Modal logics of provability predicates

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Overview of this talk

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- **There are several known conditions on provability predicates** $\mathrm{Pr}_T(x)$ **that are sufficient for the proof of the second incompleteness theorem.**
- **Such conditions (derivability conditions) can be studied through the modal logic** $\mathsf{PL}(\mathrm{Pr}_T)$ **which is called the provability logic of** $\mathrm{Pr}_T(x)$**.**
- **In this talk, I will summarize some existing results on the following two topics:**

1. **Some relationships beteen derivability conditions and the second incompleteness theorem.**
2. **Modal logics** L **such that there exists a provability predicate** $\mathrm{Pr}_T(x)$ **with** $L = \mathsf{PL}(\mathrm{Pr}_T)$**.**

Contents

Provability predicates

In this talk, T always denotes a consistent computable extension of PA.

Definition (weak representability)

A formula $\varphi(x)$ weakly represents a set $X \subseteq \mathbb{N}$ in PA
$\overset{\text{def.}}{\iff} \forall n \in \mathbb{N}(n \in X \iff \mathsf{PA} \vdash \varphi(\overline{n}))$.

Fact

For any c.e. set $X \subseteq \mathbb{N}$, there exists a formula weakly representing X in PA.

Since the set $\mathrm{Th}(T)$ of all theorems of T is c.e., we find a formula weakly representing $\mathrm{Th}(T)$ in PA.

Definition (Provability predicates)

A formula $\mathrm{Pr}_T(x)$ weakly representing $\mathrm{Th}(T)$ is called a provability predicate of T.
That is, for any formula φ, $T \vdash \varphi \iff \mathsf{PA} \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

Gödel's second incompleteness theorem (G2)

- **In his famous paper, Gödel concretely constructed a provability predicate $\mathrm{Pr}_T(x)$ and proved the second incompleteness theorem with only a sketched proof.**
- **Gödel explained that the unprovability of a consistency statement is proved by formalizing his proof of the first incompleteness theorem.**

The second incompleteness theorem (Gödel, 1931)

$T \nvdash \exists x\, (\mathrm{Fml}(x) \wedge \neg \mathrm{Pr}_T(x))$.

Hilbert and Bernays' derivability conditions

- **The first detailed proof of G2 was presented in the second volume of *Grundlagen der Mathematik* by Hilbert and Bernays (1939).**
- **They proved that if $\mathrm{Pr}_T(x)$ satisfies the following three conditions, then $T \nvdash \forall x\,(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$.**

Hilbert-Bernays' derivability conditions

- $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.
- $T \vdash \mathrm{Pr}_T(\ulcorner \neg \varphi(x) \urcorner) \to \mathrm{Pr}_T(\ulcorner \neg \varphi(\dot{x}) \urcorner)$.
- $T \vdash t(x) = 0 \to \mathrm{Pr}_T(\ulcorner t(\dot{x}) = 0 \urcorner)$ **for every primitive recursive term** $t(x)$.

$\ulcorner \varphi(\dot{x}) \urcorner$ **is a primitive recursive term corresponding to a primitive recursive function calculating the Gödel number of** $\varphi(\overline{n})$ **from** n.

Löb's derivability conditions

- **Löb (1955) proved that if** $\mathrm{Pr}_T(x)$ **satisfies the following conditions D1, D2 and D3, then Löb's theorem holds, that is, for any sentence** φ, $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi \Rightarrow T \vdash \varphi$.
- **Then** $T \nvdash \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

Löb's derivability conditions

D1: $T \vdash \varphi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

D2: $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$.

D3: $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$.

Every provability predicate automatically satisfies D1.
It is known that the formalized Löb's theorem is provable.

Fact (cf. Macintyre and Simmons, 1973)

If $\mathrm{Pr}_T(x)$ **satisfies D1, D2 and D3, then**
$T \vdash \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

Löb, Solution of a problem of Leon Henkin, 1955.

Macintyre and Simmons, Gödel's diagonalization technique and related properties of theories, 1973.

Provability predicates for which G2 does not hold

However, G2 does not generally hold for all provability predicates.

Definition (provable Σ_1-completeness)

Σ_1**C: If** φ **is a** Σ_1 **sentence, then** $T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$**.**

Theorem (Mostowski, 1966)

There exists a Σ_1 **provability predicate** $\mathrm{Pr}'_T(x)$ **satisfying** Σ_1**C such that** $T \vdash \neg \mathrm{Pr}'_T(\ulcorner 0 = 1 \urcorner)$**.**

Proof.

Let $\mathrm{Pr}_T(x)$ **be a** Σ_1 **provability predicate satisfying** Σ_1**C.**
Then, $\mathrm{Pr}'_T(x) :\equiv \mathrm{Pr}_T(x) \wedge x \neq \ulcorner 0 = 1 \urcorner$ **is a required one.** □

Jeroslow's G2

On the other hand, an alternative form of G2 holds for Σ_1 provability predicates satisfying Σ_1C.

