Lyndon interpolation property for extensions of S4 and intermediate logics

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Overview

- **We discuss Lyndon interpolation property (LIP) for several consistent normal extensions of the modal logic S4.**
- **As a result, we show that the intermediate logic LV has LIP. This completes the study of LIP for intermediate propositional logics.**
- **We also discuss uniform Lyndon interpolation property (ULIP).**

**T. Kurahashi, Lyndon interpolation property for extensions of S4 and intermediate propositional logics. *JSL*, accepted.
arXiv:2407.00505.**

1. **Interpolation properties in intermediate logic**

2. Interpolation properties in modal logic

Intermediate Logics

In this talk, we consider only propositional logics.

Definition (Intermediate logics)

A logic L is called an intermediate logic if

1. **L lies between intuitionistic propositional logic Int and classical propositional logic Cl;**
2. **L is closed under modus ponens and substitution.**

Theorem (Jankov, 1968)

There are continuum many intermediate logics.

Craig Interpolation Property (CIP)

Let $v(\varphi)$ be the set of all propositional variables occurring in φ.

Definition (CIP)

A logic L has the Craig interpolation property (CIP)
$:\iff$ for any formulas φ and ψ,
if $L \vdash \varphi \to \psi$, then there exists a formula θ such that

1. $v(\theta) \subseteq v(\varphi) \cap v(\psi)$;
2. $L \vdash \varphi \to \theta$;
3. $L \vdash \theta \to \psi$.

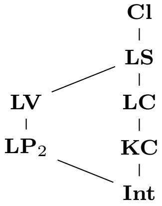
Theorem (Maksimova, 1977)

Exactly seven intermediate logics have CIP:
Int, KC, LC, $\mathbf{LP}_2$, LV, LS, and Cl.

Intermediate Logics with CIP

Each of these logics can be characterized by certain Kripke frame conditions.

- **Int: logic characterized by all intuitionistic Kripke frames;**
- **KC** $=$ **Int** $+ (\neg\varphi \vee \neg\neg\varphi)$**: directed frames;**
- **LC** $=$ **Int** $+ (\varphi \to \psi) \vee (\psi \to \varphi)$**: chain frames;**
- $\mathbf{LP}_2$ $=$ **Int** $+ (\varphi \vee (\varphi \to \psi \vee \neg\psi))$**: rooted frames of height at most 2;**
- **LV** $= \mathbf{LP}_2 + (\varphi \to \psi) \vee (\psi \to \varphi) \vee (\varphi \leftrightarrow \neg\psi)$**: rooted frames of height at most 2 and width at most 2;**
- **LS** $= \mathbf{LP}_2 +$ **LC: rooted chain frames of height at most 2;**
- **Cl** $=$ **Int** $+ (\varphi \vee \neg\varphi)$**: single-point frame.**



Lyndon Interpolation Property (LIP)

- **Let $v^+(\varphi)$ and $v^-(\varphi)$ denote the sets of all propositional variables occurring positively and negatively in φ, respectively.**
- **LIP strengthens CIP by not only requiring that interpolant contains only shared variables, but also preserving the polarity of each variable.**

Definition (LIP)

A logic L has the Lyndon interpolation property (LIP)
$:\Longleftrightarrow$ for any formulas φ and ψ,
if $L \vdash \varphi \to \psi$, then there exists a formula θ such that

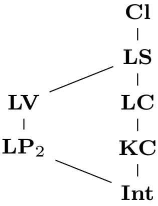
1. $v^+(\theta) \subseteq v^+(\varphi) \cap v^+(\psi)$;
2. $v^-(\theta) \subseteq v^-(\varphi) \cap v^-(\psi)$;
3. $L \vdash \varphi \to \theta$;
4. $L \vdash \theta \to \psi$.

Interpolation in Intermediate Logics

Since LIP implies CIP, it suffices to investigate LIP for the seven intermediate logics known to have CIP.

Logic	CIP	LIP
Cl	**Craig, 1957**	**Lyndon, 1959**
LS	**Maksimova, 1977**	**Maksimova, 1982**
LV	**Maksimova, 1977**	**??? (Open; Maksimova, 2014)**
$\mathbf{LP}_2$	**Maksimova, 1977**	**Shimura, 1992**
LC	**Maksimova, 1977**	**Kuznets and Lellmann, 2018**
KC	**Gabbay, 1971**	**Maksimova, 1982**
Int	**Schütte, 1962**	**Maksimova, 1982**

While LIP had already been shown for six of them, the status of LV remained unresolved.



Gödel Translation and Modal Companions

Interpolation properties of consistent normal extensions of S4 are related to those of intermediate logics via the Gödel translation.

