On the second incompleteness theorem and provability predicates

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- **In this talk, T always denotes a consistent r.e. extension of Peano Arithmetic (PA) in the language of arithmetic.**

The second incompleteness theorem (G2)

T **cannot prove a sentence Con_T asserting the consistency of T.**

- **This statement of G2 is ambiguous because there are some sentences that seem to assert the consistency of T and are provable in T.**
- **So a precise statement of G2 requires more information on Con_T.**

In this talk, I investigate relationships between several versions of G2 and derivability conditions for provability predicates.

Definition

$\mathrm{Pr}_T(x)$ **is a provability predicate of T**
$:\iff$ **it is a Σ_1 formula and for any natural number n,**
$\mathbb{N} \models \mathrm{Pr}_T(n) \iff n$ **is the Gödel number of a theorem of T.**

Outline

1. **Several versions of G2**
   1. **Gödel (1931)**
   2. **Hilbert and Bernays (1939)**
   3. **Löb (1955)**
   4. **Jeroslow (1973)**
   5. **Montagna (1979)**
   6. **Buchholz (1993)**
2. **Derivability conditions**
3. **My results**

Gödel (1931)

Gödel's second incompleteness theorem

$T \nvdash \exists x(\mathrm{Fml}(x) \land \neg \mathrm{Pr}_T(x))$

- **In his famous paper, Gödel proved G2 with only a sketched proof.**
- **Gödel explained that by formalizing his proof of the first incompleteness theorem, G2 is proved.**
- **To carry out his idea, it is desirable that the formula $\mathrm{Pr}_T(x)$ enjoys some natural properties as a formalization of the notion of T-provability.**
- **He wrote that a detailed proof would be presented in a forthcoming work, but such a paper was not published after all.**

Hilbert and Bernays (1939)

- **The first detailed proof of G2 was presented in the second volume of** *Grundlagen der Mathematik* **by Hilbert and Bernays.**
- **Especially, they proved that if $\mathrm{Pr}_T(x)$ satisfies the following conditions $\mathbf{HB1}$, $\mathbf{HB2}$ and $\mathbf{HB3}$, then $T \nvdash \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$.**

Hilbert–Bernays' derivability conditions

HB1 $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.

HB2 $T \vdash \mathrm{Pr}_T(\ulcorner \neg \varphi(x) \urcorner) \to \mathrm{Pr}_T(\ulcorner \neg \varphi(\dot{x}) \urcorner)$.

HB3 $T \vdash t(x) = 0 \to \mathrm{Pr}_T(\ulcorner t(\dot{x}) = 0 \urcorner)$ **for every primitive recursive term** $t(x)$.

$\ulcorner \varphi(\dot{x}) \urcorner$ **is a primitive recursive term corresponding to a primitive recursive function calculating the Gödel number of** $\varphi(\overline{n})$ **from** n.

Löb (1955)

- **Löb proved that if $\mathrm{Pr}_T(x)$ satisfies the following conditions $\mathbf{D1}$, $\mathbf{D2}$ and $\mathbf{D3}$, then Löb's theorem holds, that is, for any sentence φ,** $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi \Rightarrow T \vdash \varphi$.
- **Then** $T \nvdash \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.
  This is the most well-known form of G2.

Löb's derivability conditions

- **D1** $T \vdash \varphi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.
- **D2** $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$.
- **D3** $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$.

Every $\mathrm{Pr}_T(x)$ automatically satisfies $\mathbf{D1}$.

Jeroslow (1973)

- **Jeroslow proved that if $\mathrm{Pr}_T(x)$ satisfies the following condition, then $T \nvdash \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$.**

Jeroslow's condition

$T \vdash \mathrm{Pr}_T(t) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(t) \urcorner)$ **for all primitive recursive terms t.**

- **Jeroslow's argument also shows that if $\mathrm{Pr}_T(x)$ satisfies the following condition $\mathbf{\Sigma_1 C}$, then $T \nvdash \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$.**

Provable Σ_1-completeness

$\mathbf{\Sigma_1 C}$ **If φ is a Σ_1 sentence, then $T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.**

Montagna (1979)