Theorem (Jeroslow, 1973)

If a Σ_1 provability predicate $\mathrm{Pr}_T(x)$ satisfies Σ_1C, then $T \nvdash \neg(\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \land \mathrm{Pr}_T(\ulcorner \neg\varphi \urcorner))$ for some sentence φ.

Proof.

Let φ be a Σ_1 sentence such that $T \vdash \varphi \leftrightarrow \mathrm{Pr}_T(\ulcorner \neg\varphi \urcorner)$.
Suppose $T \vdash \neg(\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \land \mathrm{Pr}_T(\ulcorner \neg\varphi \urcorner))$, then

$$\begin{aligned} T \vdash \varphi &\to \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \\ &\to \neg\mathrm{Pr}_T(\ulcorner \neg\varphi \urcorner) \\ &\to \neg\varphi. \end{aligned}$$

So, we would have $T \vdash \neg\varphi$. Then, $T \vdash \mathrm{Pr}_T(\ulcorner \neg\varphi \urcorner)$, and hence $T \vdash \varphi$, a contradiction. □

Jeroslow, Redundancies in the Hilbert-Bernays derivability conditions for Gödel's second incompleteness theorem, 1973.

Two consistency statements

Let us introduce the following two different consistency statements.

Definition

1. $\mathrm{Con}_T :\equiv \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$
2. $\mathrm{Con}^{\mathrm{S}}_T := \{\neg(\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \land \mathrm{Pr}_T(\ulcorner \neg\varphi \urcorner)) \mid \varphi$ **is a sentence**$\}$

Proposition

If $\mathrm{Pr}_T(x)$ **satisifes D2, then** Con_T **and** $\mathrm{Con}^{\mathrm{S}}_T$ **are equivalent over** T**.**

Löb **D2 & D3** $\Rightarrow T \nvdash \mathrm{Con}_T$.

Jeroslow Σ_1**C** $\Rightarrow T \nvdash \mathrm{Con}^{\mathrm{S}}_T$.

Mostowski's example shows

- **There exists a** Σ_1 **provability predicate such that** $T \vdash \mathrm{Con}_T$ **and** $T \nvdash \mathrm{Con}^{\mathrm{S}}_T$.
- Σ_1**C alone is not sufficient to derive '**$T \nvdash \mathrm{Con}_T$**'.**

Two types of provability predicates

In the following, I will introduce the following two types of provability predicates.

1. **Fefermanian provability predicates**
2. **Rosser's provability predicates**

Fefermanian provability predicates

- **Let $\tau(v)$ be a formula weakly representing T in PA.**
  That is, for any formula φ, $\varphi \in T \iff \mathsf{PA} \vdash \tau(\ulcorner \varphi \urcorner)$.
- **Let $\mathrm{Pr}_\tau(x)$ be a formula naturally stating "x is provable in the theory defined by $\tau(v)$".**
- **Then, $\mathrm{Pr}_\tau(x)$ is a provability predicate of T.**

Definition (Fefermanian provability predicates)

A formula of the form $\mathrm{Pr}_\tau(x)$ is called a Fefermanian provability predicate.

Fefermanian provability predicates and G2

Theorem (Feferman, 1960)

Every Fefermaninan provability predicate satisfies D2 and Σ_1C.

D2: $\mathsf{PA} \vdash \mathrm{Pr}_\tau(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_\tau(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_\tau(\ulcorner \psi \urcorner))$.

Σ_1C: **If φ is a Σ_1 sentence, then $\mathsf{PA} \vdash \varphi \to \mathrm{Pr}_\tau(\ulcorner \varphi \urcorner)$.**

- **If $\tau(v)$ is Σ_1, then so is $\mathrm{Pr}_\tau(x)$.**
- **Since $\mathrm{Pr}_\tau(x)$ satisfies Σ_1C, in this case, $\mathrm{Pr}_\tau(x)$ also satisfies D3.**
- **By Löb's theorem, we have:**

Theorem (Feferman, 1960)

Suppose $\tau(v)$ is a Σ_1 formula weakly representing T in PA.

- $T \vdash \mathrm{Pr}_\tau(\ulcorner \mathrm{Pr}_\tau(\ulcorner \varphi \urcorner) \to \varphi \urcorner) \to \mathrm{Pr}_\tau(\ulcorner \varphi \urcorner)$.
- $T \nvdash \neg \mathrm{Pr}_\tau(\ulcorner 0 = 1 \urcorner)$.

Feferman, Arithmetization of metamathematics in a general setting, 1960.

Fefermanian provability predicates for which the second incompleteness theorem does not hold

- **If $\tau(v)$ is Σ_1, then G2 holds for $\mathrm{Pr}_\tau(x)$.**
- **On the other hand, when $\tau(v)$ is not Σ_1, G2 does not generally hold for $\mathrm{Pr}_\tau(x)$.**

Theorem (Feferman, 1960)

There exists a Π_1 formula $\pi(v)$ weakly representing T in PA such that $T \vdash \neg \mathrm{Pr}_\pi(\ulcorner 0 = 1 \urcorner)$.

- **$\mathrm{Pr}_\pi(x)$ is a Σ_2 formula that satisfies D2 and Σ_1C but not D3.**

Rosser's provability predicates

There are also (non-Fefermanian) Σ_1 provability predicates for which G2 does not hold.

