Rosser provability and the second incompleteness theorem

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An outline of this talk

In this talk, T always denotes a consistent r.e. extension of Peano Arithmetic PA in the language of arithmetic.

Gödel's second incompleteness theorem (G2)

The consistency of T cannot be proved in T.

- **This statement of G2 is ambiguous because unprovability of a consistency statement is dependent on the choice of a provability predicate.**
- **For G2, several sufficient conditions on provability predicates are known (such as the Hilbert-Bernays-Löb derivability conditions.)**
- **Arai (1990) proved that some conditions are not sufficient for G2 by showing the existence of Rosser provability predicates satisfying such conditions.**
- **In this talk, we extend Arai's results and show that several sets of conditions are not sufficient for G2.**

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Provability predicates

Provability predicates

We say a Σ_1 formula $\mathrm{Pr}_T(x)$ is a provability predicate of T if and only if for any $n \in \omega$,
$\mathrm{PA} \vdash \mathrm{Pr}_T(\overline{n}) \iff n$ is the Gödel number of some theorem of T.

Examples

Let $\mathrm{Prf}_T(x, y)$ be a Δ_1 formula saying that "y is a T-proof of x".

- $\mathrm{Pr}_T(x) \equiv \exists y \mathrm{Prf}_T(x, y)$ is a provability predicate.
- $\mathrm{Pr}^R_T(x) \equiv \exists y(\mathrm{Prf}_T(x, y) \land \forall z \leq y \neg \mathrm{Prf}_T(\dot{\neg}(x), z))$ is a provability predicate which is called a Rosser provability predicate.

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Consistency statements

In this talk, we consider the following two kinds of consistency statements based on a provability predicate $\mathrm{Pr}_T(x)$:

1. $\forall x \neg(\mathrm{Pr}_T(x) \land \mathrm{Pr}_T(\dot{\neg}(x)))$
2. $\neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$

Remark

$\mathbf{PA} \vdash \forall x \neg(\mathrm{Pr}_T(x) \land \mathrm{Pr}_T(\dot{\neg} x)) \to \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

G2 does not hold for Rosser provability predicates.

Proposition

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The Hilbert-Bernays-Löb derivability conditions and G2

The Hilbert-Bernays-Löb derivability conditions

$$\text{D1}: \boldsymbol{T} \vdash \varphi \Rightarrow \mathbf{PA} \vdash \mathbf{Pr}_T(\ulcorner \varphi \urcorner).$$

$$\text{D2}: \mathbf{PA} \vdash \mathbf{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathbf{Pr}_T(\ulcorner \varphi \urcorner) \to \mathbf{Pr}_T(\ulcorner \psi \urcorner)).$$

$$\text{D3}: \mathbf{PA} \vdash \mathbf{Pr}_T(\ulcorner \varphi \urcorner) \to \mathbf{Pr}_T(\ulcorner \mathbf{Pr}_T(\ulcorner \varphi \urcorner) \urcorner).$$

- **D1 is automatically satisfied by all provability predicates of T.**
- **D3 is a special case of the following condition.**

Formalized Σ_1-completeness

$$\Sigma_1\text{C}: \text{If } \varphi \text{ is } \Sigma_1, \text{ then } \mathbf{PA} \vdash \varphi \to \mathbf{Pr}_T(\ulcorner \varphi \urcorner).$$

Gödel's second incompleteness theorem (G2)

If $\mathbf{Pr}_T(x)$ satisfies D2 and D3, then $T \nvdash \neg \mathbf{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

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Uniform derivability conditions

It is sometimes useful to consider stronger versions of derivability conditions.

Uniform derivbility conditions

D1^U : $T \vdash \varphi(x) \Rightarrow \mathbf{PA} \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner)$.

D2^U : $\mathbf{PA} \vdash \forall x(\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner)))$.

D3^U : $\mathbf{PA} \vdash \forall x(\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \urcorner))$.

ΓC^U : **If $\varphi(x)$ is a Γ formula, then** $\mathbf{PA} \vdash \forall x(\varphi(x) \to \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner))$.

- **$\ulcorner \varphi(\dot{x}) \urcorner$ is a primitive recursive term corresponding to a primitive recursive function calculating the Gödel number of $\varphi(\overline{n})$ from n.**

Global derivability conditions

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$D2^G$: $\mathbf{PA} \vdash \forall x \forall y(\mathbf{Pr}_T(x \dot{\to} y) \to (\mathbf{Pr}_T(x) \to \mathbf{Pr}_T(y)))$.