- **Montagna proved that if** $\mathrm{Pr}_T(x)$ **satisfies the following two conditions, then Löb's theorem holds.**
- **In this case,** $T \nvdash \exists x(\mathrm{Fml}(x) \land \neg \mathrm{Pr}_T(x))$**.**

Montagna's conditions

- $T \vdash \forall x($**"**x **is a logical axiom"** $\to \mathrm{Pr}_T(x))$**.**
- $T \vdash \forall x \forall y(\mathrm{Fml}(x) \land \mathrm{Fml}(y) \to (\mathrm{Pr}_T(x \dot{\to} y) \to (\mathrm{Pr}_T(x) \to \mathrm{Pr}_T(y))))$**.**

Buchholz (1993)

- **In Buchholz's lecture note, it is proved that if $\mathrm{Pr}_T(x)$ satisfies the following condition, then it also satisfies $\mathbf{D2}$ and $\mathbf{\Sigma_1 C}$.**
- **Then $T \nvdash \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.**

Buchholz's condition

For all $m \geq 1$,

$$T \vdash \bigwedge_{0<i<m} \varphi_i(x) \to \varphi_m(x)$$

$$\Rightarrow T \vdash \bigwedge_{0<i<m} \mathrm{Pr}_T(\ulcorner \varphi_i(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi_m(\dot{x}) \urcorner).$$

$\mathrm{Pr}_T(x)$ **satisfies Buchholz's condition iff** $\mathrm{Pr}_T(x)$ **satisfies both $\mathbf{D1^U}$ and $\mathbf{D2^U}$.**

$\mathbf{D1^U}$ $T \vdash \varphi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathbf{D2^U}$ $T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner))$.

These different versions of G2 have different consequences.

Different consistency statements

- $\mathrm{Con}_T^H :\equiv \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$
- $\mathrm{Con}_T^L :\equiv \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$
- $\mathrm{Con}_T^G :\equiv \exists x(\mathrm{Fml}(x) \land \neg \mathrm{Pr}_T(x))$

Different consequences

Gödel $T \nvdash \mathrm{Con}_T^G$

Hilbert–Bernays $T \nvdash \mathrm{Con}_T^H$

Löb $T \nvdash \mathrm{Con}_T^L$

Jeroslow $T \nvdash \mathrm{Con}_T^H$

Montagna $T \nvdash \mathrm{Con}_T^G$

Buchholz $T \nvdash \mathrm{Con}_T^L$

- $\mathrm{PA} \vdash \mathrm{Con}_T^H \to \mathrm{Con}_T^L$ **and** $\mathrm{PA} \vdash \mathrm{Con}_T^L \to \mathrm{Con}_T^G$.
- **I wanted to clarify the situation.**

1. A brief history
2. **Derivability conditions**
3. My results

Local derivability conditions

Local derivbility conditions

D1 $T \vdash \varphi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

D2 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$.

D3 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$.

ΓC If φ is a Γ sentence, then $T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

$\mathbf{B_2}$ $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.

PC $T \vdash \mathrm{Pr}_\emptyset(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

$\mathrm{Pr}_\emptyset(x)$ is a provability predicate of pure predicate calculus.

Uniform derivability conditions

Uniform derivbility conditions

$\mathbf{D1^U}$ $T \vdash \varphi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathbf{D2^U}$ $T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner))$.

$\mathbf{D3^U}$ $T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \urcorner)$.

$\mathbf{\Gamma C^U}$ **If** $\varphi(x)$ **is a** Γ **formula, then** $T \vdash \varphi(x) \to \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathbf{B_2^U}$ $T \vdash \varphi(x) \to \psi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner)$.

