Derivability conditions and the second incompleteness theorem

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Outline

1. **A brief history**
2. **Derivability conditions**
3. **Our results**

1. **A brief history**
   1. **Gödel (1931)**
   2. **Hilbert and Bernays (1939)**
   3. **Löb (1955)**
   4. **Jeroslow (1973)**
   5. **Montagna (1979)**
   6. **Buchholz (1993)**
2. **Derivability conditions**
3. **Our results**

- **In this talk, T always denotes a consistent r.e. extension of Peano Arithmetic PA in the language of arithmetic.**
- **$\mathrm{Pr}_T(x)$ always denotes a Σ_1 provability predicate of T, that is, for any $n \in \omega$,**
  $\mathrm{PA} \vdash \mathrm{Pr}_T(\overline{n}) \iff n$ is the Gödel number of some theorem of T.

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Gödel (1931)

- **In his famous paper, Gödel proved the second incompleteness theorem (G2) with only a sketched proof.**
- **Gödel explained that by formalizing his proof of the first incompleteness theorem, the consistency statement $\exists x(\mathrm{Fml}(x) \land \neg \mathrm{Pr}_T(x))$ cannot be proved in T.**

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Hilbert and Bernays (1939)

- **The first detailed proof of G2 was presented in the second volume of *Grundlagen der Mathematik* by Hilbert and Bernays.**
- **Especially, they proved that if $\mathrm{Pr}_T(x)$ satisfies the following conditions HB1, HB2 and HB3, then $T \nvdash \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$.**

Hilbert-Bernays' derivability conditions

HB1 $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.

HB2 $T \vdash \mathrm{Pr}_T(\ulcorner \neg \varphi(x) \urcorner) \to \mathrm{Pr}_T(\ulcorner \neg \varphi(\dot{x}) \urcorner)$.

HB3 $T \vdash t(x) = 0 \to \mathrm{Pr}_T(\ulcorner t(\dot{x}) = 0 \urcorner)$ **for every primitive recursive term** $t(x)$.

$\ulcorner \varphi(\dot{x}) \urcorner$ **is a primitive recursive term corresponding to a primitive recursive function calculating the Gödel number of** $\varphi(\overline{n})$ **from** n.

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Löb (1955)

- **Löb proved that if $\mathrm{Pr}_T(x)$ satisfies the following conditions D1, D2 and D3, then Löb's theorem holds, that is, for any formula φ, $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi \Rightarrow T \vdash \varphi$.**
- **Then $T \nvdash \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.**

Löb's derivability conditions

D1 $T \vdash \varphi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

D2 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$.

D3 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$.

Every provability predicate automatically satisfies D1.

Jeroslow (1973)

- **Jeroslow proved that if $\mathrm{Pr}_T(x)$ satisfies the following condition, then $T \nvdash \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$.**

Jeroslow's condition

$T \vdash \mathrm{Pr}_T(t) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(t) \urcorner)$ for all primitive recursive terms t.

- **Jeroslow's argument also shows that if $\mathrm{Pr}_T(x)$ satisfies the following condition $\Sigma_1\mathrm{C}$, then $T \nvdash \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$.**

Provable Σ_1-completeness

$\Sigma_1\mathrm{C}$ If φ is a Σ_1 sentence, then $T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

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Provable Σ_1-completeness

Σ_1C **If φ is a Σ_1 sentence, then $T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.**

Montagna (1979)

- **Montagna proved that if $\mathrm{Pr}_T(x)$ satisfies the following two conditions, then Löb's theorem holds.**

Montagna's conditions

- $T \vdash \forall x($"x is a logical axiom" $\rightarrow \mathrm{Pr}_T(x))$.
- $T \vdash \forall x \forall y(\mathrm{Fml}(x) \land \mathrm{Fml}(y) \rightarrow (\mathrm{Pr}_T(x \dot{\rightarrow} y) \rightarrow (\mathrm{Pr}_T(x) \rightarrow \mathrm{Pr}_T(y))))$.

