Provability and consistency principles for the second incompleteness theorem

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The textbook G2: From Gödel to Hilbert–Bernays

In this talk, T always denotes a consistent c.e. extension of Peano Arithmetic (PA) in the language of arithmetic.

Gödel's second incompleteness theorem (G2)

T cannot prove a sentence Con_T asserting the consistency of T.

- **In his famous paper, Gödel presented only a sketched proof of G2.**
- **The first detailed proof of G2 was presented by Hilbert and Bernays in 1939.**
- **In particular, they proposed the derivability conditions, which are requirements on provability predicates sufficient to establish G2.**

Definition

A formula $\mathrm{Pr}_T(x)$ is a provability predicate of T
$:\iff$ $\mathrm{Pr}_T(x)$ is a Σ_1 formula
and for any formula φ, $\mathbb{N} \models \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \iff T \vdash \varphi$.

The textbook G2: Derivability conditions

Derivability conditions were later systematized by Löb in 1955 into the following well-known form.

Löb's derivability conditions

D1 $T \vdash \varphi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

D2 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$.

D3 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$.

A mathematically precise formulation of G2 is currently known as follows:

G2

If a provability predicate $\mathrm{Pr}_T(x)$ **of** T **satisfies Löb's derivability conditions, then** $T \nvdash \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

Overview of this talk

Here, I note the following facts:

1. **Besides the conditions of Hilbert and Bernays and of Löb, several other sufficient conditions for G2 are known.**
2. **For provability predicates that do not satisfy Löb's derivability conditions, G2 does not hold in general.**

Overview of this talk

- **In the light of these facts, we will present the relationships among several versions of G2 and the corresponding derivability conditions.**
- **In particular, we import certain principles that have been studied in the context of non-normal modal logic into the analysis of G2.**

This talk is based on the following papers:

- **Kurahashi, A note on derivability conditions, *JSL*, 2020.**
- **Kurahashi, Refinements of provability and consistency principles for the second incompleteness theorem, arXiv, 2025.**

Local derivability conditions

Local derivability conditions

D1 $T \vdash \varphi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

D2 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$.

D3 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$.

Σ_1C **If φ is a Σ_1 sentence, then** $T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

M $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.

- **Note that every $\mathrm{Pr}_T(x)$ automatically satisfies D1 in our setting.**
- **D2 $\Rightarrow$ M**
- **Σ_1C $\Rightarrow$ D3**

Uniform derivability conditions

$\ulcorner \varphi(\dot{x}) \urcorner$ is a term corresponding to a primitive recursive function calculating the Gödel number of $\varphi(\overline{n})$ from n.

Uniform derivability conditions

$\mathbf{D1}^{\mathrm{U}}$ $T \vdash \varphi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathbf{D2}^{\mathrm{U}}$ $T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner))$.

$\mathbf{\Delta_0 C}^{\mathrm{U}}$ **If** $\varphi(x)$ **is a** Δ_0 **formula, then** $T \vdash \varphi(x) \to \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathbf{\Sigma_1 C}^{\mathrm{U}}$ **If** $\varphi(x)$ **is a** Σ_1 **formula, then** $T \vdash \varphi(x) \to \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathbf{M}^{\mathrm{U}}$ $T \vdash \varphi(x) \to \psi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner)$.

$\mathbf{CB}$ $T \vdash \mathrm{Pr}_T(\ulcorner \forall x\, \varphi(x) \urcorner) \to \forall x\, \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathbf{CB}_\exists$ $T \vdash \exists x\, \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \exists x\, \varphi(x) \urcorner)$.

$$\mathbf{D2}^{\mathrm{U}} + \mathbf{D1}^{\mathrm{U}} \Rightarrow \mathbf{M}^{\mathrm{U}} \Longrightarrow \mathbf{CB} \Longrightarrow \mathbf{D1}^{\mathrm{U}}$$

$$\mathbf{M}^{\mathrm{U}} \Longrightarrow \mathbf{CB}_\exists$$

Global derivability conditions

Global derivability conditions

D2$^{\mathrm{G}}$ $T \vdash \forall x \forall y(\mathrm{Pr}_T(x \dot{\to} y) \to (\mathrm{Pr}_T(x) \to \mathrm{Pr}_T(y)))$.

PC$^{\mathrm{G}}$ $T \vdash \forall x(\mathrm{Pr}_\emptyset(x) \to \mathrm{Pr}_T(x))$.

$\mathrm{Pr}_\emptyset(x)$ **is a standard provability predicate of pure first-order predicate calculus.**

Remark

Global $\Rightarrow$ Uniform $\Rightarrow$ Local.

1. **Several versions of G2**
   1. **Gödel (1931)**
   2. **Hilbert and Bernays (1939)**
   3. **Löb (1955)**
   4. **Jeroslow (1973)**
   5. **Montagna (1979)**
   6. **Buchholz (1993)**
2. **Derivability conditions**
3. **Refinements**

Gödel (1931)

G2 (Gödel (1931))

$T \nvdash \exists x(\mathrm{Fml}(x) \land \neg \mathrm{Pr}_T(x))$.

- **Gödel explained that by formalizing his proof of the first incompleteness theorem, G2 is proved.**
- **He wrote that a detailed proof would be presented in a forthcoming work, but the paper never appeared.**

Hilbert and Bernays (1939)

The first detailed proof of G2 was presented in the second volume of *Grundlagen der Mathematik* by Hilbert and Bernays.

G2 (Hilbert and Bernays (1939))

If $\mathrm{Pr}_T(x)$ **satisfies M, CB, and** $\mathbf{\Delta_0 C^U}$**, then**
$T \nvdash \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$.

M $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.

CB $T \vdash \mathrm{Pr}_T(\ulcorner \forall x\, \varphi(x) \urcorner) \to \forall x\, \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathbf{\Delta_0 C^U}$ **If** $\varphi(x)$ **is a** Δ_0 **formula, then** $T \vdash \varphi(x) \to \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

In practice, they employed slightly different conditions.

