightarrow SE_{01}^{Ord} \Leftrightarrow S_{01}^{Ord} \Leftrightarrow S_{01}^{<\omega}$$

4

$$SE_{00} \Rightarrow E_{00} \Rightarrow E_{00}^{Ord} \Rightarrow S_{00}^{<\omega}, \quad S_{00}^{Ord} \Leftrightarrow SE_{00}^{Ord} \Rightarrow E_{00}^{Ord}.$$

Moreover,

$$E_{00} \parallel SE_{00}^{Ord}, \quad SE_{00} \parallel SE_{00}^{Ord}.$$

Here $\parallel$ denotes incomparability.

Contents

1 Introduction

2 Basic definitions

3 Eventual and strong eventual notions

4 Conclusion and future work

Conclusion

We introduced eventual and strong eventual notions for success, self-refutation, true lies, and impossible lies, as well as their transfinite versions and always informativeness when true/false. We then analyzed their logical relationships.



<https://eiyamada.github.io/>

References I

- Thomas Ågotnes, Hans van Ditmarsch, and Yanjing Wang. True lies. *Synthese*, 195 (10):4581–4615, 2018. doi: 10.1007/s11229-017-1423-y. URL <https://doi.org/10.1007/s11229-017-1423-y>.
- Jaakko Hintikka. *Knowledge and Belief: An Introduction to the Logic of the Two Notions*. Cornell University Press, Ithaca, NY, 1962.
- Wesley H. Holliday and Thomas F. Icard. Moorean phenomena in epistemic logic. In Lev Dmitrievich Beklemishev, Valentin Goranko, and Valentin Shehtman, editors, *Advances in Modal Logic 8*, pages 178–199. College Publications, 2010. URL <https://www.aiml.net/volumes/volume8/Holliday-Icard.pdf>.
- G. E. Moore. A reply to my critics. In Paul Arthur Schilpp, editor, *The Philosophy of G. E. Moore*, volume 4 of *The Library of Living Philosophers*, pages 535–677. Northwestern University Press, Evanston, IL, 1942.

References II

Ludwig Wittgenstein. *Philosophical Investigations*. Basil Blackwell, Oxford, 1953. See Part II, Section x.