- **By Montagna's argument, it can be shown that these conditions also imply "$T \nvdash \exists x(\mathrm{Fml}(x) \land \neg \mathrm{Pr}_T(x))$".**

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Montagna's conditions

- $T \vdash \forall x(\text{“}x \text{ is a logical axiom”} \to \mathrm{Pr}_T(x))$.
- $T \vdash \forall x \forall y(\mathrm{Fml}(x) \land \mathrm{Fml}(y) \to (\mathrm{Pr}_T(x \dot{\to} y) \to (\mathrm{Pr}_T(x) \to \mathrm{Pr}_T(y))))$.

- **By Montagna's argument, it can be shown that these conditions also imply “$T \nvdash \exists x(\mathrm{Fml}(x) \land \neg \mathrm{Pr}_T(x))$”.**

Buchholz (1993)

- In Buchholz's lecture note, it is proved that if $\mathrm{Pr}_T(x)$ satisfies the following condition, then it also satisfies D2 and Σ_1C.
- Then $T \nvdash \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

Buchholz's condition

For all $m \geq 1$,

$$T \vdash \bigwedge_{0<i<m} \varphi_i(x) \to \varphi_m(x)$$

$$\Rightarrow T \vdash \bigwedge_{0<i<m} \mathrm{Pr}_T(\ulcorner \varphi_i(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi_m(\dot{x}) \urcorner).$$

$\mathrm{Pr}_T(x)$ satisfies Buchholz's condition iff $\mathrm{Pr}_T(x)$ satisfies both $\mathrm{D1}^{\mathrm{U}}$ and $\mathrm{D2}^{\mathrm{U}}$.

$\mathrm{D1}^{\mathrm{U}}$ $T \vdash \varphi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

$\mathrm{D2}^{\mathrm{U}}$ $T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner))$.

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These different versions of G2 have different consequences.

Different consistency statements

- $\mathrm{Con}_T^H :\equiv \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$
- $\mathrm{Con}_T^L :\equiv \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$
- $\mathrm{Con}_T^G :\equiv \exists x(\mathrm{Fml}(x) \land \neg \mathrm{Pr}_T(x))$

Different consequences

Gödel $T \nvdash \mathrm{Con}_T^G$

Hilbert-Bernays $T \nvdash \mathrm{Con}_T^H$

Löb $T \nvdash \mathrm{Con}_T^L$

Jeroslow $T \nvdash \mathrm{Con}_T^H$

Montagna $T \nvdash \mathrm{Con}_T^G$

Buchholz $T \nvdash \mathrm{Con}_T^L$

- PA $\vdash \mathrm{Con}_T^H \to \mathrm{Con}_T^L$ and PA $\vdash \mathrm{Con}_T^L \to \mathrm{Con}_T^G$.
- We want to clarify the situation.

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Different consequences

Gödel	$T \nvdash \mathrm{Con}_T^G$
Hilbert-Bernays	$T \nvdash \mathrm{Con}_T^H$
Löb	$T \nvdash \mathrm{Con}_T^L$
Jeroslow	$T \nvdash \mathrm{Con}_T^H$
Montagna	$T \nvdash \mathrm{Con}_T^G$
Buchholz	$T \nvdash \mathrm{Con}_T^L$

- $\mathrm{PA} \vdash \mathrm{Con}_T^H \to \mathrm{Con}_T^L$ and $\mathrm{PA} \vdash \mathrm{Con}_T^L \to \mathrm{Con}_T^G$.
- We want to clarify the situation.

1. A brief history
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Local derivability conditions

Local derivbility conditions

D1 $T \vdash \varphi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

D2 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$.

D3 $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$.

ΓC If φ is a Γ sentence, then $T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

B_2 $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.

PL $T \vdash \mathrm{Pr}_\emptyset(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

Uniform derivability conditions

Uniform derivbility conditions

D1$^{\mathrm{U}}$ $T \vdash \varphi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

D2$^{\mathrm{U}}$ $T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner))$.

D3$^{\mathrm{U}}$ $T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \urcorner)$.

ΓC$^{\mathrm{U}}$ If $\varphi(x)$ is a Γ formula, then $T \vdash \varphi(x) \to \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

B$_2^{\mathrm{U}}$ $T \vdash \varphi(x) \to \psi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner)$.