$D3^G$: $\mathbf{PA} \vdash \forall x(\mathbf{Pr}_T(x) \to \mathbf{Pr}(\ulcorner \mathbf{Pr}_T(\dot{x}) \urcorner))$.

ΓC^G : $\mathbf{PA} \vdash \forall x(\mathsf{True}_\Gamma(x) \to \mathbf{Pr}_T(x))$.

- $\mathsf{True}_\Gamma(x)$ **is a formula satisfying that for any Γ sentence φ, $\mathbf{PA} \vdash \mathsf{True}_\Gamma(\ulcorner \varphi \urcorner) \leftrightarrow \varphi$.**

Remark

Global $\Rightarrow$ Uniform $\Rightarrow$ Local.

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Sufficient conditions for G2

- **{D2, D3} is sufficient for G2.**
- **Several other sets of conditions sufficient for G2 are known.**

1. **Jeroslow (1973)**
2. **Montagna (1979)**
3. **Rautenberg (?)**
4. **K.**

Jeroslow's observation

Jeroslow showed that D2 is redundant for G2 w.r.t. $\forall x \neg(\mathrm{Pr}_T(x) \land \mathrm{Pr}_T(\dot{\neg}(x)))$ if D3 is strengthened to Σ_1C.

Theorem (Jeroslow, 1973)

If $\mathrm{Pr}_T(x)$ satisfies Σ_1C, then $T \nvdash \forall x \neg(\mathrm{Pr}_T(x) \land \mathrm{Pr}_T(\dot{\neg}(x)))$.

$$\text{D2} : \mathrm{PA} \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner)).$$

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Montagna showed that D3 is also redundant for G2 if $\mathrm{Pr}_T(x)$ satisfies the global version of D2 and an additional condition.

Theorem (Montagna, 1979)

Suppose $\mathrm{Pr}_T(x)$ satisfies $\mathrm{D2}^G$ and $T \vdash \mathrm{LogAx}(x) \to \mathrm{Pr}_T(x)$.
Then $\mathrm{Pr}_T(x)$ satisfies $\mathrm{D1}^U$ and $\Sigma_1\mathrm{C}^G$.
Consequently, $T \nvdash \neg\mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

$$\mathrm{D1}^U : T \vdash \varphi(x) \Rightarrow \mathrm{PA} \vdash \mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner).$$

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In Rautenberg's textbook "A Concise Introduction to Mathematical Logic", a comprehensible proof of G2 is presented.

It shows that the uniform versions of D1 and D2 are sufficient for G2.

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$\mathrm{D1}^{U}_{+}$: $T \vdash \varphi(x) \to \psi(x)$
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Remark

If $\mathrm{Pr}_T(x)$ satisfies $\mathrm{D1}^U$ and $\mathrm{D2}^U$, then it satisfies $\mathrm{D1}^U_+$.

Theorem (K.)

Suppose $\mathrm{Pr}_T(x)$ satisfies $\mathrm{D1}^U_+$. Then $\mathrm{Pr}_T(x)$ satisfies $\Sigma_1\mathrm{C}^U$. Consequently, $T \nvdash \forall x \neg(\mathrm{Pr}_T(x) \land \mathrm{Pr}_T(\dot{\neg}(x)))$.

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There exists a provability predicate of T which satisfies $\mathrm{D1}^U_+$ but does not satisfy D2.

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There exists a provability predicate of T which satisfies $\mathrm{D1}^U_+$ but does not satisfy D2.

Negative results for Rosser provability predicates 1

Proposition

For Rosser provability predicates $\mathrm{Pr}_T^R(x)$, PA $\vdash \neg\mathrm{Pr}_T^R(\ulcorner 0 = 1 \urcorner)$.

Corollary

There exists no Rosser provability predicate of T satisfying both D2 and D3.

Question (Kreisel and Takeuti, 1974)

Is D2 valid for Rosser provability predicates?

Guaspari and Solovay (1979)

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Negative results for Rosser provability predicates 2

Theorem (Jeroslow 1973)

There exists no Rosser provability predicate of T satisfying Σ_1C.

Corollary

There exists no Rosser provability predicate of T satisfying $\mathrm{D1}^U_+$.

$$\begin{aligned} \mathrm{D1}^U_+ : \ & T \vdash \varphi(x) \to \psi(x) \\ & \Rightarrow \mathbf{PA} \vdash \forall x(\mathrm{Pr}_T(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi(\dot{x}) \urcorner)). \end{aligned}$$

Negative results for Rosser provability predicates 2

Theorem (Jeroslow 1973)

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There exists no Rosser provability predicate of T satisfying $\mathrm{D1}^U_+$.