$\mathbf{PC^U}$ $T \vdash \mathrm{Pr}_\emptyset(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

CB $T \vdash \mathrm{Pr}_T(\ulcorner \forall x\, \varphi(x) \urcorner) \to \forall x\, \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

Global derivability conditions

Global derivbility conditions

$$\mathbf{D2^G} \quad T \vdash \forall x \forall y (\mathrm{Fml}(x) \land \mathrm{Fml}(y) \to (\mathrm{Pr}_T(x \dot{\to} y) \to (\mathrm{Pr}_T(x) \to \mathrm{Pr}_T(y)))).$$

$$\mathbf{\Gamma C^G} \quad T \vdash \forall x (\mathsf{True}_\Gamma(x) \to \mathrm{Pr}_T(x)).$$

$$\mathbf{PC^G} \quad T \vdash \forall x (\mathrm{Fml}(x) \to (\mathrm{Pr}_\emptyset(x) \to \mathrm{Pr}_T(x))).$$

$\mathsf{True}_\Gamma(x)$ **is a formula saying that "x is a true Γ sentence".**

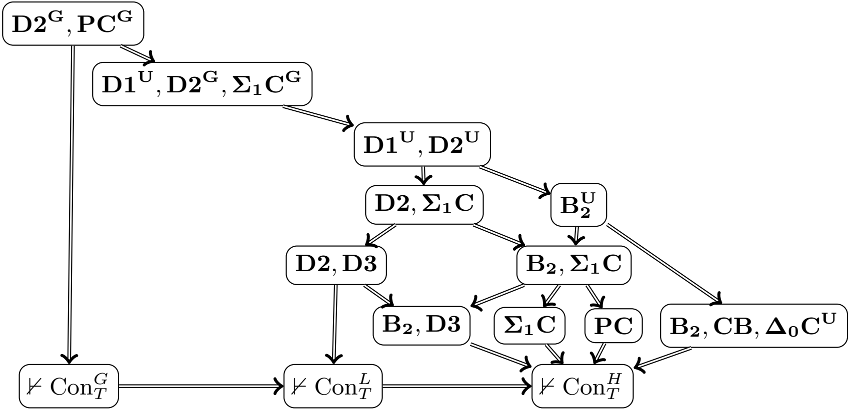
Remark

Global $\Rightarrow$ Uniform $\Rightarrow$ Local.

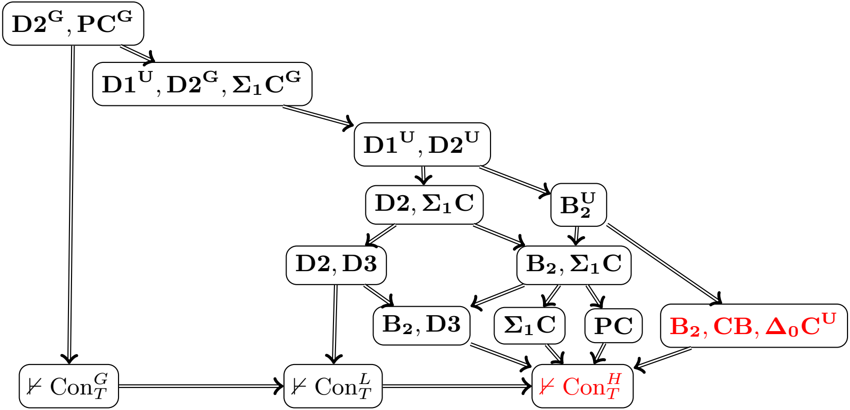
Known results

Hilbert–Bernays	$\mathbf{B_2}, \mathbf{CB}, \mathbf{\Delta_0 C^U} \Rightarrow T \nvdash \mathrm{Con}_T^H$
Löb	$\mathbf{D2}, \mathbf{D3} \Rightarrow T \nvdash \mathrm{Con}_T^L$
Jeroslow	$\mathbf{\Sigma_1 C} \Rightarrow T \nvdash \mathrm{Con}_T^H$
Montagna	$\mathbf{D2^G}, \mathbf{PC^G} \Rightarrow T \nvdash \mathrm{Con}_T^G$
Buchholz	$\mathbf{D1^U}, \mathbf{D2^U} \Rightarrow \mathbf{\Sigma_1 C^U}$