PL$^{\mathrm{U}}$ $T \vdash \mathrm{Pr}_\emptyset(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

CB $T \vdash \mathrm{Pr}_T(\ulcorner \forall x\, \varphi(x) \urcorner) \to \forall x\, \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

Global derivability conditions

Global derivbility conditions

D2$^{\mathbf{G}}$ $T \vdash \forall x \forall y(\mathrm{Fml}(x) \land \mathrm{Fml}(y) \to (\mathrm{Pr}_T(x \dot{\to} y) \to (\mathrm{Pr}_T(x) \to \mathrm{Pr}_T(y))))$.

ΓC$^{\mathbf{G}}$ $T \vdash \forall x(\mathsf{True}_\Gamma(x) \to \mathrm{Pr}_T(x))$.

PL$^{\mathbf{G}}$ $T \vdash \forall x(\mathrm{Fml}(x) \to (\mathrm{Pr}_\emptyset(x) \to \mathrm{Pr}_T(x)))$.

$\mathsf{True}_\Gamma(x)$ is a formula saying that "x is a true Γ sentence".

Remark

Global ⇒ Uniform ⇒ Local.

Global derivability conditions

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Remark

Global $\Rightarrow$ Uniform $\Rightarrow$ Local.

Known results

Hilbert-Bernays $\mathbf{B_2, CB, \Delta_0C^U} \Rightarrow T \nvdash \mathrm{Con}_T^H$

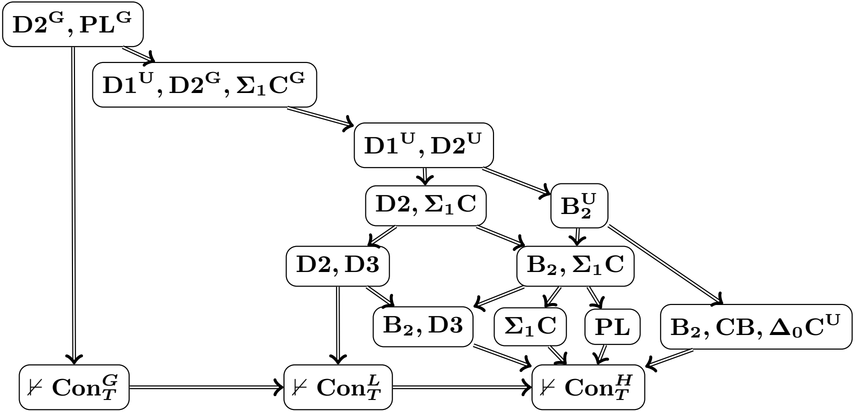
Löb $\mathbf{D2, D3} \Rightarrow T \nvdash \mathrm{Con}_T^L$

Jeroslow $\mathbf{\Sigma_1C} \Rightarrow T \nvdash \mathrm{Con}_T^H$

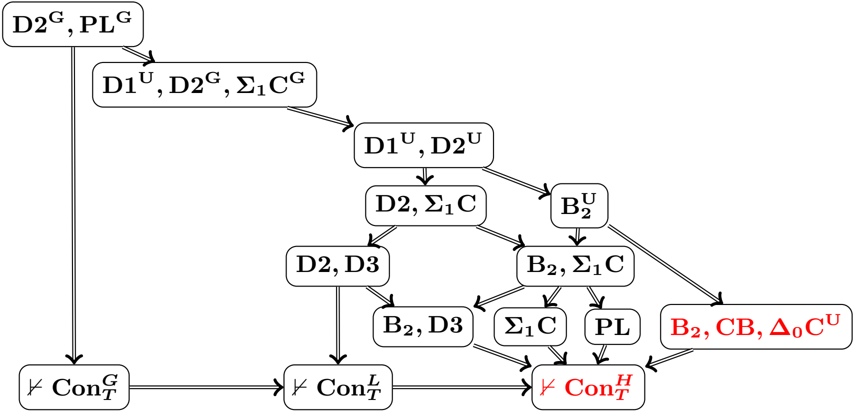
Montagna $\mathbf{D2^G, PL^G} \Rightarrow T \nvdash \mathrm{Con}_T^G$

