# Normal modal logics and provability predicates

**Taishi Kurahashi**
**(National Institute of Technology, Kisarazu College)**

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**Kanazawa**
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## Outline

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1. Provability predicates
2. Arithmetical interpretations and provability logics
3. Our results

## Provability predicates

- $\mathcal{L}_A$: **the language of first-order arithmetic**
- $\overline{n}$: **the numeral for** $n \in \omega$
- $\ulcorner \varphi \urcorner$: **the numeral for the Gödel number of** $\varphi$

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**In the usual proof of Gödel's incompleteness theorems, a provability predicate plays an important role.**

### Provability predicates

**A formula** $\mathsf{Pr}(x)$ **is a provability predicate of** $\mathsf{PA}$
$\stackrel{\text{def.}}{\iff}$ **for any** $n \in \omega$**,**
$\mathsf{PA} \vdash \mathsf{Pr}(\overline{n}) \iff n$ **is the Gödel number of some theorem of** $\mathsf{PA}$**.**

## Standard construction of provability predicates

**Gödel-Feferman's standard construction of provability predicates of PA is as follows.**

### Numerations

**A formula $\tau(v)$ is a numeration of PA**
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- **Let $\tau(v)$ be a numeration of PA.**
- **The relation "$x$ is the Gödel number of an $\mathcal{L}_A$-fomula provable in the theory defined by $\tau(v)$" is naturally expressed in the language $\mathcal{L}_A$.**

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- **The resulting $\mathcal{L}_A$-formula is denoted by $\mathsf{Pr}_\tau(x)$.**
- **If $\tau(v)$ is $\Sigma_{n+1}$, then $\mathsf{Pr}_\tau(x)$ is also $\Sigma_{n+1}$.**

## Properties of standard provability predicates

**Theorem (Hilbert-Bernays-Löb-Feferman)**

**Let $\tau(v)$ be any numeration of PA.**

- $\mathsf{Pr}_\tau(x)$ **is a provability predicate of PA.**
- $\mathsf{PA} \vdash \mathsf{Pr}_\tau(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \psi \urcorner))$.
- $\mathsf{PA} \vdash \varphi \to \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner)$ **for any $\Sigma_1$ sentence $\varphi$.**

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**Theorem**

**Let $\tau(v)$ be any $\Sigma_1$ numeration of PA.**

- $\mathsf{PA} \vdash \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \urcorner)$.
- **(Gödel's second incompleteness theorem)**
  $\mathsf{PA} \nvdash \mathsf{Con}_\tau$,
  **where $\mathsf{Con}_\tau$ is the consistency statement $\neg \mathsf{Pr}_\tau(\ulcorner \overline{0} = \overline{1} \urcorner)$ of $\tau(v)$.**
- **(Löb's theorem)**
  $\mathsf{PA} \vdash \mathsf{Pr}_\tau(\ulcorner \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \to \varphi \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \varphi \urcorner)$.

## Nonstandard provability predicates

**There are many nonstandard provability predicates.**

- **Rosser's provability predicate**
  $\mathsf{Pr}^R(x) \equiv \exists y(\mathsf{Prf}(x, y) \land \forall z \leq y \neg \mathsf{Prf}(\dot{\neg} x, z))$,
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**Problem**

**What are the PA-provable principles of each provability predicate?**

**This problem is investigated in the framework of modal logic.**

## Outline

## Modal logics

### Axioms and Rules of the modal logic K

Axioms **Tautologies and** $\Box(p \to q) \to (\Box p \to \Box q)$**.**

Rules **Modus ponens** $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$**, Necessitation** $\dfrac{\varphi}{\Box\varphi}$**, and Substitution.**

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**A modal logic $L$ is normal**
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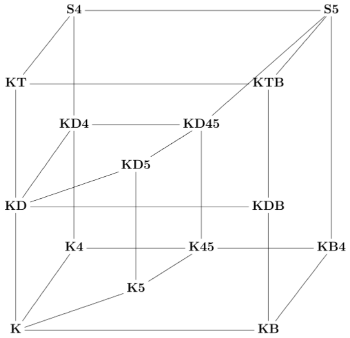
**Normal modal logics**

**A modal logic $L$ is normal**
$\overset{\text{def.}}{\iff}$ **$L$ includes K and is closed under three rules of K.**

**For each modal formula $A$, $L + A$ denotes the smallest normal modal logic including $L$ and $A$.**

- $\mathsf{KT} = \mathsf{K} + \Box p \to p$
- $\mathsf{KD} = \mathsf{K} + \neg\Box\bot$
- $\mathsf{K4} = \mathsf{K} + \Box p \to \Box\Box p$
- $\mathsf{K5} = \mathsf{K} + \Diamond p \to \Box\Diamond p$
- $\mathsf{KB} = \mathsf{K} + p \to \Box\Diamond p$
- $\mathsf{GL} = \mathsf{K} + \Box(\Box p \to p) \to \Box p$
- $\cdots$



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**A mapping** $f$ **from modal formulas to** $\mathcal{L}_A$**-sentences is an arithmetical interpretation based on** $\mathsf{Pr}(x)$

$\overset{\text{def.}}{\Longleftrightarrow}$ $f$ **satisfies the following conditions:**

- $f(\bot) \equiv \overline{0} = \overline{1}$;
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$\mathsf{PL}(\mathsf{Pr}) := \{A : \mathsf{PA} \vdash f(A)$ **for all arithmetical interpretations $f$ based on** $\mathsf{Pr}(x)\}$.