Löb (1955)

Löb's Theorem (1955)

If $\mathrm{Pr}_T(x)$ satisfies the following conditions D2 and D3, then for any sentence φ,

$$T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi \Rightarrow T \vdash \varphi.$$

Corollary (G2)

If $\mathrm{Pr}_T(x)$ satisfies D2 and D3, then $T \nvdash \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

- **This is the most well-known form of G2.**
- **Löb's derivability conditions also provide the basis for the study of provability predicates in modal logic.**

D1 $T \vdash \varphi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

D2 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$.

D3 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$.

Löb, Solution of a problem of Leon Henkin, 1955.

Jeroslow (1973)

Jeroslow proved that an alternative form of G2 holds.

G2 (Jeroslow (1973))

If $\mathrm{Pr}_T(x)$ **satisfies** $\mathbf{\Sigma_1 C}$**, then** $T \nvdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \neg \mathrm{Pr}_T(\ulcorner \neg \varphi \urcorner)$ **for some sentence** φ**.**

$\mathbf{\Sigma_1 C}$ **If** φ **is a** Σ_1 **sentence, then** $T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$**.**

Jeroslow, Redundancies in the Hilbert-Bernays derivability conditions for Gödel's second incompleteness theorem, 1973.

Montagna (1979)

G2 (Montagna (1979))

If $\mathrm{Pr}_T(x)$ **satisfies** $\mathbf{D2}^{\mathrm{G}}$ **and** $\mathbf{PC}^{\mathrm{G}}$**, then** $T \nvdash \exists x(\mathrm{Fml}(x) \land \neg \mathrm{Pr}_T(x))$.

$\mathbf{D2}^{\mathrm{G}}$ $T \vdash \forall x \forall y(\mathrm{Pr}_T(x \dot{\to} y) \to (\mathrm{Pr}_T(x) \to \mathrm{Pr}_T(y)))$.

$\mathbf{PC}^{\mathrm{G}}$ $T \vdash \forall x(\mathrm{Pr}_\emptyset(x) \to \mathrm{Pr}_T(x))$.

Montagna, On the formulas of Peano arithmetic which are provably closed under modus ponens, 1979.

Buchholz (1993)

Buchholz proved the following theorem in his lecture note.

Theorem (Buchholz (1993))

If $\mathrm{Pr}_T(x)$ satisfies the conditions $\mathbf{D1}^{\mathrm{U}}$ and $\mathbf{D2}^{\mathrm{U}}$, then it also satisfies $\mathbf{\Sigma_1 C}^{\mathrm{U}}$.

This provides a clear proof of formalized Σ_1-completeness.

Corollary (G2)

If $\mathrm{Pr}_T(x)$ satisfies the conditions $\mathbf{D1}^{\mathrm{U}}$ and $\mathbf{D2}^{\mathrm{U}}$, then $T \nvdash \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

$\mathbf{D1}^{\mathrm{U}}$ $T \vdash \varphi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathbf{D2}^{\mathrm{U}}$ $T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner))$.

Buchholz, *Mathematische Logik II*, 1993.

These different versions of G2 have different consequences.

Different consistency statements

- $\mathrm{Con}_T^{\mathrm{H}} :\equiv \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg\mathrm{Pr}_T(\dot{\neg} x))$
- $\mathrm{Con}_T^{\mathrm{S}} := \{\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \neg\mathrm{Pr}_T(\ulcorner \neg\varphi \urcorner) \mid \varphi$ **is a sentence**$\}$
- $\mathrm{Con}_T^{\mathrm{L}} :\equiv \neg\mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$
- $\mathrm{Con}_T^{\mathrm{G}} :\equiv \exists x(\mathrm{Fml}(x) \land \neg\mathrm{Pr}_T(x))$

Different consequences

Gödel $T \nvdash \mathrm{Con}_T^{\mathrm{G}}$

Hilbert and Bernays $\{\mathbf{M}, \mathbf{CB}, \mathbf{\Delta_0 C}^{\mathrm{U}}\} \Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{H}}$

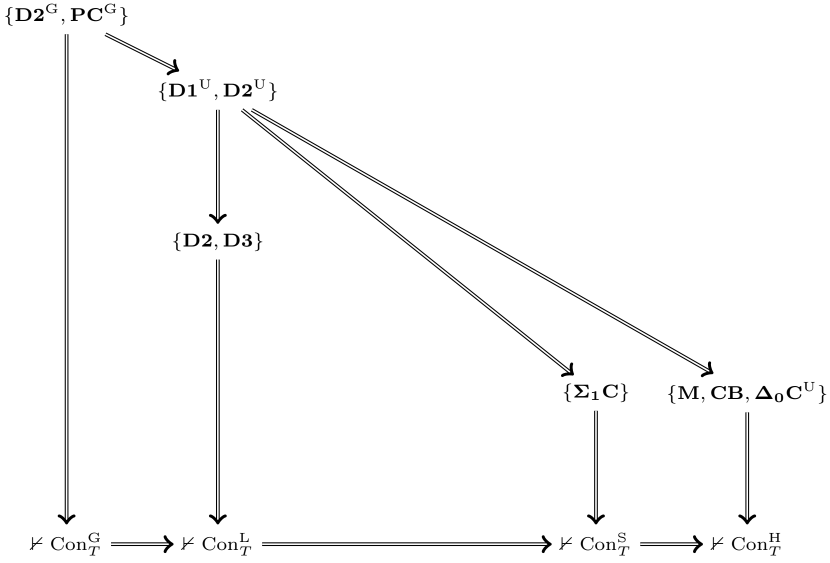
Löb $\{\mathbf{D2}, \mathbf{D3}\} \Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{L}}$

Jeroslow $\{\mathbf{\Sigma_1 C}\} \Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{S}}$

Montagna $\{\mathbf{D2}^{\mathrm{G}}, \mathbf{PC}^{\mathrm{G}}\} \Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{G}}$

Buchholz $\{\mathbf{D1}^{\mathrm{U}}, \mathbf{D2}^{\mathrm{U}}\} \Rightarrow \mathbf{\Sigma_1 C}^{\mathrm{U}}$