Definition (Rosser's provability predicates)

For a Δ_1(PA) formula $\mathrm{Prf}_T(x, y)$ saying "y is a T-proof of x", the following formula $\mathrm{Pr}^{\mathrm{R}}_T(x)$ is called a Rosser's provability predicate of T:

$$\mathrm{Pr}^{\mathrm{R}}_T(x) \equiv \exists y\, (\mathrm{Prf}_T(x, y) \wedge \forall z < y\, \neg \mathrm{Prf}_T(\neg x, z))$$

Fact

- $\mathrm{Pr}^{\mathrm{R}}_T(x)$ **is a** Σ_1 **provability predicate of** T**.**
- **For any sentence** φ, $T \vdash \neg\varphi \Rightarrow T \vdash \neg \mathrm{Pr}^{\mathrm{R}}_T(\ulcorner \varphi \urcorner)$.

Since $T \vdash \neg 0 = 1$, **we have:**

Fact (cf. Kreisel, 1960)

$T \vdash \neg \mathrm{Pr}^{\mathrm{R}}_T(\ulcorner 0 = 1 \urcorner)$.

Kreisel, Ordinal logics and the characterization of informal concepts of proof, 1960.

Rosser's provability predicates and derivability conditions

Since G2 does not hold for $\mathrm{Pr}^{\mathrm{R}}_{T}(x)$, at least one of the derivability conditions D2 and D3 does not hold for $\mathrm{Pr}^{\mathrm{R}}_{T}(x)$.

Theorem (cf. Guaspari and Solovay, 1979)

There exists a Rosser's provability predicate which satisfies neither D2 nor D3.

Theorem (Bernardi and Montagna, 1984; Arai, 1990)

There exists a Rosser's provability predicate satisfying D2.

Theorem (Arai, 1990)

There exists a Rosser's provability predicate satisfying D3.

Whether $\mathrm{Pr}^{\mathrm{R}}_{T}(x)$ satisfies D2 or D3 depends on its construction.

Guaspari and Solovay, Rosser sentences, 1979.
Bernardi and Montagna, Equivalence relations induced by extensional formulae, 1984.
Arai, Derivability conditions on Rosser's provability predicates, 1990.

Derivability conditions and the second incompleteness theorem

- **In particular, the existence of $\mathrm{Pr}^{\mathrm{R}}_{T}(x)$ satisfying D2 or D3 is important in the following sense.**

Corollary

For Σ_1 provability predicates $\mathrm{Pr}_T(x)$,

- **D2 alone is not sufficient to derive '$T \nvdash \mathrm{Con}_T$'.**
- **D3 alone is not sufficient to derive '$T \nvdash \mathrm{Con}_T$'.**

By showing the existence of Rosser provability predicates that satisfy various conditions, it can be shown that G2 cannot be derived from those conditions alone.

Monotonicity

If $\mathrm{Pr}_T(x)$ satisfies D2, then it also satisfies the following condition M of monotonicity, which is the first condition of Hilbert and Bernays.

Definition (monotonicity)

M: $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.

Theorem (K., 2021)

There exists a Rosser's provability predicate satisfing both M and D3.

Corollary

The combination of M and D3 is not sufficient to derive '$T \nvdash \mathrm{Con}_T$'.

On the other hand, for the combination of M and D3, the following form of G2 holds:

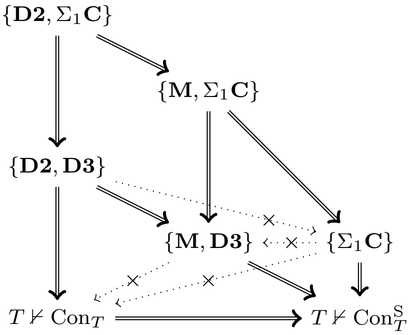
Theorem (K., 2021)

If $\mathrm{Pr}_T(x)$ satisfies both M and D3,
then there exists a sentence φ such that $T \nvdash \mathrm{Con}^{\mathrm{S}}_T$.

K., Rosser provability and the second incompleteness theorem, 2021.

Derivability conditions and the second incompleteness theorem

The following figure summarizes the situation for Σ_1 provability predicates.



Problem

Is there a Σ_1 provability predicate satisfying both M and Σ_1C such that $T \vdash \mathrm{Con}_T$?

K., A note on derivability conditions, 2020.