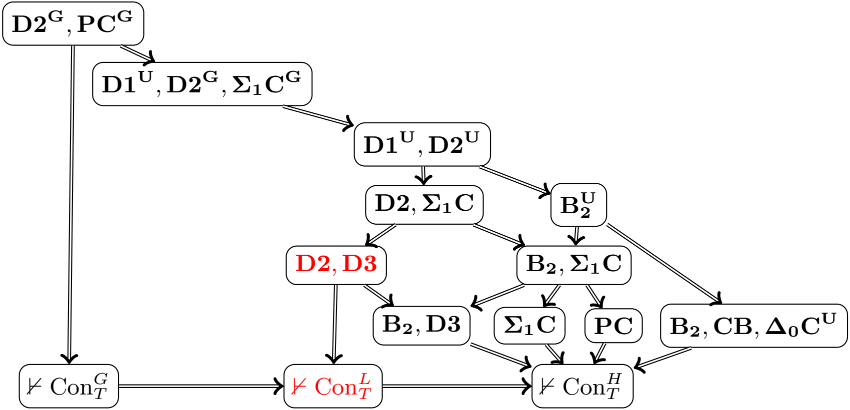
Implications between prominent sets of conditions



Hilbert and Bernays (1939)

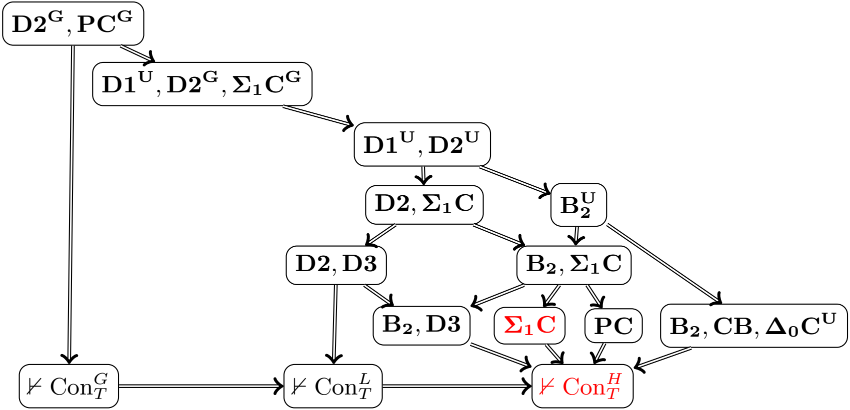


Löb (1955)

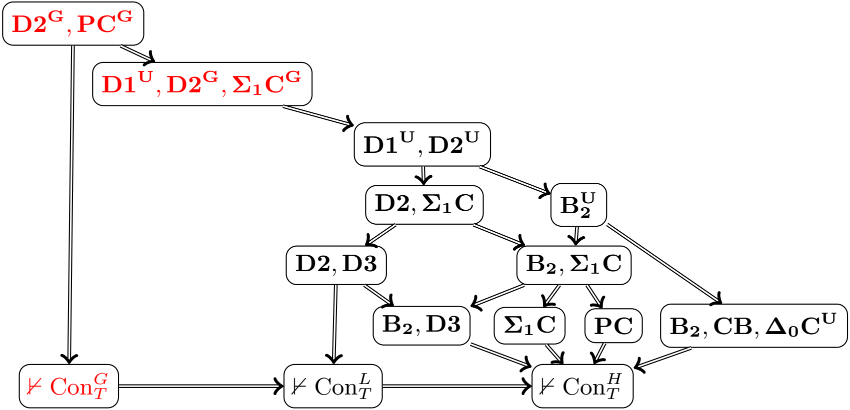


$\mathbf{D2^G, PC^G}$
$\mathbf{D1^U, D2^G, \Sigma_1 C^G}$
$\mathbf{D1^U, D2^U}$
$\mathbf{D2, \Sigma_1 C}$
$\mathbf{B_2^U}$
$\mathbf{D2, D3}$
$\mathbf{B_2, \Sigma_1 C}$
$\mathbf{B_2, D3}$
$\mathbf{\Sigma_1 C}$
$\mathbf{PC}$
$\mathbf{B_2, CB, \Delta_0 C^U}$
$\nvdash \mathrm{Con}_T^G$
$\nvdash \mathrm{Con}_T^L$
$\nvdash \mathrm{Con}_T^H$