Buchholz $\mathbf{D1^U, D2^U} \Rightarrow \mathbf{\Sigma_1C^U}$

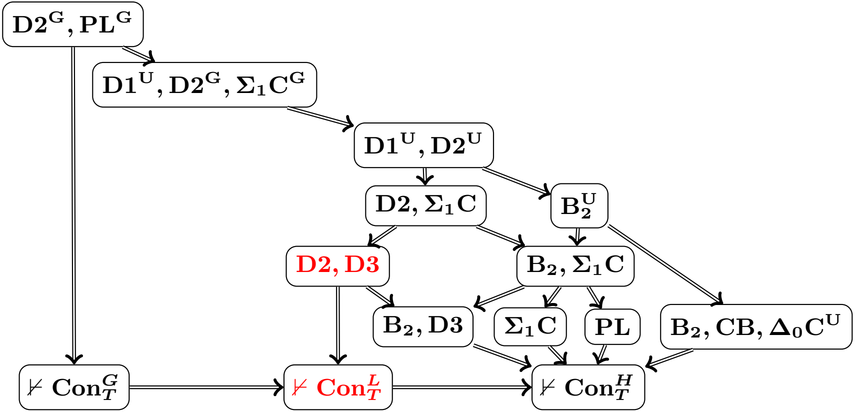
Implications between prominent sets of conditions



Hilbert and Bernays



Löb


D2^G, PL^G
D1^U, D2^G, Σ1C^G
D1^U, D2^U
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B2^U
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B2, CB, Δ0C^U
⊬ Con_T^G
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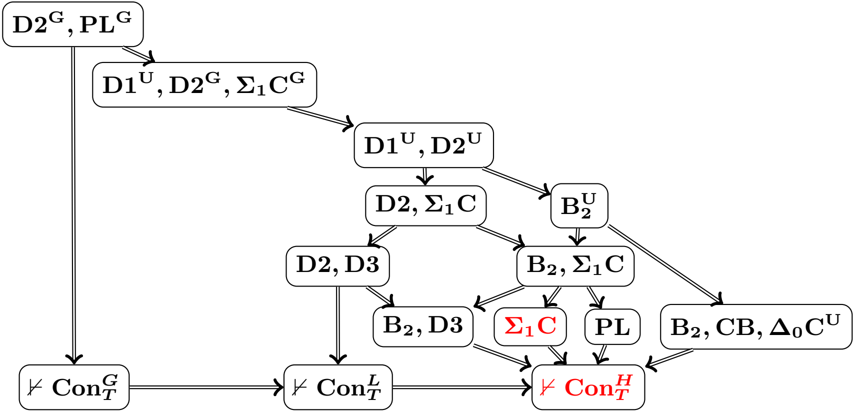
Jeroslow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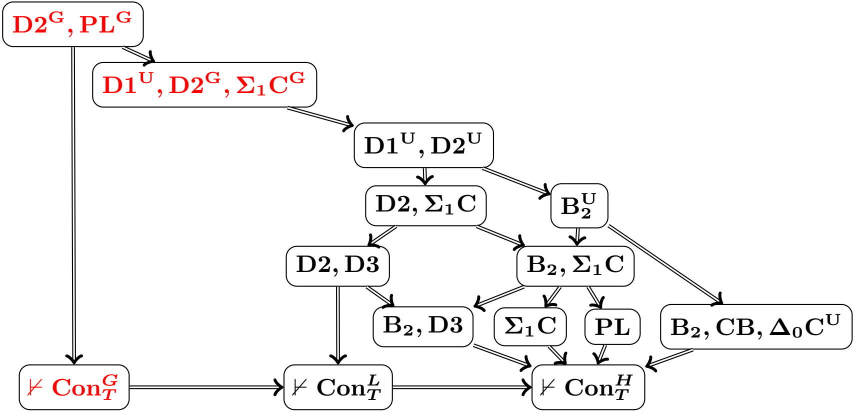
Montagna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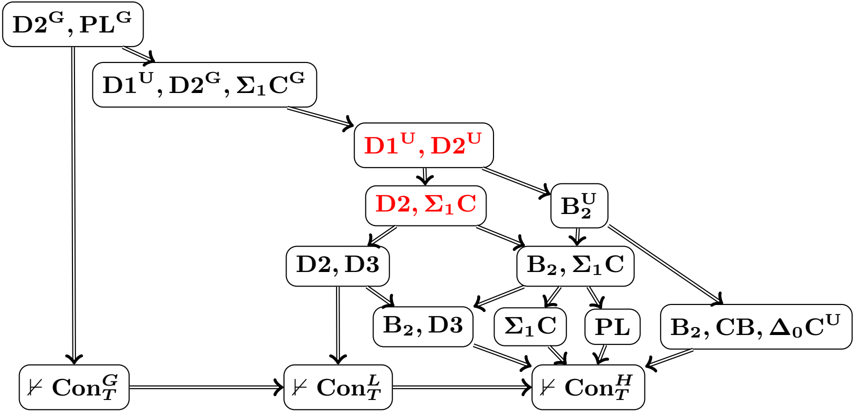
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3. **Results**