Definition (Gödel Translation)

The Gödel translation T maps formulas of intermediate logic to modal formulas as follows:

- $\mathsf{T}(p) \equiv \Box p$
- $\mathsf{T}(\bot) \equiv \bot$
- $\mathsf{T}(\varphi \land \psi) \equiv \mathsf{T}(\varphi) \land \mathsf{T}(\psi)$
- $\mathsf{T}(\varphi \lor \psi) \equiv \mathsf{T}(\varphi) \lor \mathsf{T}(\psi)$
- $\mathsf{T}(\varphi \to \psi) \equiv \Box(\mathsf{T}(\varphi) \to \mathsf{T}(\psi))$
- $\mathsf{T}(\neg\varphi) \equiv \Box\neg\mathsf{T}(\varphi)$

Definition (Modal Companion)

A normal extension M of S4 is a modal companion of an intermediate logic $L :\iff$ for every formula φ of intermediate logic,

$$L \vdash \varphi \iff M \vdash \mathsf{T}(\varphi).$$

Relation Between Interpolation in Intermediate and Modal Logics

Theorem (Gödel, 1933; McKinsey and Tarski, 1948; Grzegorczyk, 1967)

Both S4 and Grz are modal companions of Int.

Fact

If a modal companion M of an intermediate logic L has CIP (resp. LIP), then L also has CIP (resp. LIP).

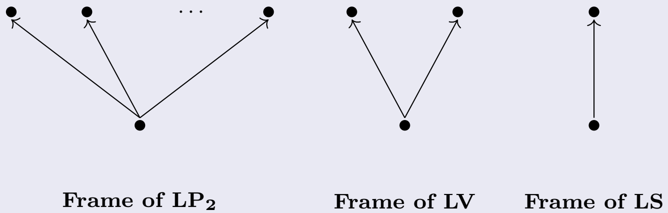
- **Thus, CIP of S4 or Grz implies CIP of Int.**
- **Conversely, to study CIP in consistent normal extensions of S4, it suffices to consider the modal companions of the seven logics having CIP: Int, KC, LC, LP_2, LV, LS, and Cl.**

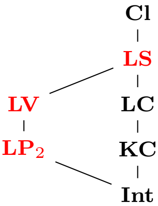
Theorem (Maksimova, 1979–1987)

- **There are at most 36 consistent normal extensions of S4 that have CIP.**
- **Among them, each of LP_2, LV, and LS has exactly 5 modal companions having CIP.**

Modal Companions of $\mathbf{LP_2}$, $\mathbf{LV}$, and $\mathbf{LS}$

- **We focus on 15 modal companions of LP_2, LV, and LS.**
- **The diagram illustrates the typical Kripke frames corresponding to each of these logics.**





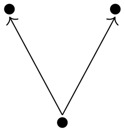
Modal Logic $\Gamma(L, m, n)$

To systematically describe these modal companions, we introduce the notation $\Gamma(L, m, n)$.

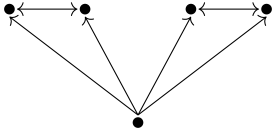
Definition

Given an intermediate logic L, we define $\Gamma(L, m, n)$ to be the modal logic characterized by the class of S4-frames satisfying the following:

- **The quotient frame obtained by collapsing clusters is a finite Kripke frame of L;**
- **Every final cluster contains at most m points;**
- **All other clusters contain at most n points.**



Frame of intermediate logic LV



Frame of modal logic $\Gamma(\mathbf{LV}, 2, 1)$

Modal Companions of $\mathbf{LP}_2$, $\mathbf{LV}$, and $\mathbf{LS}$

Logic	**CIP**	**LIP**
$\Gamma(\mathbf{LP}_2, \omega, 1)$	**Schumm, 1976**	**Shimura, 1992**
$\Gamma(\mathbf{LP}_2, 2, 1)$	**Maksimova, 1980**	???
$\Gamma(\mathbf{LP}_2, 1, \omega)$	**Maksimova, 1980**	???
$\Gamma(\mathbf{LP}_2, 1, 2)$	**Maksimova, 1980**	**(Fails) Maksimova, 1982**
$\Gamma(\mathbf{LP}_2, 1, 1) = \mathbf{GW}$	**Schumm, 1976**	**Shimura, 1992**
$\Gamma(\mathbf{LV}, \omega, 1)$	**Maksimova, 1980**	???
$\Gamma(\mathbf{LV}, 2, 1)$	**Maksimova, 1980**	???
$\Gamma(\mathbf{LV}, 1, \omega)$	**Maksimova, 1980**	???
$\Gamma(\mathbf{LV}, 1, 2)$	**Maksimova, 1980**	**(Fails) Maksimova, 1982**
$\Gamma(\mathbf{LV}, 1, 1) = \mathbf{GV}$	**Maksimova, 1980**	???
$\Gamma(\mathbf{LS}, \omega, 1) = \mathbf{S4.4}$	**Schumm, 1976**	**Shimura, 1992**
$\Gamma(\mathbf{LS}, 2, 1)$	**Maksimova, 1980**	???
$\Gamma(\mathbf{LS}, 1, \omega)$	**Maksimova, 1982**	???
$\Gamma(\mathbf{LS}, 1, 2)$	**Maksimova, 1980**	**(Fails) Maksimova, 1982**
$\Gamma(\mathbf{LS}, 1, 1) = \mathbf{GW.2}$	**Schumm, 1976**	**Maksimova, 1982**

There were 8 logics whose LIP status was previously unknown.