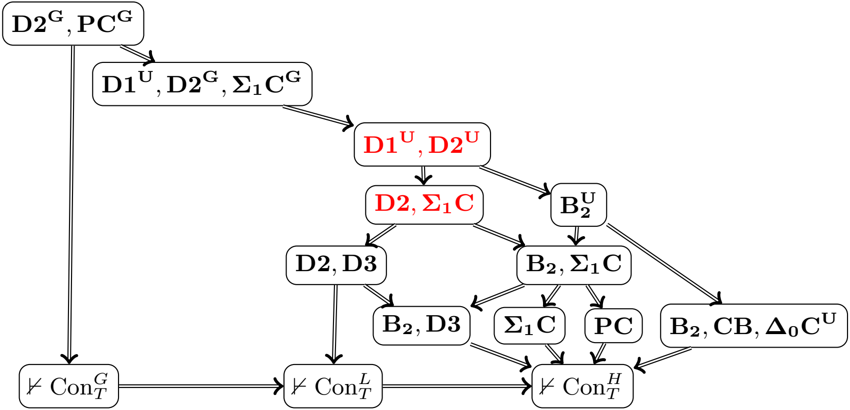
Jeroslow (1973)



Montagna (1979)



Buchholz (1993)



1. **A brief history**
2. **Derivability conditions**
3. **My results**

New sufficient conditions

I found two sets of sufficient conditions for the unprovability of Con_T^H.

Theorem

- $\mathbf{B_2}, \mathbf{D3} \Rightarrow T \nvdash \mathrm{Con}_T^H$
- $\mathbf{PC} \Rightarrow T \nvdash \mathrm{Con}_T^H$

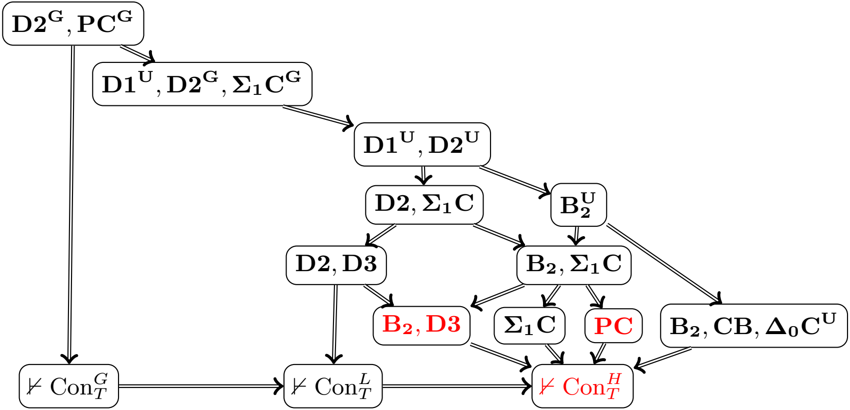
$\mathbf{B_2}$ $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.

$\mathbf{D3}$ $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$.

$\mathbf{PC}$ $T \vdash \mathrm{Pr}_\emptyset(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

$$\mathrm{Con}_T^H \equiv \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$$

New sufficient conditions



An improvement of Buchholz's result

Buchholz's result is stated precisely as follows:

Theorem (Buchholz)

$\mathbf{D1^U}, \mathbf{D2^U} \Rightarrow \mathbf{\Sigma_1 C^U}$

I proved that only the $m = 2$ case of Buchholz's conditions is sufficient for $\mathbf{\Sigma_1 C^U}$.

Theorem

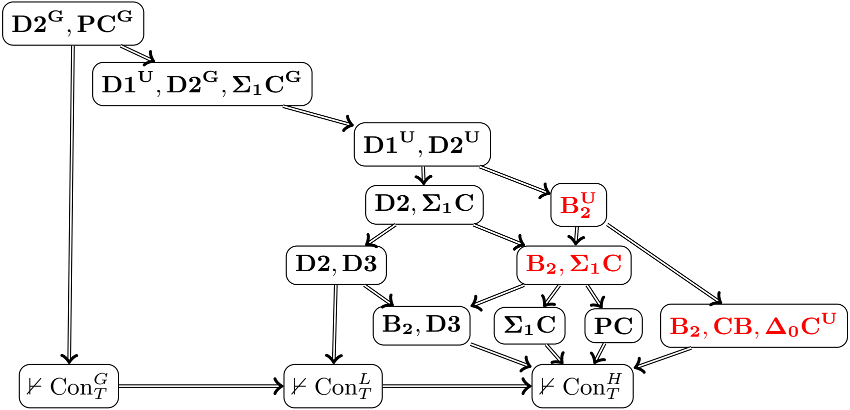
$\mathbf{B_2^U} \Rightarrow \mathbf{\Sigma_1 C^U}$