New sufficient conditions

Theorem (K.)

- $\mathbf{B_2}, \mathbf{D3} \Rightarrow T \nvdash \mathrm{Con}_T^H$
- $\mathbf{PL} \Rightarrow T \nvdash \mathrm{Con}_T^H$

B₂ $T \vdash \varphi \to \psi \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)$.

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$$\mathrm{Con}_T^H \equiv \forall x(\mathrm{Fml}(x) \land \mathrm{Pr}_T(x) \to \neg \mathrm{Pr}_T(\dot{\neg} x))$$

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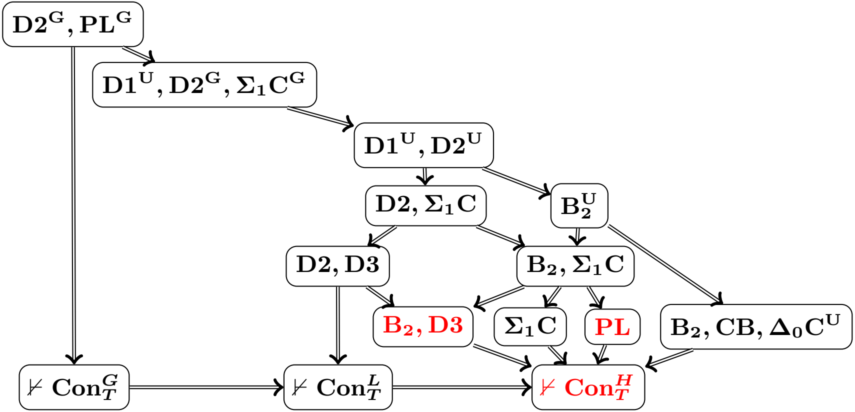

An improvement of Buchholz's result

Theorem (Buchholz)

$\mathbf{D1^U}, \mathbf{D2^U} \Rightarrow \mathbf{\Sigma_1 C^U}$

Notice $\mathbf{D1^U}, \mathbf{D2^U} \Rightarrow \mathbf{B_2^U}$.

Theorem (K.)

$\mathbf{B_2^U} \Rightarrow \mathbf{\Sigma_1 C^U}$

$$\mathbf{D2^U}\quad T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner)).$$

$$\mathbf{B_2^U}\quad T \vdash \varphi(x) \to \psi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner).$$

This is actually an improvement of Buchholz's theorem.

Theorem (K.)

$\mathbf{B_2^U} \not\Rightarrow \mathbf{D2}$

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Theorem (K.)