- $\mathsf{PA} + \mathrm{Con}_T^{\mathrm{H}} \vdash \mathrm{Con}_T^{\mathrm{S}}$, $\mathsf{PA} + \mathrm{Con}_T^{\mathrm{S}} \vdash \mathrm{Con}_T^{\mathrm{L}}$, **and** $\mathsf{PA} + \mathrm{Con}_T^{\mathrm{L}} \vdash \mathrm{Con}_T^{\mathrm{G}}$**.**
- **In what follows, we clarify the situation.**


$\{\mathbf{D2}^{\mathrm{G}}, \mathbf{PC}^{\mathrm{G}}\}$
$\{\mathbf{D1}^{\mathrm{U}}, \mathbf{D2}^{\mathrm{U}}\}$
$\{\mathbf{D2}, \mathbf{D3}\}$
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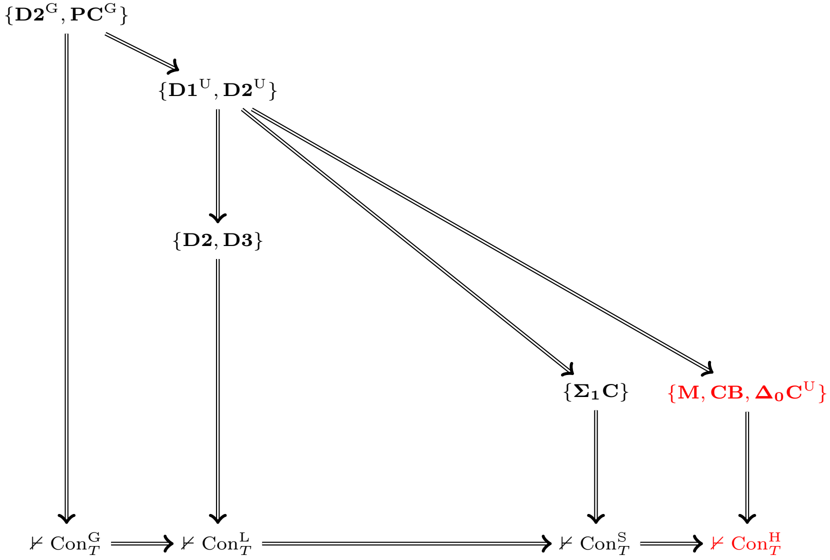
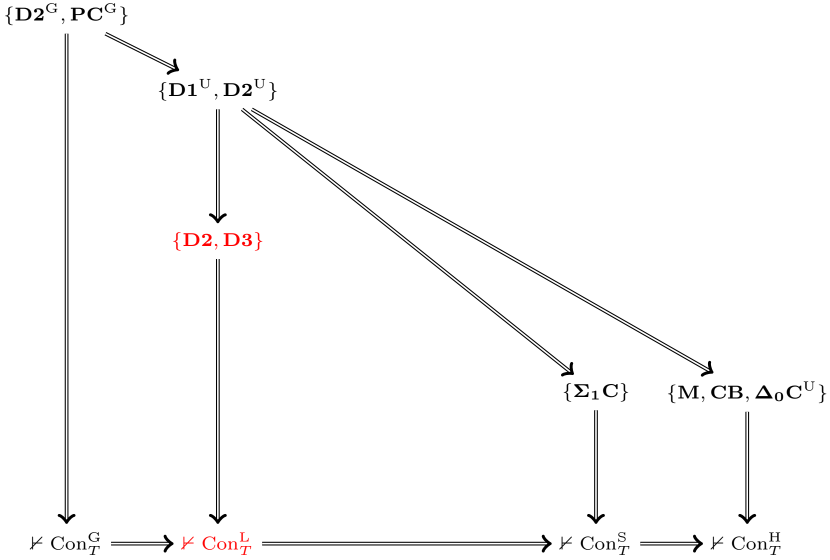
$\{\mathbf{M}, \mathbf{CB}, \mathbf{\Delta_0 C}^{\mathrm{U}}\}$
$\nvdash \mathrm{Con}_T^{\mathrm{G}}$
$\nvdash \mathrm{Con}_T^{\mathrm{L}}$
$\nvdash \mathrm{Con}_T^{\mathrm{S}}$
$\nvdash \mathrm{Con}_T^{\mathrm{H}}$

Hilbert and Bernays (1939)