$$\mathbf{B_2^U} \quad T \vdash \varphi(x) \to \psi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner).$$

Corollary

$\mathbf{B_2^U} \Rightarrow T \nvdash \mathrm{Con}_T^H$

An improvement of Buchholz's result



Non-implications

- **I am also interested in non-implications between sets of condtions.**
- **For example, I pay attention to Rosser's provability predicate:**

$$\mathrm{Pr}_T(x) \equiv \exists y(\mathrm{Prf}_T(x, y) \land \forall z < y \neg \mathrm{Prf}_T(\dot{\neg} x, z)).$$

- **This is because** PA **proves** Con_T^L **of Rosser's provability predicates.**

Theorem (Arai, 1990)

- **There exists a Rosser provability predicate of** T **satisfying** $\mathbf{D2^G}$**.**
- **There exists a Rosser provability predicate of** T **satisfying** $\mathbf{D3^G}$**.**

- **Therefore each of** $\mathbf{D2^G}$ **and** $\mathbf{D3^G}$ **is not sufficient for** $T \nvdash \mathrm{Con}_T^L$**.**

I extended Arai's results and showed that some sets of conditions are not sufficient for $T \nvdash \mathrm{Con}_T^L$**.**

Theorem A

This is an improvement of Arai's first result.

Theorem A

There exists a Rosser provability predicate $\mathrm{Pr}_T(x)$ of T satisfying $\mathbf{D2^G}$, $\mathbf{\Delta_0 C^G}$ and $\mathrm{PA} \vdash \mathrm{Con}_T^H$. That is,

- $\mathrm{PA} \vdash \forall x \forall y (\mathrm{Pr}_T(x \dot{\to} y) \to (\mathrm{Pr}_T(x) \to \mathrm{Pr}_T(y)))$.
- $\mathrm{PA} \vdash \forall x (\mathsf{True}_{\Delta_0}(x) \to \mathrm{Pr}_T(x))$.
- $\mathrm{PA} \vdash \forall x (\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg}(x)))$.

- $\{\mathbf{D2}, \mathbf{D3}\} \Rightarrow \quad T \nvdash \mathrm{Con}_T^L$ **(Löb)**
- $\{\mathbf{D2^G}, \mathbf{\Delta_0 C^G}\} \not\Rightarrow \quad T \nvdash \mathrm{Con}_T^L$ **(From Theorem A)**

Theorem B

There exists a Rosser provability predicate satisfying Hilbert–Bernays' derivability conditions.

Theorem B

There exists a Rosser provability predicate $\mathrm{Pr}_T(x)$ **of** T **satisfying** $\mathbf{CB}$, $\mathbf{D2}$ **and** $\mathbf{\Delta_0 C^G}$. **That is,**

- $T \vdash \mathrm{Pr}_T(\ulcorner \forall x \varphi(x) \urcorner) \to \forall x \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.
- $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$.
- $\mathrm{PA} \vdash \forall x (\mathsf{True}_{\Delta_0}(x) \to \mathrm{Pr}_T(x))$.

- $\{\mathbf{CB}, \mathbf{B_2}, \mathbf{\Delta_0 C^U}\} \Rightarrow \quad T \nvdash \mathrm{Con}_T^H$ **(Hilbert–Bernays)**
- $\{\mathbf{CB}, \mathbf{D2}, \mathbf{\Delta_0 C^G}\} \not\Rightarrow \quad T \nvdash \mathrm{Con}_T^L$ **(From Theorem B)**

- $\{\mathbf{D1^U}, \mathbf{D2^U}\} \Rightarrow \mathbf{\Sigma_1 C^U}$ **(Buchholz)**
- $\{\mathbf{D1^U}, \mathbf{D2}\} \not\Rightarrow \quad \mathbf{\Sigma_1 C}$ **(From Theorem B)**

Theorem C

This is an improvement of Arai's second result.