$\mathbf{B_2^U} \Rightarrow \mathbf{\Sigma_1 C^U}$

$$\mathbf{D2^U} \quad T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner)).$$

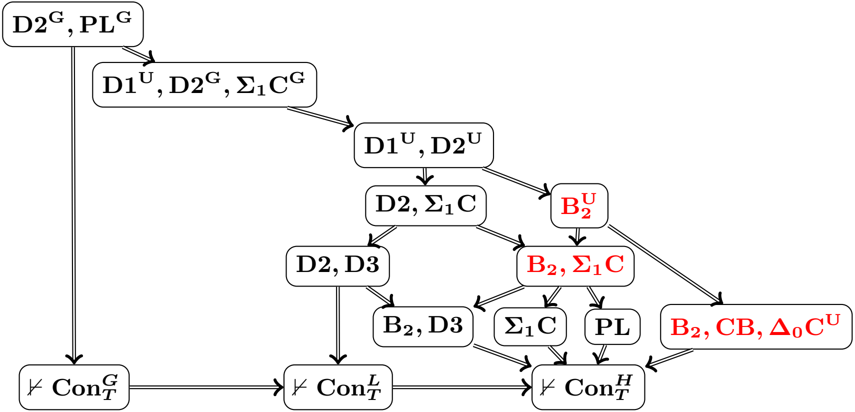
$$\mathbf{B_2^U} \quad T \vdash \varphi(x) \to \psi(x) \Rightarrow T \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner).$$

This is actually an improvement of Buchholz's theorem.

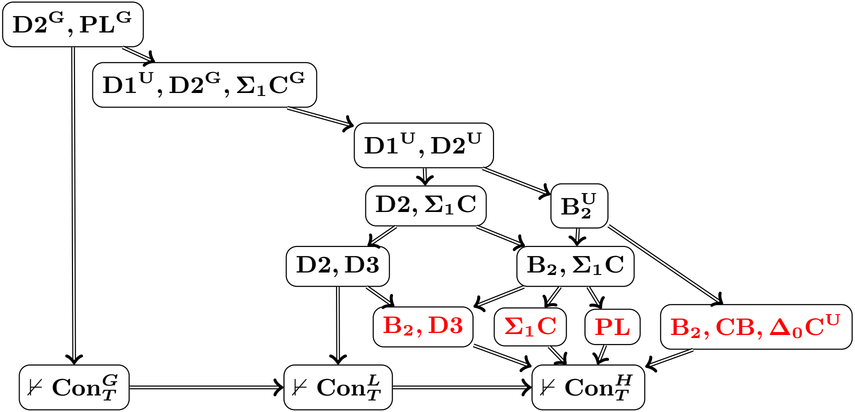
Theorem (K.)

$\mathbf{B_2^U} \not\Rightarrow \mathbf{D2}$

An improvement of Buchholz's result



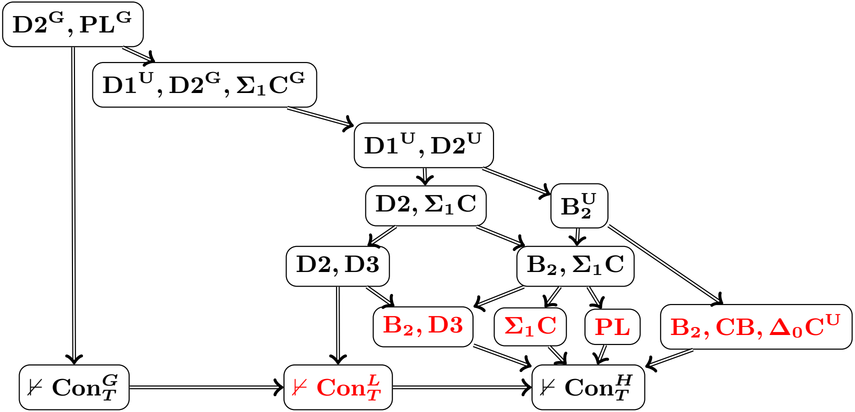
Non-implications 1



Theorem (K.)

$\{\mathbf{B_2, D3}\}$, $\{\mathbf{\Sigma_1 C}\}$, $\{\mathbf{PL}\}$ **and** $\{\mathbf{B_2, CB, \Delta_0 C^U}\}$ **are pairwise incomparable.**