Modal operator based on a provability predicate

The formalized Löb's theorem (repeated)

Suppose a provability predicate $\mathrm{Pr}_T(x)$ satisfies the following D2 and D3:

D2: $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$

D3: $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$

Then, for any sentence φ,

$$T \vdash \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner).$$

D2, D3 and the formalized Löb's theorem correspond to the following axioms K, 4 and GL of modal logic, respectively:

K: $\Box(A \to B) \to (\Box A \to \Box B)$

4: $\Box A \to \Box\Box A$

GL: $\Box(\Box A \to A) \to \Box A$

Arithmetical interpretations and provability logic

Definition (arithmetical interprertations)

A mapping f from modal formulas to formulas of arithmetic is an arithmetical interpretation based on $\mathrm{Pr}_T(x)$ $\overset{\text{def.}}{\Longleftrightarrow}$

- $f(\bot)$ **is** $0 = 1$
- $f(\neg A)$ **is** $\neg f(A)$
- $f(A \circ B)$ **is** $f(A) \circ f(B)$ **for** $\circ \in \{\wedge, \vee, \rightarrow\}$
- $f(\Box A)$ **is** $\mathrm{Pr}_T(\ulcorner f(A) \urcorner)$

Definition (provability logic)

Let $\mathrm{Pr}_T(x)$ **be a provability predicate of** T**.**

$$\mathsf{PL}(\mathrm{Pr}_T) := \{A \mid A \text{ is a modal formua } \& \ \forall f\text{: arithmetical interpretation based on } \mathrm{Pr}_T(x),\ T \vdash f(A)\}$$

is called the provability logic of $\mathrm{Pr}_T(x)$**.**

- **Can we axiomatize the modal logic** $\mathsf{PL}(\mathrm{Pr}_T)$**?**

Several modal logics

The axioms and rules of the modal logic **K**

Axioms **Tautologies and** $\mathrm{K}: \Box(A \to B) \to (\Box A \to \Box B)$

Rules **Modus Ponens** $\dfrac{A \quad A \to B}{B}$ **and Necessitation** $\dfrac{A}{\Box A}$

We obtain several modal logics from K by adding several axioms:

Several modal logics

- $\mathbf{KD} = \mathbf{K} + \neg\Box\bot$
- $\mathbf{KT} = \mathbf{K} + (\Box A \to A)$
- $\mathbf{K4} = \mathbf{K} + (\Box A \to \Box\Box A)$
- $\mathbf{K5} = \mathbf{K} + (\Diamond A \to \Box\Diamond A)$
- $\mathbf{KB} = \mathbf{K} + (A \to \Box\Diamond A)$
- $\mathbf{GL} = \mathbf{K} + (\Box(\Box A \to A) \to \Box A)$

Solovay's theorem

Since the modal logic GL has the axioms and rules corresponding to the derivability conditions and the formalized Löb's theorem, the following proposition immediately follows:

Proposition (arithmetical soundness)

For any Σ_1 formula $\tau(v)$ weakly representing T, we have GL $\subseteq \mathsf{PL}(\mathrm{Pr}_\tau)$.

Solovay proved that the converse inclusion holds for Σ_1-sound theories.

Solovay's arithmetical completeness theorem (Solovay, 1976)

If T is Σ_1-sound and $\tau(v)$ is a Σ_1 formula weakly representing T, then $\mathsf{PL}(\mathrm{Pr}_\tau) =$ GL.

Solovay, Provability interpretations of modal logic, 1976.

Theories that are not Σ_1-sound

- **In the statement of Solovay's theorem, the assumption of the Σ_1-soundness of T cannot be removed.**
- **For example, if $T \vdash \mathrm{Pr}_\tau(\ulcorner 0 = 1 \urcorner)$, then $\Box\bot \in \mathsf{PL}(\mathrm{Pr}_\tau)$ and hence $\mathsf{PL}(\mathrm{Pr}_\tau) = \mathbf{GL} + \Box\bot$.**

Theorem (Visser, 1984)

Suppose T is not Σ_1-sound and let $\tau(v)$ be a Σ_1 formula weakly representing T.
Then, $\mathsf{PL}(\mathrm{Pr}_\tau)$ is either $\mathbf{GL}$ or $\mathbf{GL} + \Box^n\bot$ for some $n \geq 1$.

Theorem (Beklemishev, 1989)

Suppose T is not Σ_1-sound.
For each $L \in \{\mathbf{GL}\} \cup \{\mathbf{GL} + \Box^n\bot \mid n \geq 1\}$, there exists a Σ_1 formula $\tau(v)$ weakly representing T such that $L = \mathsf{PL}(\mathrm{Pr}_\tau)$.

Visser, The provability logics of recursively enumerable theories extending Peano Arithmetic at arbitrary theories extending Peano Arithmetic, 1984.

Beklemishev, On the classification of propositional provability logics, 1989.

Questions

The situation of provability logics for Fefermanian $\mathrm{Pr}_\tau(x)$ based on a Σ_1 formula $\tau(v)$ is clarified.

- **However, this does not tell us everything about the modal logical behavior of provability predicates.**

Two questions

1. **What is the provability logic of each Fefermanian $\mathrm{Pr}_\tau(x)$ based on a non-Σ_1 formula $\tau(v)$?**
2. **For which modal logics L is there a provability predicate $\mathrm{Pr}_T(x)$ such that $L = \mathsf{PL}(\mathrm{Pr}_T)$?**

Fefermanian provability predicates (1/4)

Let's start with the first question.

The first question

What is the provability logic of each Fefermanian $\mathrm{Pr}_\tau(x)$ based on a non-Σ_1 formula $\tau(v)$?

Recall that every Fefermanian provability predicate satisfies D2.