Main Theorem

Main Theorem

Logic	CIP	LIP
$\Gamma(\mathbf{LP}_2, \omega, 1)$	**Schumm, 1976**	**Shimura, 1992**
$\Gamma(\mathbf{LP}_2, 2, 1)$	**Maksimova, 1980**	✓
$\Gamma(\mathbf{LP}_2, 1, \omega)$	**Maksimova, 1980**	**(Fails)**
$\Gamma(\mathbf{LP}_2, 1, 2)$	**Maksimova, 1980**	**(Fails) Maksimova, 1982**
$\Gamma(\mathbf{LP}_2, 1, 1) = \mathbf{GW}$	**Schumm, 1976**	**Shimura, 1992**
$\Gamma(\mathbf{LV}, \omega, 1)$	**Maksimova, 1980**	✓
$\Gamma(\mathbf{LV}, 2, 1)$	**Maksimova, 1980**	✓
$\Gamma(\mathbf{LV}, 1, \omega)$	**Maksimova, 1980**	**(Fails)**
$\Gamma(\mathbf{LV}, 1, 2)$	**Maksimova, 1980**	**(Fails) Maksimova, 1982**
$\Gamma(\mathbf{LV}, 1, 1) = \mathbf{GV}$	**Maksimova, 1980**	✓
$\Gamma(\mathbf{LS}, \omega, 1) = \mathbf{S4.4}$	**Schumm, 1976**	**Shimura, 1992**
$\Gamma(\mathbf{LS}, 2, 1)$	**Maksimova, 1980**	✓
$\Gamma(\mathbf{LS}, 1, \omega)$	**Maksimova, 1980**	**(Fails)**
$\Gamma(\mathbf{LS}, 1, 2)$	**Maksimova, 1980**	**(Fails) Maksimova, 1982**
$\Gamma(\mathbf{LS}, 1, 1) = \mathbf{GW.2}$	**Schumm, 1976**	**Maksimova, 1982**

Proof Strategy

For logics with LIP

For each logic with LIP, we investigated the structure of corresponding Kripke frames by a case-by-case analysis, making use of properties such as height and the form of clusters.

For logics without LIP

For $\Gamma(\mathbf{LP}_2, 1, \omega)$, $\Gamma(\mathbf{LV}, 1, \omega)$, and $\Gamma(\mathbf{LS}, 1, \omega)$, the following formula is provable:

$$p \wedge \Box(\Box\neg p \vee p) \rightarrow \Box(p \vee q \vee \Box\neg q),$$

but it does not admit a Lyndon interpolant.

Corollary of the Main Theorem

One key outcome of the main theorem is that a modal companion GV of LV has LIP.

Corollary

The intermediate logic LV has LIP.

Thus, the analysis of LIP for all intermediate propositional logics is now complete.

Logic	CIP	LIP
Cl	Craig, 1957	Lyndon, 1959
LS	Maksimova, 1977	Maksimova, 1982
LV	Maksimova, 1977	✓
LP_2	Maksimova, 1977	Shimura, 1992
LC	Maksimova, 1977	Kuznets and Lellmann, 2018
KC	Gabbay, 1971	Maksimova, 1982
Int	Schütte, 1962	Maksimova, 1982

UIP and ULIP

Finally, we discuss the uniform versions of interpolation properties.

Fact

For each $L \in \{\mathbf{Cl}, \mathbf{LS}, \mathbf{LV}, \mathbf{LP}_2, \mathbf{LC}\}$**,**

- L **has CIP** $\iff$ L **has UIP (Uniform Interpolation Property);**
- L **has LIP** $\iff$ L **has ULIP (Uniform Lyndon Interpolation Property).**

Logic	UIP	ULIP
Cl	✓	✓
LS	✓	✓
LV	✓	✓
$\mathbf{LP}_2$	✓	✓
LC	✓	✓
KC	**Maksimova, 2014**	???
Int	**Pitts, 1992**	???

Proposition

If a modal companion M of an intermediate logic L has ULIP, then L also has ULIP.

Theorem

- **Grz (a modal companion of Int) has ULIP.** **(K., 2020)**
- **Grz.2 (a modal companion of KC) has ULIP.**

Corollary

Both Int and KC have ULIP.

This completes the analysis of ULIP for all intermediate propositional logics.

Logic	UIP	ULIP
Cl	✓	✓
LS	✓	✓
LV	✓	✓
$\mathbf{LP}_2$	✓	✓
LC	✓	✓
KC	Maksimova, 2014	✓
Int	Pitts, 1992	✓