Löb (1955)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$\nvdash \mathrm{Con}_T^{\mathrm{G}}$
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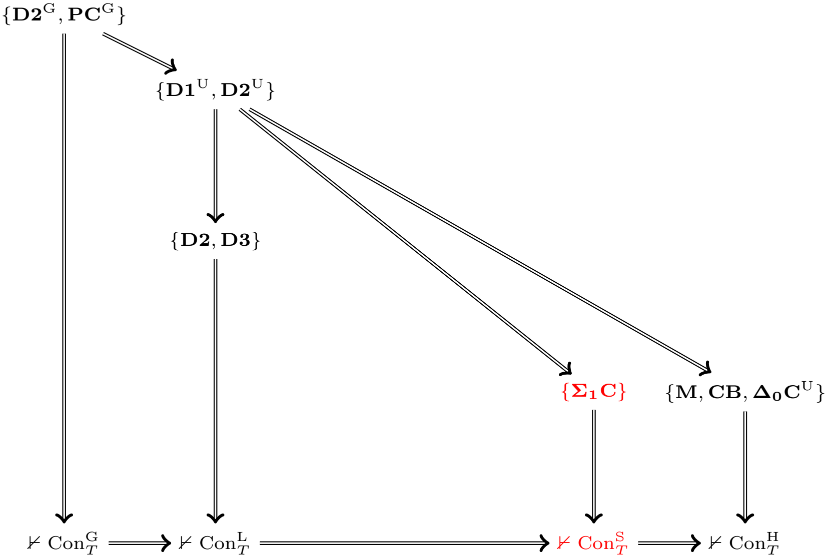
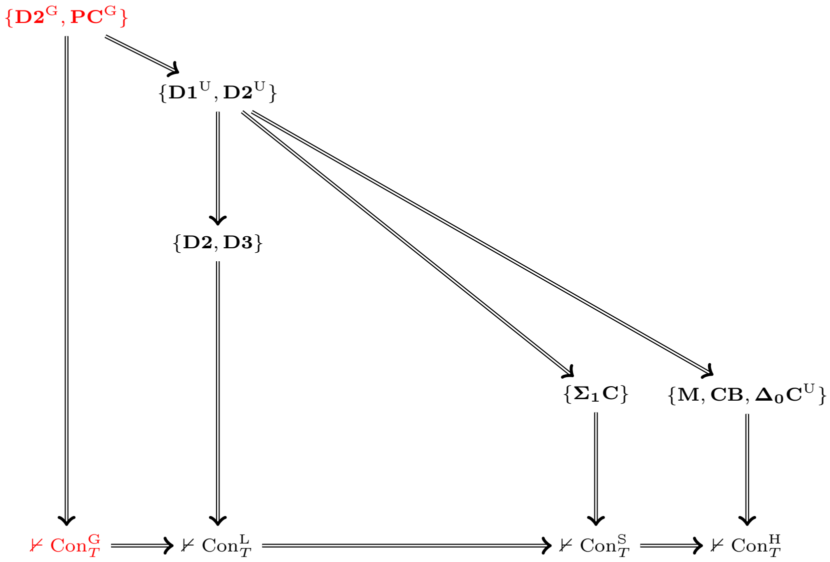
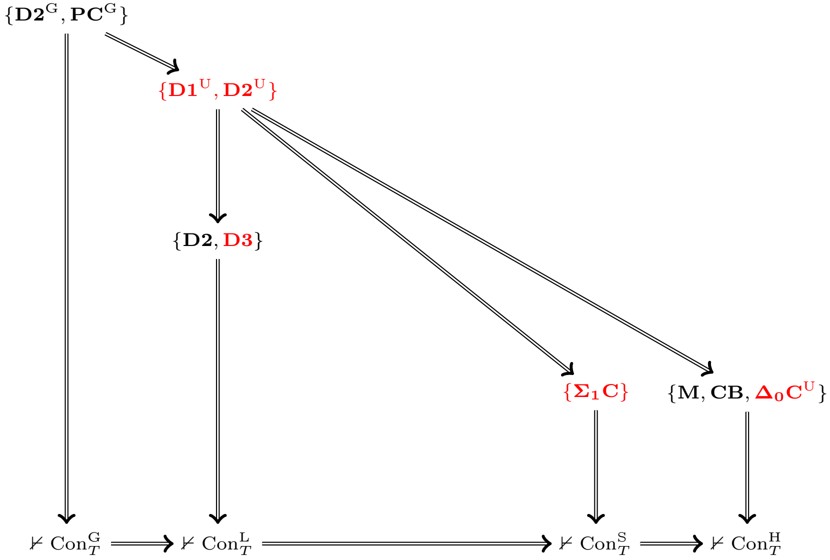
Jeroslow (1973)



Montagna (1979)



Buchholz (1993)



Some refinements

The picture surrounding G2 and the derivability conditions can be refined as follows.

1. **We refine Hilbert and Bernays' formulation of G2.**
2. **We import certain principles studied in the context of non-normal modal logic into the analysis of G2.**
3. **We sharpen Buchholz's sufficient conditions for $\Sigma_1 C$ and $\Sigma_1 C^{U}$.**

A refinement of Hilbert–Bernays' G2

G2 (Hilbert and Bernays) restated

$\{\mathbf{M}, \mathbf{CB}, \mathbf{\Delta_0 C^U}\} \Rightarrow T \nvdash \mathrm{Con}^{\mathrm{H}}_T$.

By using the uniform Δ_0 reflection principle

$$\mathrm{RFN}_T(\Delta_0) := \{\forall x(\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x)) \mid \varphi(x) \in \Delta_0\},$$

their proof can be decomposed as follows.

Theorem (K., 2025+)

1. $\{\mathbf{CB}\} \Rightarrow T \nvdash \mathrm{RFN}_T(\Delta_0)$.
2. $\{\mathbf{\Delta_0 C^U}\} \Rightarrow T + \mathrm{Con}^{\mathrm{H}}_T \vdash \mathrm{RFN}_T(\Delta_0)$.

The first clause is itself a version of G2.

Corollary (K., 2025+)

$\{\mathbf{CB}, \mathbf{\Delta_0 C^U}\} \Rightarrow T \nvdash \mathrm{Con}^{\mathrm{H}}_T$.

M is redundant.

A refinement of Hilbert–Bernays' G2

On the other hand, Hilbert–Bernays' conditions are not sufficient for the unprovability of $\mathrm{Con}_T^{\mathrm{S}}$**.**

Theorem (K., 2021)

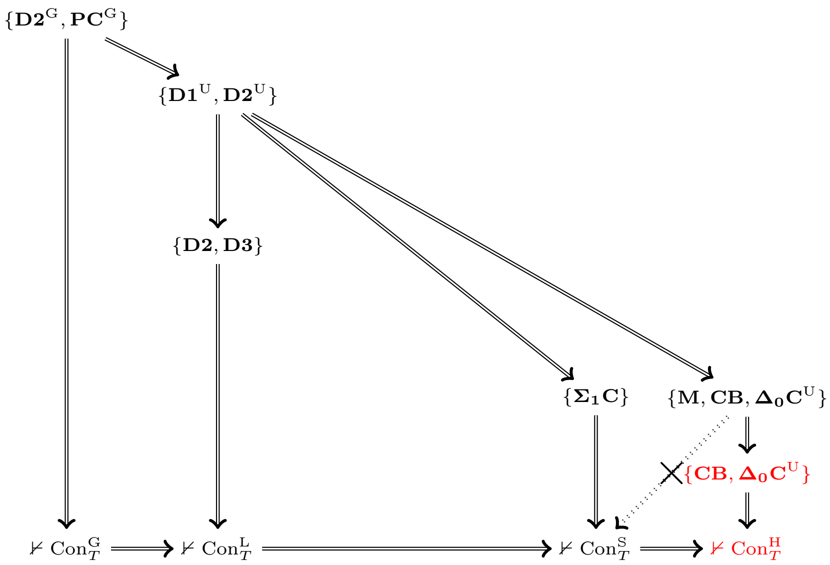
There exists $\mathrm{Pr}_T(x)$ **satisfying M, CB, and** $\mathbf{\Delta_0 C^U}$ **such that** $T \vdash \mathrm{Con}_T^{\mathrm{S}}$**.**

The statement is abbreviated as "$\{\mathbf{M}, \mathbf{CB}, \mathbf{\Delta_0 C^U}\} \not\Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{S}}$**".**

K., Rosser provability and the second incompleteness theorem, 2021.