Proposition

For any formula $\tau(v)$ weakly representing T, we have that $\mathsf{PL}(\mathrm{Pr}_\tau)$ is a normal modal logic. That is,

- $\mathbf{K} \subseteq \mathsf{PL}(\mathrm{Pr}_\tau)$,
- $\mathsf{PL}(\mathrm{Pr}_\tau)$ **is closed under Modus Ponens, Necessitation, and uniform substitution.**

Fefermanian provability predicates (2/4)

The following theorem is an immediate collorary to Feferman's theorem on the existence of a Π_1 formula $\pi(v)$ such that $T \vdash \neg \mathrm{Pr}_\pi(\ulcorner 0 = 1 \urcorner)$.

Corollary

There exists a Π_1 formula $\pi(v)$ weakly representing T such that $\mathsf{PL}(\mathrm{Pr}_\pi) \supseteq \mathbf{KD} = \mathbf{K} + \neg\Box\bot$.

Problems

1. **Axiomatize the modal logic $\mathsf{PL}(\mathrm{Pr}_\pi)$ (cf. Visser, 1989).**
2. **Does there exist a $\tau(v)$ such that $\mathsf{PL}(\mathrm{Pr}_\tau) = \mathbf{KD}$?**

Visser, Peano's smart children: A provability logical study of systems with built-in consistency, 1989.

Fefermanian provability predicates (3/4)

- **There are some known results about provability logics of Fefermanian provability predicates.**
- **The following theorem says that the only modal logical property common to all Fefermanian provability predicates is D2.**

Theorem (K., 2018)

$$\mathbf{K} = \bigcap \{\mathsf{PL}(\mathrm{Pr}_\tau) \mid \tau(v) \text{ weakly represents } T\}.$$

In particular, there exists a Σ_2 formula $\tau(v)$ weakly representing T such that $\mathsf{PL}(\mathrm{Pr}_\tau) = \mathbf{K}$.

K., Arithmetical completeness theorem for modal logic K, 2018.

Fefermanian provability predicates (4/4)

The following are sublogics of GL, introduced by Sacchetti (2001).

- $\mathbf{wGL}_n := \mathbf{K} + (\Box(\Box^n A \to A) \to \Box A)$ **(**$n \geq 2$**)**

Theorem (K., 2018)

For each $n \geq 2$**, there exists a** Σ_2 **formula** $\tau(v)$ **weakly representing** T **such that** $\mathbf{wGL}_n = \mathsf{PL}(\mathrm{Pr}_\tau)$**.**

Definition

For each $n \geq 1$**, let**
$\mathcal{PL}_n(T) := \{\mathsf{PL}(\mathrm{Pr}_\tau) \mid \tau(v)$ **is a** Σ_n **formula weakly representing** $T\}$**.**

- **If** T **is** Σ_1**-sound, then** $\mathcal{PL}_1(T) = \{\mathbf{GL}\}$**.**
- **If** T **is not** Σ_1**-sound, then** $\mathcal{PL}_1(T) = \{\mathbf{GL}\} \cup \{\mathbf{GL} + \Box^n \bot \mid n \geq 1\}$**.**
- $\mathcal{PL}_2(T) \supseteq \{\mathbf{K}\} \cup \{\mathbf{wGL}_n \mid n \geq 2\}$**.**

Problem

For each $n \geq 2$**, do we have** $\mathcal{PL}_n(T) = \mathcal{PL}_{n+1}(T)$**?**

Sacchetti, The fixed point property in modal logic, 2001.
K., Arithmetical soundness and completeness for Σ_2 numerations, 2018.

Shavrukov's provability predicate

Next, let's consider the second question.

The second question

For which modal logics L is there a provability predicate $\mathrm{Pr}_T(x)$ such that $L = \mathsf{PL}(\mathrm{Pr}_T)$?

The following is an interesting result by Shavrukov.

Definition

Let $\mathrm{Pr}^{\mathrm{Sh}}_{\mathsf{PA}}(x)$ be the formula $\exists y \, (\mathrm{Pr}_{\mathsf{I}\Sigma_y}(x) \land \neg \mathrm{Pr}_{\mathsf{I}\Sigma_y}(\ulcorner 0 = 1 \urcorner))$.

$\mathrm{Pr}^{\mathrm{Sh}}_{\mathsf{PA}}(x)$ **is a Σ_2 provability predicate of PA.**

Theorem (Shavrukov, 1994)

$\mathsf{PL}(\mathrm{Pr}^{\mathrm{Sh}}_{\mathsf{PA}}) = \mathbf{KD} + (\Box A \to \Box((\Box B \to B) \lor \Box A))$.

- **It is interesting that such a logic appears, which probably would not have been considered in the context of pure modal logic.**
- **However, this also indicates that the second question is difficult to clarify.**

Shavrukov, A smart child of Peano's, 1994.

Modal logics which cannot be a provability logic (1/2)

There are modal logics which cannot be a provability logic.

Theorem

Let $\mathrm{Pr}_T(x)$ **be a provability predicate of** T**.**

1. $\mathsf{PL}(\mathrm{Pr}_T) \neq \mathbf{K4} = \mathbf{K} + (\Box A \to \Box\Box A)$.
2. $\mathsf{PL}(\mathrm{Pr}_T) \not\supseteq \mathbf{KT} = \mathbf{K} + (\Box A \to A)$. **(Montague, 1963)**

Proof.