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Arai's positive results for Rosser provability predicates

Theorem (Arai 1990)

- **There exists a Rosser provability predicate of T satisfying $\mathbf{D2}^G$.**
- **There exists a Rosser provability predicate of T satisfying $\mathbf{D3}^G$.**

$$\mathrm{D2}^G : \mathbf{PA} \vdash \forall x \forall y (\mathrm{Pr}_T(x \dot{\to} y) \to (\mathrm{Pr}_T(x) \to \mathrm{Pr}_T(y))).$$

$$\mathrm{D3}^G : \mathbf{PA} \vdash \forall x (\mathrm{Pr}_T(x) \to \mathrm{Pr}(\ulcorner \mathrm{Pr}_T(\dot{x}) \urcorner)).$$

Corollary

- $\mathrm{D2}^G$ is not sufficient for G2.
- $\mathrm{D3}^G$ is not sufficient for G2.

Arai's positive results for Rosser provability predicates

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$$\mathrm{D3}^G : \mathbf{PA} \vdash \forall x (\mathbf{Pr}_T(x) \to \mathbf{Pr}(\ulcorner \mathbf{Pr}_T(\dot{x}) \urcorner)).$$

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- **$\mathbf{D2}^G$ is not sufficient for G2.**
- **$\mathbf{D3}^G$ is not sufficient for G2.**

Theorem 1

Theorem 1

There exists a Rosser provability predicate $\mathrm{Pr}_T^R(x)$ of T satisfying $\mathbf{D2}^G$ and $\mathbf{\Delta_0 C}^G$. That is,

- $\mathrm{PA} \vdash \forall x \forall y (\mathrm{Pr}_T^R(x \dot{\to} y) \to (\mathrm{Pr}_T^R(x) \to \mathrm{Pr}_T^R(y)))$.
- $\mathrm{PA} \vdash \forall x (\mathsf{True}_{\Delta_0}(x) \to \mathrm{Pr}_T^R(x))$.

$$\{\mathrm{D2}, \mathrm{D3}\} \Rightarrow \mathrm{G2}$$
$$\{\mathrm{D2}^G, \Delta_0\mathrm{C}^G\} \not\Rightarrow \mathrm{G2}$$

Corollary

$\{\mathrm{D2}^G, \Delta_0\mathrm{C}^G\} \not\Rightarrow \mathrm{D3}$

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Theorem 2

Theorem 2

There exists a Rosser provability predicate $\mathrm{Pr}_T^R(x)$ of T satisfying D1^U, D2 and Δ_0C^G. That is,

- $T \vdash \varphi(x) \Rightarrow \mathrm{PA} \vdash \mathrm{Pr}_T^R(\ulcorner \varphi(\dot{x}) \urcorner)$.
- $\mathrm{PA} \vdash \mathrm{Pr}_T^R(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T^R(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T^R(\ulcorner \psi \urcorner))$.
- $\mathrm{PA} \vdash \forall x(\mathsf{True}_{\Delta_0}(x) \to \mathrm{Pr}_T^R(x))$.

$\{$D2, D3$\} \Rightarrow$ G2

$\{$D1^U, D2$^U\} \Rightarrow \Sigma_1$C^U and G2

$\{$D1^U, D2, Δ_0C$^G\} \not\Rightarrow$ G2

Corollary

- $\{$D1^U, D2, Δ_0C$^G\} \not\Rightarrow$ D3
- $\{$D1^U, D2, Δ_0C$^G\} \not\Rightarrow$ D2^U

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- $\mathrm{PA} \vdash \forall x(\mathsf{True}_{\Delta_0}(x) \to \mathrm{Pr}_T^R(x))$.

$$\{\mathrm{D2}, \mathrm{D3}\} \Rightarrow \mathrm{G2}$$
$$\{\mathrm{D1}^U, \mathrm{D2}^U\} \Rightarrow \Sigma_1\mathrm{C}^U \text{ and } \mathrm{G2}$$
$$\{\mathrm{D1}^U, \mathrm{D2}, \Delta_0\mathrm{C}^G\} \not\Rightarrow \mathrm{G2}$$