Theorem C

There exists a Rosser provability predicate $\mathrm{Pr}_T(x)$ of T satisfying $\mathbf{CB}$, $\mathbf{B_2}$, $\mathbf{D3^G}$ and $\mathbf{\Delta_0 C^G}$. That is,

- $T \vdash \mathrm{Pr}_T(\ulcorner \forall x \varphi(x) \urcorner) \to \forall x \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.
- $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.
- $\mathrm{PA} \vdash \forall x(\mathrm{Pr}_T(x) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\dot{x}) \urcorner))$.
- $\mathrm{PA} \vdash \forall x(\mathrm{True}_{\Delta_0}(x) \to \mathrm{Pr}_T(x))$.

- $\{\mathbf{D2}, \mathbf{D3}\} \Rightarrow \quad T \nvdash \mathrm{Con}_T^L$ **(Löb)**
- $\{\mathbf{B_2}, \mathbf{D3^G}\} \not\Rightarrow \quad T \nvdash \mathrm{Con}_T^L$ **(From Therem C)**

- **Moreover, I also contructed some (artificial) provability predicates satisfying some conditions but not satisfying others.**

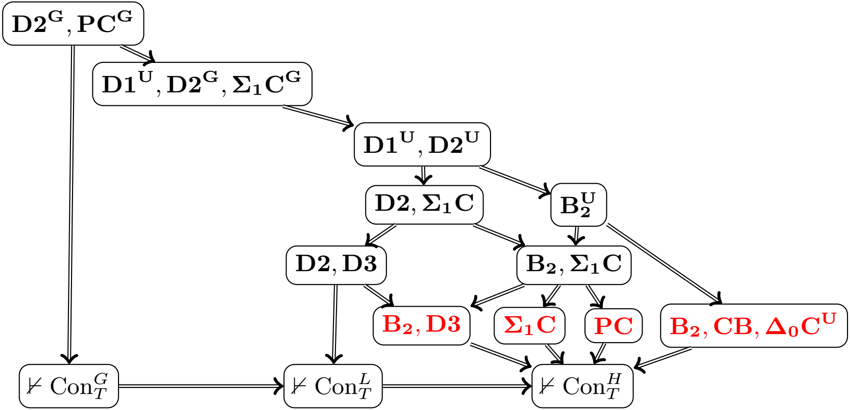
For example,

Theorem

There exists a provability predicate $\mathrm{Pr}_T(x)$ of T which satisfies $\mathbf{\Sigma_1 C^G}$, but does not satisfy any of $\mathbf{D1^U}$ and $\mathbf{PC}$.

- **I present some non-implications in these that relate to the previous figure.**

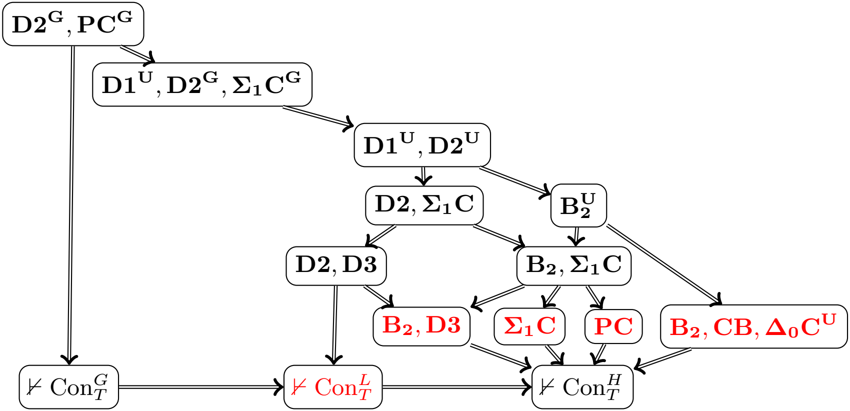
Non-implications 1



Theorem

$\{\mathbf{B_2}, \mathbf{D3}\}$, $\{\mathbf{\Sigma_1 C}\}$, $\{\mathbf{PC}\}$ **and** $\{\mathbf{B_2}, \mathbf{CB}, \mathbf{\Delta_0 C}^{\mathbf{U}}\}$ **are pairwise incomparable.**