1. Suppose $\mathbf{K4} \subseteq \mathsf{PL}(\mathrm{Pr}_T)$**, then** $\mathrm{Pr}_T(x)$ **satisfies both D2 and D3. By Löb's theorem,** $\mathbf{GL} \subseteq \mathsf{PL}(\mathrm{Pr}_T)$ **and hence** $\mathbf{K4} \neq \mathsf{PL}(\mathrm{Pr}_T)$**.**

2. Suppose $\mathbf{KT} \subseteq \mathsf{PL}(\mathrm{Pr}_T)$**, then** $T \vdash \varphi$ **for any Gödel sentence** φ **of** $\mathrm{Pr}_T(x)$**, a contradiction.** □

Since the fixed point theorem holds for arithmetic, arithmetic can do some things that modal logic cannot do.

Montague, Syntactical treatments of modality, with corollaries on reflexion principles and finite axiomatizability, 1963.

Modal logics which cannot be a provability logic (2/2)

Theorem (K., 2018)

If $T \nvdash \neg \mathrm{Con}_T$, then $\mathsf{PL}(\mathrm{Pr}_T) \nsupseteq \mathbf{KB} \cap \mathbf{K5}$.

Proof.

Consider φ and ψ such that $T \vdash \varphi \leftrightarrow \mathrm{Pr}_T(\ulcorner \neg \varphi \urcorner)$ and $T \vdash \psi \leftrightarrow \neg \mathrm{Pr}_T(\ulcorner \neg \mathrm{Pr}_T(\ulcorner \neg \psi \urcorner) \urcorner)$. □

The following theorem is a refinement of Montague's theorem.

Theorem (K., 2020)

For any $\mathrm{Pr}_T(x)$, we have $\mathsf{PL}(\mathrm{Pr}_T) \nsupseteq \mathbf{KD4} \cap \mathbf{KD5} \cap \mathbf{KT}$.

Remark that $\mathbf{KD4} \cap \mathbf{KT}$ contains $\mathsf{PL}(\mathrm{Pr}^{\mathrm{Sh}}_{\mathsf{PA}})$.

K., Arithmetical soundness and completeness for Σ_2 numerations, 2018.

K., Rosser provability and normal modal logics, 2020.

Modal logics that can be a provability logic (1/2)

- **I already introduced the existence of a Fefermanian Σ_2 provability predicate $\mathrm{Pr}_\tau(x)$ such that $\mathbf{K} = \mathsf{PL}(\mathrm{Pr}_\tau)$.**
- **The same holds for a Σ_1 provability predicate.**

Theorem (K., 202x)

There exists a Σ_1 provability predicate $\mathrm{Pr}_T(x)$ of T such that $\mathbf{K} = \mathsf{PL}(\mathrm{Pr}_T)$.

Problem

For each $n \geq 2$, does there exist a Σ_1 provability predicate $\mathrm{Pr}_T(x)$ such that $\mathsf{PL}(\mathrm{Pr}_T) = \mathbf{wGL}_n$?

K., The provability logic of all provability predicates, to appear.

Modal logics that can be a provability logic (2/2)

- **It is known that there exists a Rosser's provability predicate satisfying D2.**
- **For such a $\mathrm{Pr}^{\mathrm{R}}_T(x)$, we have $\mathbf{KD} \subseteq \mathsf{PL}(\mathrm{Pr}^{\mathrm{R}}_T)$.**

Theorem (K., 2020)

There exists a Rosser's provability predicate $\mathrm{Pr}^{\mathrm{R}}_T(x)$ such that $\mathsf{PL}(\mathrm{Pr}^{\mathrm{R}}_T) = \mathbf{KD}$.

Also, there exists a Rosser's provability predicate whose provability logic properly contains KD.

Theorem (K., 2020)

There exists a Rosser's provability predicate $\mathrm{Pr}^{\mathrm{R}}_T(x)$ such that $\mathsf{PL}(\mathrm{Pr}^{\mathrm{R}}_T) \supseteq \mathbf{KD} + (\Box A \to \Box\Diamond A)$.

Problem

Does there exist a $\mathrm{Pr}^{\mathrm{R}}_T(x)$ such that $\mathsf{PL}(\mathrm{Pr}^{\mathrm{R}}_T) = \mathbf{KD} + (\Box A \to \Box\Diamond A)$?

K., Rosser provability and normal modal logics, 2020.

Provability predicates which does not satisfy **D2**

- **So far, I have only introduced provability predicates satisfying D2.**
- **There is a serious problem when considering provability logics of provability predicates that do not satisfy D2.**

- **If** $\mathrm{Pr}_T(x)$ **does not satisfy D2, then K** $\not\subseteq \mathsf{PL}(\mathrm{Pr}_T)$ **and so** $\mathsf{PL}(\mathrm{Pr}_T)$ **is a non-normal modal logic.**
- **This means that Kripke semantics cannot be used to investigate** $\mathsf{PL}(\mathrm{Pr}_T)$**.**
- **The proof of Solovay's theorem embeds corresponding Kripke models into arithmetic, and this proof method cannot be directly used as is.**

Recently, Kogure and I extended Solovay's method to non-normal modal logics having a relational semantics similar to Kripke semantics.