K., Refinements of provability and consistency principles for the second incompleteness theorem, 2025+.

Refinements



Principles studied in non-normal modal logic

- **D2 corresponds to the axiom scheme** $\mathsf{K} : \Box(A \to B) \to (\Box A \to \Box B)$ **of normal modal logic.**
- **In the context of non-normal modal logics that do not validate** K**, the following inference rules and axiom scheme have been studied.**

$$\textbf{RE} \quad \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

$$\textbf{RM} \quad \frac{A \to B}{\Box A \to \Box B}$$

$$\textbf{C} \quad \Box A \wedge \Box B \to \Box(A \wedge B)$$

- **We have already introduced the derivability conditions M and** M^{U} **corresponding to RM.**
- **Similarly, we introduce the following derivability conditions corresponding to RE and C.**

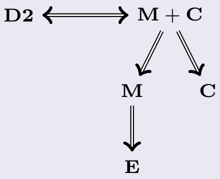
$$\mathbf{E} \quad T \vdash \varphi \leftrightarrow \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \leftrightarrow \mathrm{Pr}_T(\ulcorner \psi \urcorner).$$

$$\mathbf{E}^{\mathrm{U}} \quad T \vdash \varphi(x) \leftrightarrow \psi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \leftrightarrow \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner).$$

$$\mathbf{C} \quad T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \wedge \mathrm{Pr}_T(\ulcorner \psi \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi \wedge \psi \urcorner).$$

A new G2

The following implications are easily verified.

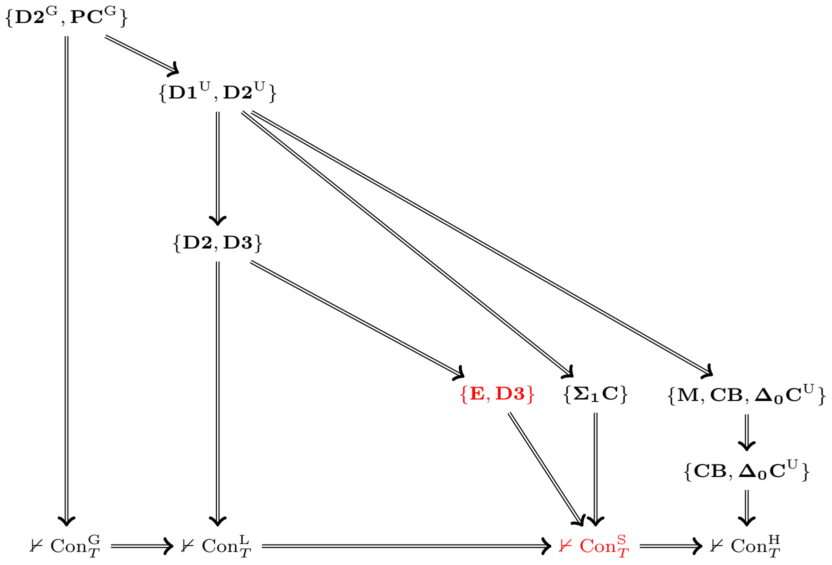


Then, we have the following new version of G2 for $\mathrm{Con}_T^{\mathrm{S}}$**.**

Theorem (K., 2025+)

$\{\mathbf{E}, \mathbf{D3}\} \Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{S}}$.

Refinements



Principles D and P

The following modal principles D and P correspond to $\mathrm{Con}_T^{\mathrm{S}}$ and $\mathrm{Con}_T^{\mathrm{L}}$, respectively.

- $\mathsf{D} : \Box A \to \neg\Box\neg A$
- $\mathsf{P} : \neg\Box\bot$

- **These are equivalent over normal modal logic and characterize seriality: $\forall x \exists y (xRy)$ on Kripke frames.**
- **However, these are not equivalent over non-normal modal logics that do not validate K.**
- **Moreover, seriality exactly corresponds to the following rule Ros which is strictly intermediate between D and P in the context of the pure logic of necessitation.**

$$\mathsf{Ros} : \frac{\neg A}{\neg\Box A}$$

Fitting, Marek and Truszczyński, The pure logic of necessitation, 1992.

K., The provability logic of all provability predicates, 2024.

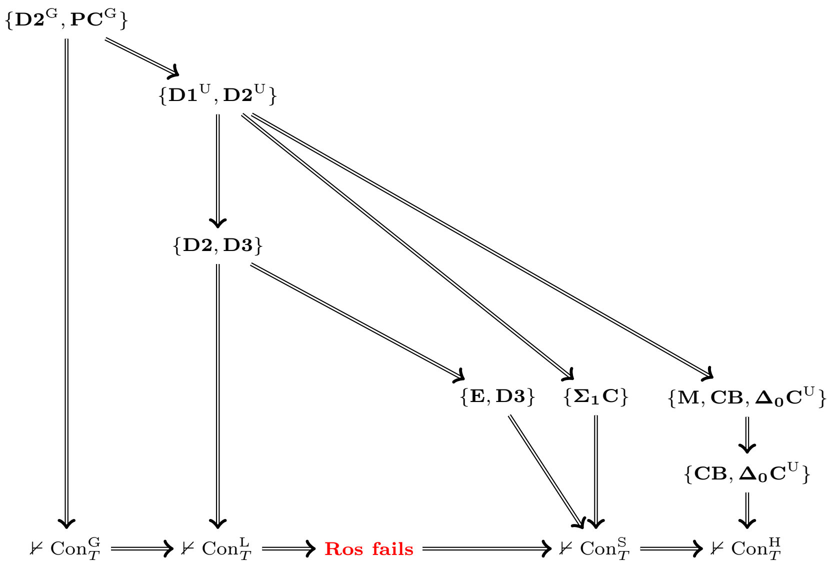
The condition **Ros**

We introduce the following arithmetical condition Ros corresponding to Ros**.**

Ros $T \vdash \neg\varphi \Rightarrow T \vdash \neg\mathrm{Pr}_T(\ulcorner\varphi\urcorner)$