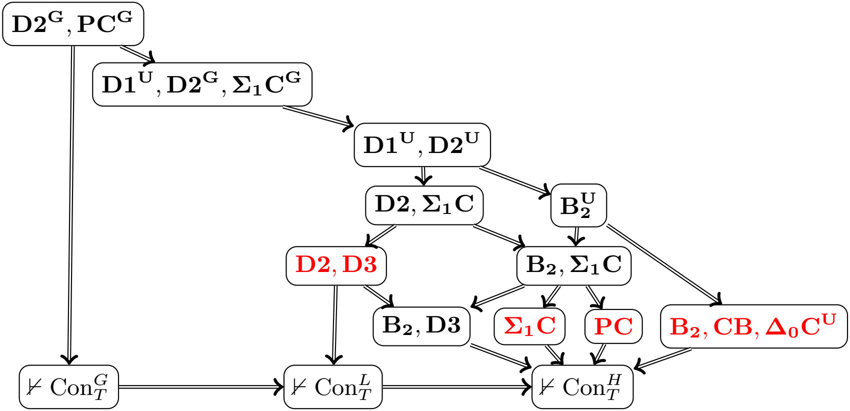
Non-implications 2



Theorem

Each of $\{\mathrm{B_2, D3}\}$, $\{\Sigma_1\mathrm{C}\}$, $\{\mathrm{PC}\}$ and $\{\mathrm{B_2, CB}, \Delta_0\mathrm{C^U}\}$ is not sufficient for $T \nvdash \mathrm{Con}_T^L$.

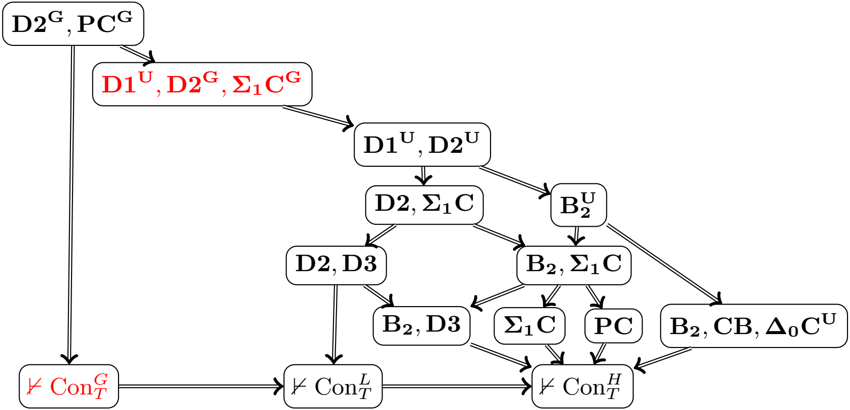
Non-implications 3



Theorem

$\{\mathbf{D2}, \mathbf{D3}\}$ **does not imply any of** $\{\mathbf{\Sigma_1 C}\}$, $\{\mathbf{PC}\}$ **and** $\{\mathbf{B_2}, \mathbf{CB}, \mathbf{\Delta_0 C^U}\}$.

Non-implication 4



Theorem

$\{\mathbf{D1^U}, \mathbf{D2^G}, \mathbf{\Sigma_1 C^G}\}$ **is not sufficient for** $T \nvdash \mathrm{Con}_T^G$.

This shows that both of Hilbert–Bernays' conditions and Löb's conditions do not accomplish Gödel's original statement of G2.

- **G2 is a collection of theorems that claims the unprovability of Con_T.**
- **I constructed several artificial provability predicates, and it is not easy to specify the range of provability predicates to be treated in G2.**
- **Thus, the problem of what is an exact statement of G2 is still unclear.**
- **Is there any general principle behind these different versions of G2?**

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