1. **The modal logic MN of provability predicates satisfying the monotonicity condition M.**
2. **The modal logic N of all provability predicates**

Monotonic modal logics of provability (1/5)

The axioms and the rules of the logic **MN**

Axioms **Tautologies.**

Rules **Modus Ponens** $\dfrac{A \quad A \to B}{B}$, **Monotonicity** $\dfrac{A \to B}{\Box A \to \Box B}$

and Necessitation $\dfrac{A}{\Box A}$.

Proposition

For any provability predicate $\mathrm{Pr}_T(x)$ **satisfying the monotonicy condition M** $\dfrac{\varphi \to \psi}{\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)}$, **we have** $\mathbf{MN} \subseteq \mathsf{PL}(\mathrm{Pr}_T)$.

Monotonic modal logics of provability (2/5)

Monotonic modal logics have a natural relational semantics.

Definition (MN-models)

A triple $(W, R, \Vdash)$ is called an MN-model $\overset{\text{def.}}{\iff}$

- W **is a non-empty set.**
- $R \subseteq W \times (\mathcal{P}(W) \setminus \{\emptyset\})$ **and** R **satisfies the following condition:**

$$w\ R\ V\ \&\ V \subseteq U \Rightarrow w\ R\ U.$$

- $\Vdash$ **is a binary relation beteen elements of** W **and modal formulas fulfilling the usual conditions for satisfaction and**

$$w \Vdash \Box A \iff \forall V \in \mathcal{P}(W)(w\ R\ V \Rightarrow \exists y \in V(y \Vdash A)).$$

This resembles Verbrugge semantics in the study of interpretability logic.

Monotonic modal logics of provability (3/5)

This semantics is essentially the same as neibourhood semantics. So, the following fact is known.

Fact (cf. Chellas, 1980)

MN is sound and complete w.r.t. the class of all MN-models. Also, MN has the finite model property w.r.t. MN-models.

By using this fact, we proved the arithmetical completeness theorem of MN.

Theorem (Kogure and K., 2023)

$$\mathbf{MN} = \bigcap \{\mathsf{PL}(\mathrm{Pr}_T) \mid \mathrm{Pr}_T \text{ is a provability predicate of } T \text{ satisfying } \mathbf{M}\}.$$

Moreover, there exists a Σ_1 provability predicate $\mathrm{Pr}_T(x)$ of T satisfying M such that $\mathbf{MN} = \mathsf{PL}(\mathrm{Pr}_T)$.

Kogure and K., Arithmetical completeness theorems for monotonic modal logics, 2023.

Monotonic modal logics of provability (4/5)

We also introduced the logic MN4.

Definition

$\mathbf{MN4} := \mathbf{MN} + (\Box A \to \Box\Box A)$.

By showing the finite model property of MN4 w.r.t. transitive MN-models, we also proved the following theorem.

Theorem (Kogure and K., 2023)

$$\mathbf{MN4} = \bigcap \{\mathsf{PL}(\mathrm{Pr}_T) \mid \mathrm{Pr}_T \textbf{ satisfies M and D3}\}.$$

Moreover, there exists a Σ_1 **provability predicate** $\mathrm{Pr}_T(x)$ **of** T **satisfying M such that** $\mathbf{MN4} = \mathsf{PL}(\mathrm{Pr}_T)$.

Monotonic modal logics of provability (5/5)

- **For provability predicates without D2, the consistency statements** Con_T **and** $\mathrm{Con}^{\mathrm{S}}_T$ **are not generally equivalent.**
- **The axioms** P **and** D **correspond to each of them, respectively.**

Definition (MNP and MND)

- $\mathbf{MNP} := \mathbf{MN} + \neg\Box\bot$
- $\mathbf{MND} := \mathbf{MN} + \neg(\Box A \land \Box\neg A)$

We could also distinguish these consistency statements in terms of arithmetical completeness.

Theorem (Kogure and K., 2023)

Let $L \in \{\mathbf{MNP}, \mathbf{MND}, \mathbf{MNP4}\}$**.**
Then, there exists a Rosser's provability predicate $\mathrm{Pr}^{\mathrm{R}}_T(x)$ **such that** $L = \mathsf{PL}(\mathrm{Pr}^{\mathrm{R}}_T)$**.**

Recall that if $\mathrm{Pr}_T(x)$ **satisfies M and D3, then** $T \nvdash \mathrm{Con}^{\mathrm{S}}_T$**.**

Problem

Does there exist a $\mathrm{Pr}_T(x)$ **such that** $\mathsf{PL}(\mathrm{Pr}_T) \supseteq \mathbf{MND4}$**?**

Modal logic of all provability predicates (1/5)

- **The only property common to all provability predicates is probably "$T \vdash \varphi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$".**
- **The pure logic of necessitation N was introduced and studied by Fitting, Marek, and Truszczyński (1992).**

The axioms and rules of the modal logic N

Axioms **Tautologies.**

Rules **Modus Ponens** $\dfrac{A \quad A \to B}{B}$ **and Necessitation** $\dfrac{A}{\Box A}$.