Proposition

- **Every Rosser provability predicate satisfies Ros.**
- $T \vdash \mathrm{Con}^{\mathrm{S}}_T \Rightarrow \{\mathbf{Ros}\}$.
- $\{\mathbf{Ros}\} \Rightarrow T \vdash \mathrm{Con}^{\mathrm{L}}_T$.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$\{\mathbf{\Sigma_1 C}\}$
$\{\mathbf{M}, \mathbf{CB}, \mathbf{\Delta_0 C}^{\mathrm{U}}\}$
$\{\mathbf{CB}, \mathbf{\Delta_0 C}^{\mathrm{U}}\}$
$\nvdash \mathrm{Con}_T^{\mathrm{G}}$
$\nvdash \mathrm{Con}_T^{\mathrm{L}}$
Ros fails
$\nvdash \mathrm{Con}_T^{\mathrm{S}}$
$\nvdash \mathrm{Con}_T^{\mathrm{H}}$

Non-implications

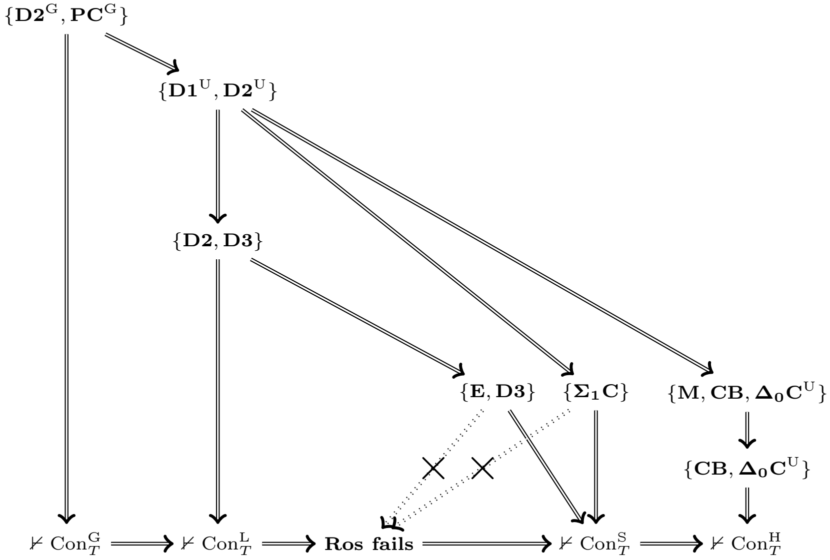
In the context of arithmetic, the failure of Ros is strictly stronger than the unprovability of $\mathrm{Con}_T^{\mathrm{S}}$.

Theorem (K., 2021)

$\{\mathbf{E}, \mathbf{D3}\} \not\Rightarrow$ **Ros fails.**

Theorem (K., 2025+)

$\{\mathbf{\Sigma_1 C}\} \not\Rightarrow$ **Ros fails.**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Ros fails
$\nvdash \mathrm{Con}_T^{\mathrm{S}}$
$\nvdash \mathrm{Con}_T^{\mathrm{H}}$

G2 for **Ros**

The condition D3 is generalized as follows.

For $m, n \in \omega$**,**

$$\mathbf{D3}^n_m \quad T \vdash \mathrm{Pr}^n_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}^m_T(\ulcorner \varphi \urcorner).$$

For $0 < n < m$**, we have** $\mathbf{D3} \Rightarrow \mathbf{D3}^n_m$**.**

Theorem (K., 2025+)

Suppose $0 < n < m$**.**
$\{\mathbf{C}, \mathbf{D3}^n_m\} \Rightarrow$ **Ros fails.**

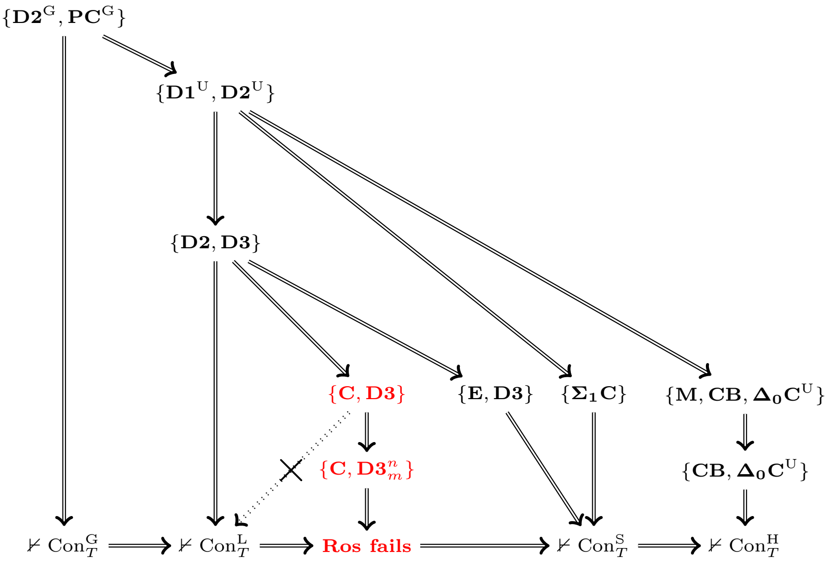
On the other hand, we have:

Theorem (Mostowski, 1965)

$\{\mathbf{C}, \mathbf{D3}\} \not\Rightarrow T \nvdash \mathrm{Con}^{\mathrm{L}}_T$.

Mostowski, Thirty years of foundational studies, 1965.

