On partial disjunction properties of theories containing PA

Taishi Kurahashi (倉橋 太志)
National Institute of Technology, Kisarazu College
（木更津工業高等専門学校）

Workshop “Logic and Philosophy of Mathematics”
Waseda University
July 15, 2017

Introduction

Definition

A theory T has the disjunction property (DP) $\overset{\text{def.}}{\iff}$ for any sentences φ, ψ, $T \vdash \varphi \vee \psi \Rightarrow T \vdash \varphi$ or $T \vdash \psi$.

Gödel, 1932

Intuitionistic propositional logic has DP.

Gentzen, 1934-35

Intuitionistic predicate logic has DP.

Kleene, 1945

Heyting arithmetic HA has DP.

DP seems to reflect their constructivity.

Introduction

In classical logic, DP plays a different role.

Fact

Let T be a consistent theory in classical logic. Then

$$T \text{ has DP} \iff T \text{ is complete.}$$

Proof.

Let φ, ψ be any sentences.

($\Rightarrow$): $T \vdash \varphi \vee \neg\varphi$ by the law of excluded middle.

Then $T \vdash \varphi$ or $T \vdash \neg\varphi$ by DP.

($\Leftarrow$): If $T \vdash \varphi \vee \psi$, then $T \nvdash \neg\varphi \wedge \neg\psi$ by consistency.

Then $T \nvdash \neg\varphi$ or $T \nvdash \neg\psi$.

By completeness, $T \vdash \varphi$ or $T \vdash \psi$. □

Introduction

The first incompleteness theorem (Gödel, 1931; Rosser 1936)

If T is a recursively enumerable consistent extension of PA, then T is incomplete.

The first incompleteness theorem (rephrased)

If T is a recursively enumerable consistent extension of PA, then T does not have DP.

In this talk, we present our results contained in the following papers.

1. **Kikuchi, M. and Kurahashi, T.: Generalizations of Gödel's incompleteness theorems for Σ_n-definable theories of arithmetic. Submitted.**
2. **Kurahashi, T.: On partial disjunction properties of theories containing Peano arithmetic. Submitted.**

Contents

$\mathcal{L}_A = \{0, 1, +, \times\}$: the language of first-order arithmetic.
We consider only $\mathcal{L}_A$-formulas.

Axioms of PA (Peano Arithmetic)

- $\forall x(0 \neq x + 1)$
- $\forall x \forall y(x + 1 = y + 1 \to x = y)$
- $\forall x(x + 0 = x)$
- $\forall x \forall y(x + (y + 1) = (x + y) + 1)$
- $\forall x(x \times 0 = 0)$
- $\forall x \forall y(x \times (y + 1) = (x \times y) + x)$
- For every formula φ,
  $\forall y_0 \cdots \forall y_{k-1}((\varphi(0) \land \forall x(\varphi(x) \to \varphi(x + 1))) \to \forall x \varphi(x))$

In this talk, T always denotes an $\mathcal{L}_A$-theory containing **PA**.

Some classes of formulas

Definition

Let φ be a formula.

- φ is Σ_0 or Π_0 $\overset{\text{def.}}{\iff}$ every quantifier contained in φ is of the form $\forall x \leq t$ or $\exists x \leq t$ for some term t.
- φ is Σ_{n+1} $\overset{\text{def.}}{\iff}$ φ is of the form $\exists x_0 \cdots \exists x_{k-1} \psi$ for some Π_n formula ψ.
- φ is Π_{n+1} $\overset{\text{def.}}{\iff}$ φ is of the form $\forall x_0 \cdots \forall x_{k-1} \psi$ for some Σ_n formula ψ.
- φ is $\mathcal{B}(\Sigma_n)$ $\overset{\text{def.}}{\iff}$ φ is a Boolean combination of Σ_n formulas.

In this talk, we assume $n \geq 1$.
Also we assume Γ denotes one of Σ_n, Π_n and $\mathcal{B}(\Sigma_n)$.

Definability of sets

$\mathbb{N}$: the standard model of arithmetic

Definition

Let X be a set of natural numbers.

- **X is Γ-definable $\overset{\text{def.}}{\iff}$ there exists a Γ formula $\varphi(x)$ such that $X = \{n : \mathbb{N} \models \varphi(n)\}$.**
- **Such a formula $\varphi(x)$ is said to be a Γ definition of X.**

Fact

- **X is recursively enumerable $\iff$ X is Σ_1-definable.**
- **X is recursive $\iff$ X is Σ_1-definable and Π_1-definable.**

The first incompleteness theorem

The first incompleteness theorem (Gödel, 1931; Rosser, 1936)

If T is Σ_1-definable and consistent, then there exists a Σ_1 sentence φ such that $T \nvdash \varphi$ and $T \nvdash \neg\varphi$.

For each Σ_1 sentence φ, both φ and $\neg\varphi$ are $\mathcal{B}(\Sigma_1)$.

Corollary

If T is consistent and Σ_1-definable, then there are $\mathcal{B}(\Sigma_1)$ sentences φ, ψ such that $T \vdash \varphi \vee \psi$, $T \nvdash \varphi$ and $T \nvdash \psi$.

On the other hand, PA enjoys a partial disjunction property.

Fact

For any Σ_1 sentences φ, ψ,
$\mathsf{PA} \vdash \varphi \lor \psi \Rightarrow \mathsf{PA} \vdash \varphi$ or $\mathsf{PA} \vdash \psi$.

Proof.