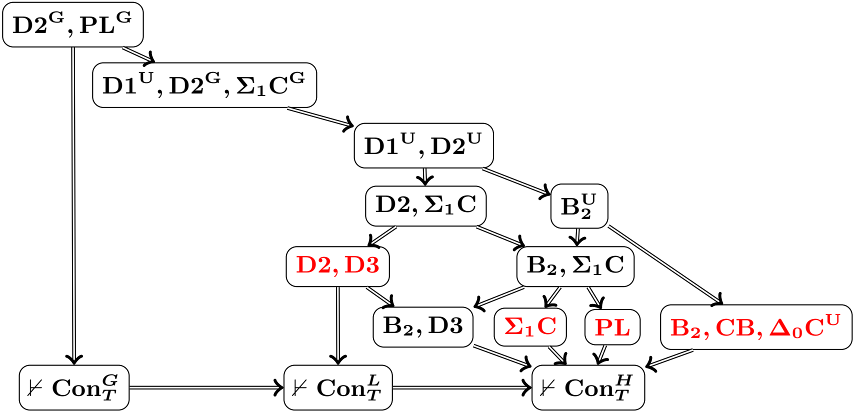
Non-implications 2



Theorem (K.)

Each of $\{B_2, D3\}$, $\{\Sigma_1 C\}$, $\{PL\}$ and $\{B_2, CB, \Delta_0 C^U\}$ is not sufficient for $T \nvdash \mathrm{Con}_T^L$.

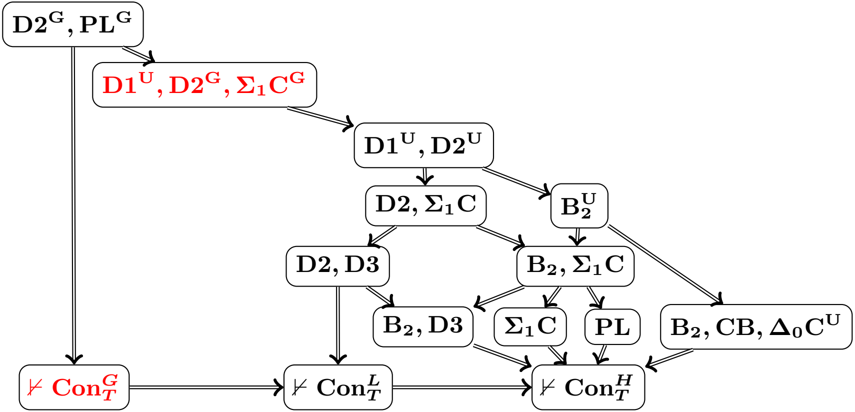
Non-implications 3



Theorem (K.)

{D2, D3} does not imply any of {Σ_1C}, {PL} and {B_2, CB, $\Delta_0 C^U$}.

Non-implication 4

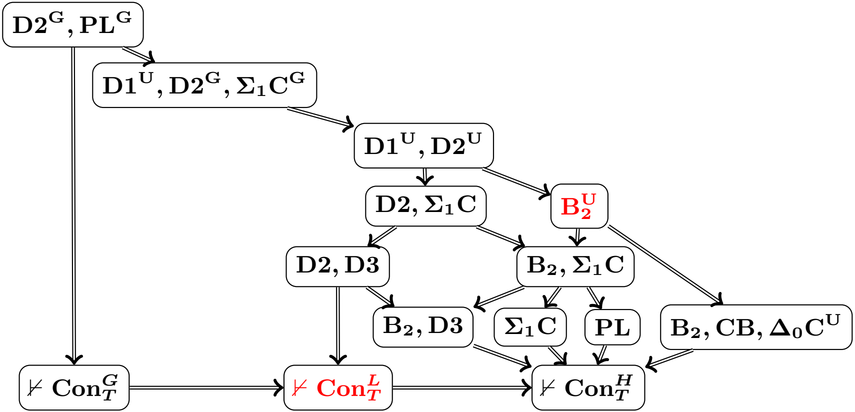


Theorem (K.)

$\{\mathbf{D1^U, D2^G, \Sigma_1 C^G}\}$ **is not sufficient for** $T \nvdash \mathrm{Con}_T^G$.

This shows that both of Hilbert-Bernays' conditions and Löb's conditions do not accomplish Gödel's original statement of G2.

Problem



Problem

Is there a Σ_1 provability predicate $\mathrm{Pr}_T(x)$ satisfying $\mathrm{B}_2^{\mathrm{U}}$ such that $T \vdash \mathrm{Con}_T^L$?

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