Refinements



A refinement of G2 for $\mathrm{Con}_T^{\mathrm{L}}$

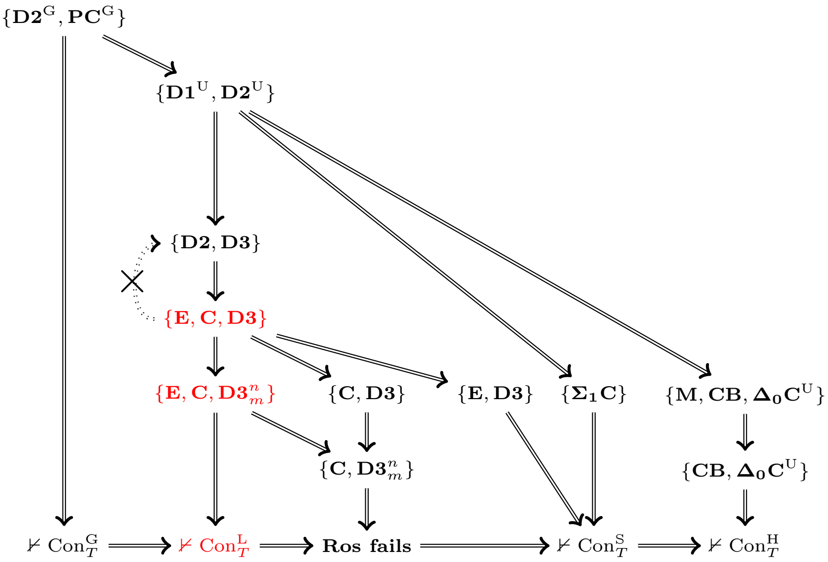
Theorem (K., 2025+)

Suppose $0 < n < m$**.**
$\{\mathbf{E}, \mathbf{C}, \mathbf{D3}_m^n\} \Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{L}}$**.**

This theorem is actually a strengthening of the well-known form "$\{\mathbf{D2}, \mathbf{D3}\} \Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{L}}$**" of G2.**

Theorem (K., 2025+)

$\{\mathbf{E}, \mathbf{C}, \mathbf{D3}\} \not\Rightarrow \mathbf{D2}$**.**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$\{\mathbf{D2}, \mathbf{D3}\}$
$\{\mathbf{E}, \mathbf{C}, \mathbf{D3}\}$
$\{\mathbf{E}, \mathbf{C}, \mathbf{D3}^n_m\}$
$\{\mathbf{C}, \mathbf{D3}\}$
$\{\mathbf{E}, \mathbf{D3}\}$
$\{\mathbf{\Sigma_1 C}\}$
$\{\mathbf{M}, \mathbf{CB}, \mathbf{\Delta_0 C}^{\mathrm{U}}\}$
$\{\mathbf{C}, \mathbf{D3}^n_m\}$
$\{\mathbf{CB}, \mathbf{\Delta_0 C}^{\mathrm{U}}\}$
$\nvdash \mathrm{Con}^{\mathrm{G}}_T$
$\nvdash \mathrm{Con}^{\mathrm{L}}_T$
Ros fails
$\nvdash \mathrm{Con}^{\mathrm{S}}_T$
$\nvdash \mathrm{Con}^{\mathrm{H}}_T$

Buchholz's result

Finally, we consider Buchholz's result.

Theorem (Buchholz) restated

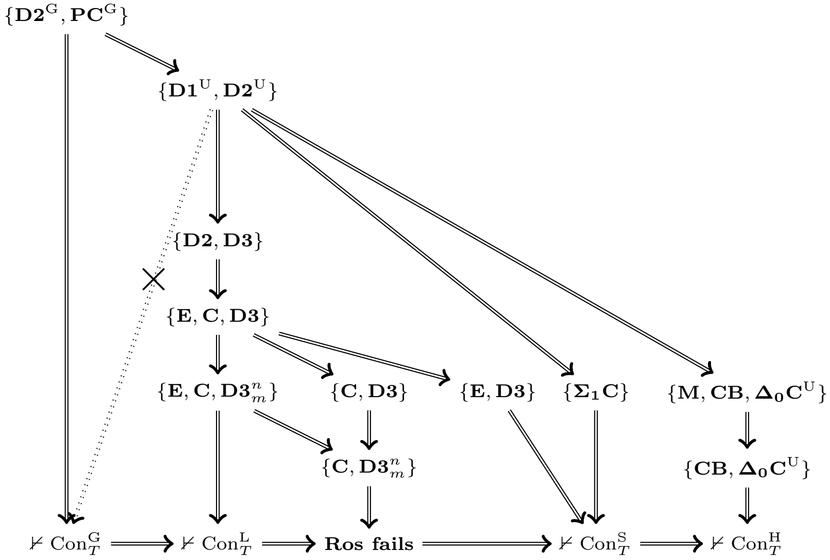
$\{\mathbf{D1}^{\mathrm{U}}, \mathbf{D2}^{\mathrm{U}}\} \Rightarrow \mathbf{\Sigma_1 C}^{\mathrm{U}}$.

- **Buchholz's result shows that $\{\mathbf{D1}^{\mathrm{U}}, \mathbf{D2}^{\mathrm{U}}\}$ is so powerful that all known uniform derivability conditions follow from it.**
- **On the other hand, $\{\mathbf{D1}^{\mathrm{U}}, \mathbf{D2}^{\mathrm{U}}\}$ is not even sufficient for the unprovability of $\mathrm{Con}_T^{\mathrm{G}}$.**

Theorem (K. 2020)

$\{\mathbf{D1}^{\mathrm{U}}, \mathbf{D2}^{\mathrm{U}}\} \not\Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{G}}$.

K., A note on derivability conditions, 2020.


$\{\mathbf{D2}^{\mathrm{G}}, \mathbf{PC}^{\mathrm{G}}\}$
$\{\mathbf{D1}^{\mathrm{U}}, \mathbf{D2}^{\mathrm{U}}\}$
$\{\mathbf{D2}, \mathbf{D3}\}$
$\{\mathbf{E}, \mathbf{C}, \mathbf{D3}\}$
$\{\mathbf{E}, \mathbf{C}, \mathbf{D3}^n_m\}$
$\{\mathbf{C}, \mathbf{D3}\}$
$\{\mathbf{E}, \mathbf{D3}\}$
$\{\mathbf{\Sigma_1 C}\}$
$\{\mathbf{M}, \mathbf{CB}, \mathbf{\Delta_0 C}^{\mathrm{U}}\}$
$\{\mathbf{C}, \mathbf{D3}^n_m\}$
$\{\mathbf{CB}, \mathbf{\Delta_0 C}^{\mathrm{U}}\}$
$\nvdash \mathrm{Con}^{\mathrm{G}}_T$
$\nvdash \mathrm{Con}^{\mathrm{L}}_T$
Ros fails
$\nvdash \mathrm{Con}^{\mathrm{S}}_T$
$\nvdash \mathrm{Con}^{\mathrm{H}}_T$

An improvement of Buchholz's result

We sharpen Buchholz's result as follows:

Theorem (K. 2020)

$\{\mathbf{M}^{\mathrm{U}}\} \Rightarrow \mathbf{\Sigma_1 C}^{\mathrm{U}}$.