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$\mathsf{PL}(\mathsf{Pr}) := \{A : \mathsf{PA} \vdash f(A)$ **for all arithmetical interpretations $f$ based on** $\mathsf{Pr}(x)\}$.
**The set $\mathsf{PL}(\mathsf{Pr})$ is said to be the provability logic of $\mathsf{Pr}(x)$.**

## Solovay's arithmetical completeness theorem

- **Recall that for each $\Sigma_1$ numeration $\tau(v)$ of PA,**
  - $\mathsf{PA} \vdash \mathsf{Pr}_\tau(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}_\tau(\ulcorner \varphi \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \psi \urcorner))$,
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- **Corresponding modal formulas $\Box(p \to q) \to (\Box p \to \Box q)$ and $\Box(\Box p \to p) \to \Box p$ are axioms of GL.**
- **In fact, GL is exactly the provability logic of standard $\Sigma_1$ provability predicates.**

**Arithmetical completeness theorem (Solovay, 1976)**

**For any $\Sigma_1$ numeration $\tau(v)$ of PA, $\mathsf{PL}(\mathsf{Pr}_\tau)$ coincides with GL.**

## Feferman's predicate

**On the other hand, there are provability predicates whose provability logics are completely different from GL.**

**Theorem (Feferman, 1960)**

**There exists a $\Pi_1$ numeration $\tau(v)$ of PA such that $\mathsf{PA} \vdash \mathsf{Con}_\tau$. Consequently, $\mathsf{KD} \subseteq \mathsf{PL}(\mathsf{Pr}_\tau)$ ($\mathsf{KD} = \mathsf{K} + \neg\Box\bot$).**

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**Consequently, $\mathsf{KD} \subseteq \mathsf{PL}(\mathsf{Pr}_\tau)$ ($\mathsf{KD} = \mathsf{K} + \neg\Box\bot$).**

**Shavrukov found a nonstandard provability predicate whose provability logic is strictly stronger than KD.**

**Theorem (Shavrukov, 1994)**

**Let $\mathsf{Pr}^S(x) \equiv \exists y(\mathsf{Pr}_{I\Sigma_y}(x) \land \mathsf{Con}_{I\Sigma_y})$.**
**Then $\mathsf{PL}(\mathsf{Pr}^S) = \mathsf{KD} + \Box p \to \Box((\Box q \to q) \lor \Box p)$.**

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- **We are interested in the following general problem.**

General Problem

**Which normal modal logic is the provability logic $\mathsf{PL}(\mathsf{Pr})$ of some provability predicate $\mathsf{Pr}(x)$ of $\mathsf{PA}$?**

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- **Kurahashi, T., Arithmetical completeness theorem for modal logic $\mathsf{K}$, *Studia Logica*, to appear.**
- **Kurahashi, T., Arithmetical soundness and completeness for $\Sigma_2$ numerations, *Studia Logica*, to appear.**
- **Kurahashi, T., Rosser provability and normal modal logics, submitted.**

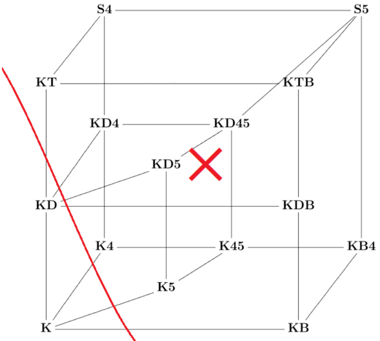
## Outline

**Several normal modal logics cannot be of the form PL(Pr).**

### Proposition (K., 201x)

**Let $L$ be a normal modal logic satisfying one of the following conditions. Then $L \neq \mathsf{PL}(\mathsf{Pr})$ for all provability predicates $\mathsf{Pr}(x)$ of PA.**

1. $\mathsf{KT} \subseteq L$.
2. $\mathsf{K4} \subseteq L$ and $\mathsf{GL} \not\subseteq L$.
3. $\mathsf{K5} \subseteq L$.
4. $\mathsf{KB} \subseteq L$.