Corollary

- $\{\mathrm{D1}^U, \mathrm{D2}, \Delta_0\mathrm{C}^G\} \not\Rightarrow \mathrm{D3}$
- $\{\mathrm{D1}^U, \mathrm{D2}, \Delta_0\mathrm{C}^G\} \not\Rightarrow \mathrm{D2}^U$

Theorem 3

Theorem 3

There exists a Rosser provability predicate $\mathrm{Pr}_T^R(x)$ of T satisfying $\mathbf{D1}^U$, $\mathbf{D1}_+$ and $\mathbf{D3}^G$. That is,

- $\mathbf{PA} \vdash \varphi(x) \Rightarrow \mathbf{PA} \vdash \mathrm{Pr}_T^R(\ulcorner \varphi(\dot{x}) \urcorner)$.
- $\mathbf{PA} \vdash \varphi \to \psi \Rightarrow \mathbf{PA} \vdash \mathrm{Pr}_T^R(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T^R(\ulcorner \psi \urcorner)$.
- $\mathbf{PA} \vdash \forall x(\mathrm{Pr}_T^R(x) \to \mathrm{Pr}_T^R(\ulcorner \mathrm{Pr}_T^R(\dot{x}) \urcorner))$.

$\{\mathrm{D2}, \mathrm{D3}\} \Rightarrow \mathrm{G2}$

$\{\mathrm{D1}_+^U\} \Rightarrow \Sigma_1\mathrm{C}^U$

$\{\mathrm{D1}^U, \mathrm{D1}_+, \mathrm{D3}^G\} \not\Rightarrow \mathrm{G2}$

Corollary

- $\{\mathrm{D1}^U, \mathrm{D1}_+, \mathrm{D3}^G\} \not\Rightarrow \mathrm{D2}$
- $\{\mathrm{D1}^U, \mathrm{D1}_+, \mathrm{D3}^G\} \not\Rightarrow \mathrm{D1}_+^U$

Theorem 3

Theorem 3

There exists a Rosser provability predicate $\mathrm{Pr}^R_T(x)$ of T satisfying D1^U, D1$_+$ and D3^G. That is,

- $\mathbf{PA} \vdash \varphi(x) \Rightarrow \mathbf{PA} \vdash \mathrm{Pr}^R_T(\ulcorner \varphi(\dot{x}) \urcorner)$.
- $\mathbf{PA} \vdash \varphi \to \psi \Rightarrow \mathbf{PA} \vdash \mathrm{Pr}^R_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}^R_T(\ulcorner \psi \urcorner)$.
- $\mathbf{PA} \vdash \forall x(\mathrm{Pr}^R_T(x) \to \mathrm{Pr}^R_T(\ulcorner \mathrm{Pr}^R_T(\dot{x}) \urcorner))$.

$$\{\mathbf{D2}, \mathbf{D3}\} \Rightarrow \mathbf{G2}$$
$$\{\mathbf{D1}^U_+\} \Rightarrow \Sigma_1\mathbf{C}^U$$
$$\{\mathbf{D1}^U, \mathbf{D1}_+, \mathbf{D3}^G\} \not\Rightarrow \mathbf{G2}$$

Corollary

- $\{\mathbf{D1}^U, \mathbf{D1}_+, \mathbf{D3}^G\} \not\Rightarrow \mathbf{D2}$
- $\{\mathbf{D1}^U, \mathbf{D1}_+, \mathbf{D3}^G\} \not\Rightarrow \mathbf{D1}^U_+$

Conclusion

Sufficient for G2 w.r.t. $\neg \mathbf{Pr}_T(\ulcorner 0 = 1 \urcorner)$

- $\{$D2, D3$\}$
- $\{$D2^G, $T \vdash \mathrm{LogAx}(x) \to \mathbf{Pr}_T(x)\}$ (Montagna)
- $\{$D1^U, D2$^U\}$ (Rautenberg)

Sufficient for G2 w.r.t. $\forall x \neg(\mathbf{Pr}_T(x) \wedge \mathbf{Pr}_T(\dot{\neg}(x)))$

- $\{\Sigma_1$C$\}$ (Jeroslow)
- $\{$D1$^U_+\}$ (K.)

Not sufficient for G2 w.r.t. $\neg \mathbf{Pr}_T(\ulcorner 0 = 1 \urcorner)$

- $\{$D2^G, Δ_0C$^G\}$
- $\{$D1^U, D2, Δ_0C$^G\}$
- $\{$D1^U, D1$_+$, D3$^G\}$