Theorem (K. 2025+)

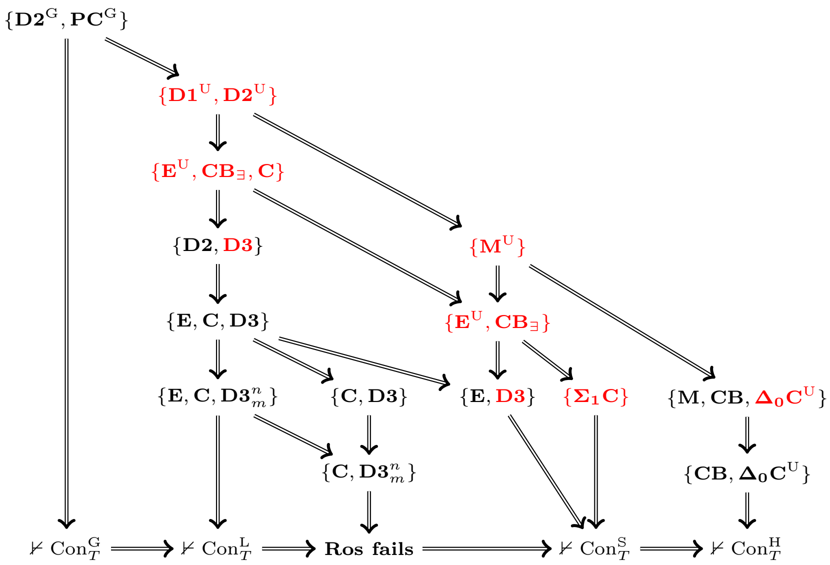
$\{\mathbf{E}^{\mathrm{U}}, \mathbf{CB}_{\exists}\} \Rightarrow \mathbf{\Sigma_1 C}$.

My proofs use the formalized MRDP theorem.

$$\begin{aligned}
\mathbf{M}^{\mathrm{U}} \quad & T \vdash \varphi(x) \to \psi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner). \\
\mathbf{E}^{\mathrm{U}} \quad & T \vdash \varphi(x) \leftrightarrow \psi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \leftrightarrow \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner). \\
\mathbf{CB}_{\exists} \quad & T \vdash \exists x \, \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \exists x \, \varphi(x) \urcorner).
\end{aligned}$$

$$\begin{array}{ccccc}
\mathbf{D1}^{\mathrm{U}} + \mathbf{D2}^{\mathrm{U}} & \Longrightarrow & \mathbf{M}^{\mathrm{U}} & \Longrightarrow & \mathbf{E}^{\mathrm{U}} + \mathbf{CB}_{\exists} \\
 & & \Downarrow & & \Downarrow \\
 & & \mathbf{\Sigma_1 C}^{\mathrm{U}} & \Longrightarrow & \mathbf{\Sigma_1 C}
\end{array}$$

Refinements



Non-implication

$\mathbf{M}^{\mathrm{U}}$ **is strong enough to yield** $\mathbf{\Sigma_1 C}^{\mathrm{U}}$**; however, it does not yield C.**

Theorem (K. 2020)

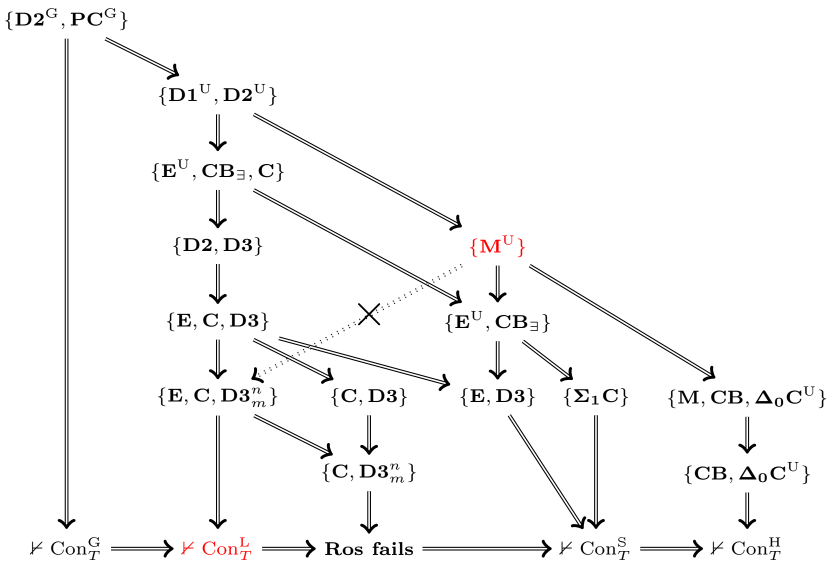
$\{\mathbf{M}^{\mathrm{U}}\} \not\Rightarrow \mathbf{C}$.

I have been considering the following problem for some years; however, it remains open.

Open problem

$\{\mathbf{M}^{\mathrm{U}}\} \Rightarrow T \nvdash \mathrm{Con}_T^{\mathrm{L}}$**?**

Refinements



Conclusion

- **There is no single mathematically precise formulation of G2.**
- **I have constructed various artificial provability predicates, but one may argue that such predicates should be excluded from the scope of G2.**
- **However, the notion of a "naturally defined provability predicate" is not mathematically well-defined.**
- **Accordingly, I do not fix a precise admissible class of provability predicates for G2 at this time.**
- **At present, I view G2 as a family of theorems asserting the unprovability of suitably formulated consistency statements.**

A related modal logical analysis of provabily predicates has been carried out by Haruka Kogure, who will give a talk on Friday.