## Theorem 1

**There exists a numeration of PA whose provability logic is minimum.**

**Theorem 1 (K., 201x)**

**There exists a $\Sigma_2$ numeration $\tau(v)$ of PA such that $\mathsf{PL}(\mathsf{Pr}_\tau) = \mathsf{K}$.**

## Theorem 2

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- **For $n \geq 2$, $\mathsf{K} + \Box(\Box^n p \to p) \to \Box p \subsetneq \mathsf{GL}$.**
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- **He conjectured that these logics are provability logics of some nonstandard provability predicates.**

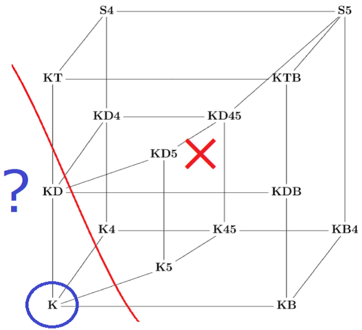
**We gave a proof of this conjecture.**

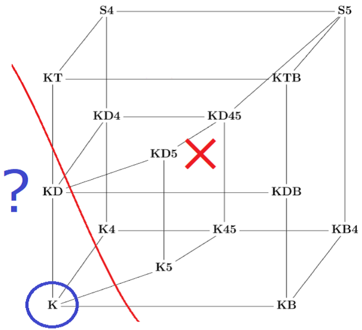
**Theorem 2 (K., 201x)**

**For each** $n \geq 2$**, there exists a** $\Sigma_2$ **numeration** $\tau(v)$ **of PA such that** $\mathsf{PL}(\mathsf{Pr}_\tau) = \mathsf{K} + \Box(\Box^n p \to p) \to \Box p$**.**

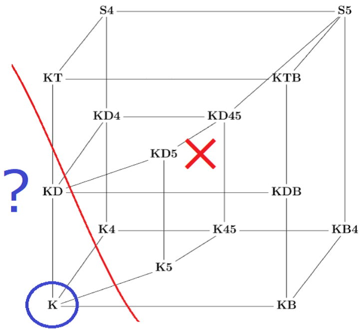
**Therefore there are infinitely many provability logics.**

## Our results






- **How about** KD?




- **How about KD?**
- **We paid attention to Rosser's provability predicates $\mathsf{Pr}^R(x)$ because PA always proves the consistency statements $\mathsf{Con}^R$ defined by using $\mathsf{Pr}^R(x)$.**

## Rosser's provability predicates

**However, provability logics of Rosser's provability predicates are sometimes not normal.**

**Theorem (Guaspari and Solovay, 1979)**

**There exists a Rosser provability predicate** $\mathsf{Pr}^R(x)$ **such that** $\mathsf{PA} \nvdash \mathsf{Pr}^R(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}^R(\ulcorner \varphi \urcorner) \to \mathsf{Pr}^R(\ulcorner \psi \urcorner))$ **for some** $\varphi$ **and** $\psi$**.**

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**On the other hand, there exists a Rosser provability predicate whose provability logic is normal.**

**Theorem (Arai, 1990)**

**There exists a Rosser provability predicate** $\mathsf{Pr}^R(x)$ **such that** $\mathsf{PA} \vdash \mathsf{Pr}^R(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}^R(\ulcorner \varphi \urcorner) \to \mathsf{Pr}^R(\ulcorner \psi \urcorner))$ **for any** $\varphi$ **and** $\psi$**.**

**Then** $\mathsf{KD} \subseteq \mathsf{PL}(\mathsf{Pr}^R)$ **for Arai's predicate** $\mathsf{Pr}^R(x)$**.**

## Theorems 3 and 4

**We proved that there exists $\mathsf{Pr}^R(x)$ whose provability logic coincides with $\mathsf{KD}$.**

**Theorem 3 (K., 201x)**

**There exists a Rosser provability predicate $\mathsf{Pr}^R(x)$ such that $\mathsf{PL}(\mathsf{Pr}^R) = \mathsf{KD}$.**

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**Moreover, there exists a Rosser provability predicate whose provability logic is strictly stronger than $\mathsf{KD}$.**

**Theorem 4 (K., 201x)**

**There exists a Rosser provability predicate $\mathsf{Pr}^R(x)$ such that $\mathsf{KD} + \Box\neg p \to \Box\neg\Box p \subseteq \mathsf{PL}(\mathsf{Pr}^R)$.**

## Open Problems

### Open Problem 1

**Are there any others logics $L$ such that $\mathsf{K} \subsetneq L \subsetneq \mathsf{GL}$ and $L = \mathsf{PL}(\mathsf{Pr})$ for some $\mathsf{Pr}(x)$?**

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### General Problem

**Which (normal) modal logic is in the set $\{\mathsf{PL}(\mathsf{Pr}) : \mathsf{Pr}(x)$ is a provability predicate of $\mathsf{PA}\}$?**