Proposition

For any provability predicate $\mathrm{Pr}_T(x)$ **of** T**, we have** $\mathbf{N} \subseteq \mathsf{PL}(\mathrm{Pr}_T)$.

Fitting, Marek and Truszczyński, The pure logic of necessitation, 1992.

Modal logic of all provability predicates (2/5)

The logic N has the following relational semantics.
Let MF denote the set of all modal formulas.

Definition (N-models)

A triple $(W, \{R_A\}_{A \in \mathsf{MF}}, \Vdash)$ is an N-model $\overset{\text{def.}}{\iff}$

- **W is a non-empty set.**
- **For each $A \in \mathsf{MF}$, R_A is a binary relation on W.**
- **$\Vdash$ a satisfaction relation on $W \times \mathsf{MF}$ fulfilling**

$$w \Vdash \Box A \iff \forall x \in W (w \; R_A \; x \Rightarrow x \Vdash A).$$

Theorem (Fitting, Marek, and Truszczyński, 1992)

N is sound and complete w.r.t. the class of all N-models.
Also, N has the finite model property w.r.t. N-models.

Modal logic of all provability predicates (3/5)

By using the finite model property of N, the following theorem saying that the logic N is the modal logic of all provability predicates is proved:

Theorem (K., 202x)

$$\mathbf{N} = \bigcap\{\mathsf{PL}(\mathrm{Pr}_T) \mid \mathrm{Pr}_T \text{ is a provability predicate of } T\}.$$

Moreover, there exists a Σ_1 provability predicate $\mathrm{Pr}_T(x)$ of T such that $\mathbf{N} = \mathsf{PL}(\mathrm{Pr}_T)$.

K., The provability logic of all provability predicates, to appear.

Modal logic of all provability predicates (4/5)

Definition

- $\mathbf{N4} := \mathbf{N} + (\Box A \to \Box\Box A)$
- $\mathbf{NR} := \mathbf{N} + \dfrac{\neg A}{\neg\Box A}$
- $\mathbf{NR4} := \mathbf{NR} + (\Box A \to \Box\Box A)$

By using the finite model property of each of these logics, the following arithmetical completeness theorem is also proved.

Theorem (K., 202x)

- **There exists a** Σ_1 **provability predicate** $\mathrm{Pr}_T(x)$ **such that** $\mathbf{N4} = \mathsf{PL}(\mathrm{Pr}_T)$.
- **For** $L \in \{\mathbf{NR}, \mathbf{NR4}\}$, **there exists a Rosser's provability predicate** $\mathrm{Pr}^{\mathrm{R}}_T(x)$ **such that** $L = \mathsf{PL}(\mathrm{Pr}^{\mathrm{R}}_T)$.

In particular, NR is the modal logic of all Rosser's provability predicates.

Modal logic of all provability predicates (5/5)

Recently, we have developed further extensions of N.

Definition

Let $m, n \geq 0$**.**

$$\mathbf{N}^+\mathbf{A}_{m,n} := \mathbf{N} + (\Box^n A \to \Box^m A) + \frac{\neg\Box A}{\neg\Box\Box A}$$

Theorem (K. and Sato)

For $m, n \geq 0$**,** $\mathbf{N}^+\mathbf{A}_{m,n}$ **has the finite model property w.r.t. the corresponding class of N-frames.**

The following theorem is a part of Kogure's ongoing work.

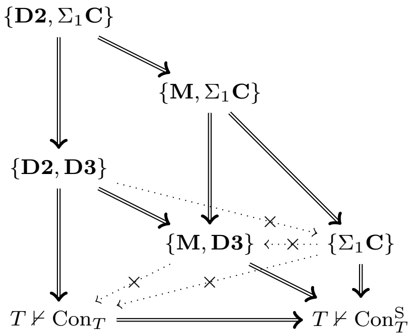
Theorem (Kogure)

For $m, n \geq 1$**, there exists a** Σ_1 **provability predicate** $\mathrm{Pr}_T(x)$ **such that** $\mathsf{PL}(\mathrm{Pr}_T) = \mathbf{N}^+\mathbf{A}_{m,n}$**.**

K. and Sato, The finite frame property of some extensions of the pure logic of necessitation, submitted.

Future Work (1/2)

I would like to understand the following diagram more precisely, both in terms of G2 and in terms of modal logic.



Future work (2/2)

Recently, I am trying to think about the following problems.

Future work 1

Can we prove arithmetical completeness for logics without relational semantics (e.g. EN)?

The axioms and the rules of **EN**

Axioms **Tautologies.**

Rules **Modus Ponens** $\dfrac{A \quad A \to B}{B}$, **Necessitation** $\dfrac{A}{\Box A}$ **and** $\dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$.

The condition Σ_1C is important in the study of G2.

Future work 2

Can we develop a modal logical study of provability predicates satisfying D3 but not Σ_1C?