Suppose $\mathsf{PA} \vdash \varphi \lor \psi$ for Σ_1 sentences φ, ψ.
Notice $\varphi \lor \psi$ is equivalent to a Σ_1 sentence.
Since every Σ_1 sentence provable in PA is true, $\mathbb{N} \models \varphi \lor \psi$.
Then $\mathbb{N} \models \varphi$ or $\mathbb{N} \models \psi$.
Since every true Σ_1 sentence is provable in PA, $\mathsf{PA} \vdash \varphi$ or $\mathsf{PA} \vdash \psi$. □

Partial disjunction properties

Definition

A theory T has the Γ-disjunction property (Γ-DP)
$\overset{\text{def.}}{\Longleftrightarrow}$ **for any Γ sentences φ, ψ,**
$T \vdash \varphi \lor \psi \Rightarrow T \vdash \varphi$ **or** $T \vdash \psi$.

Proposition

If $\Gamma \subseteq \Gamma'$ and T has Γ'-DP, then T has Γ-DP.

$$\begin{array}{ccccc} \mathcal{B}(\Sigma_n)\text{-DP} & \Rightarrow & \Sigma_n\text{-DP} & \Rightarrow & \mathcal{B}(\Sigma_{n-1})\text{-DP} \\ & \searrow & & \nearrow & \\ & & \Pi_n\text{-DP} & & \end{array}$$

- **PA does not have $\mathcal{B}(\Sigma_1)$-DP.**
- **PA has Σ_1-DP.**

These results can be improved.

Theorem (Macintyre and Simmons, 1975)

If T is Σ_1-definable and consistent, then T does not have Π_1-DP.

Definition

A theory T is Γ-sound
$\overset{\text{def.}}{\Longleftrightarrow}$ for any Γ sentence φ ($T \vdash \varphi \Rightarrow \mathbb{N} \vdash \varphi$).

Theorem (Guaspari, 1979)

Let T be a Σ_1-definable consistent theory. T.F.A.E.:

1. **T is Σ_1-sound.**
2. **T has Σ_1-DP.**

Problem

Problem

What are the interrelationships between the following conditions?

1. **T is complete.**
2. **T has Γ-DP.**
3. **T is Σ_n-sound.**
4. **T is Σ_n-definable.**

- **The situation for Σ_1-definable theories has already been clarified.**
- **We investigate theories which are not necessarily Σ_1-definable.**

1. The first incompleteness theorem and DP
2. DP and related properties
3. Σ_n-definable theories
4. Unprovability of formalized DP

Before investigating Σ_n-theories, we show general interrelationships between DP and related properties.

Proposition

T.F.A.E.:

1. T has $\mathcal{B}(\Sigma_n)$-DP.
2. For any Σ_n sentence φ, $T \vdash \varphi$ or $T \vdash \neg\varphi$.
3. For any $\mathcal{B}(\Sigma_n)$ sentence φ, $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Proof.

(1 $\Rightarrow$ 2):By the law of excluded middle.
(2 $\Rightarrow$ 3): Let φ be a $\mathcal{B}(\Sigma_n)$ sentence such that $T \nvdash \varphi$.
φ is logically equivalent to $\psi_0 \wedge \cdots \wedge \psi_{l-1}$ such that each ψ_i is of the form $\gamma_0^i \vee \cdots \vee \gamma_{k_i-1}^i$ where each γ_j^i is Σ_n or Π_n.
Then $T \nvdash \psi_i$ for some $i < l$.
$T \nvdash \gamma_j^i$ for all $j < k_i$.
$T \vdash \neg\gamma_j^i$ by 2.
$T \vdash \neg\psi_i$ and $T \vdash \neg\varphi$.
(3 $\Rightarrow$ 1): Suppose $T \vdash \varphi_0 \vee \varphi_1$ for $\varphi_0, \varphi_1 \in \mathcal{B}(\Sigma_1)$. If $T \nvdash \varphi_0$, then $T \vdash \neg\varphi_0$. We obtain $T \vdash \varphi_1$. Thus T has $\mathcal{B}(\Sigma_n)$-DP. □

Proposition

If T has Π_n-DP, then T has Σ_n-DP.

Proof.

Suppose $T \vdash \varphi_0 \lor \varphi_1$ for some $\varphi_0, \varphi_1 \in \Sigma_n$.
There are Π_{n-1} formulas $\psi_0(x)$ and $\psi_1(x)$ such that $T \vdash \varphi_i \leftrightarrow \exists x \psi_i(x)$.
Let σ_0 and σ_1 be the following Σ_n sentences:

- $\sigma_0 \equiv \exists x(\psi_0(x) \land \forall y \leq x \neg \psi_1(y))$,
- $\sigma_1 \equiv \exists x(\psi_1(x) \land \forall y < x \neg \psi_0(y))$.

Then $T \vdash \neg \sigma_0 \lor \neg \sigma_1$.
By Π_n-DP, we have $T \vdash \neg \sigma_0$ or $T \vdash \neg \sigma_1$.
Since $T \vdash \varphi_0 \lor \varphi_1 \to \sigma_0 \lor \sigma_1$, we have $T \vdash \sigma_0 \lor \sigma_1$.
In the case of $T \vdash \neg \sigma_i$, we have $T \vdash \sigma_{1-i}$.
Then $T \vdash \exists x \psi_{1-i}(x)$ and hence $T \vdash \varphi_{1-i}$.
T has Σ_n-DP. $\square$

$$\mathcal{B}(\Sigma_n)\text{-DP} \Rightarrow \Pi_n\text{-DP} \Rightarrow \Sigma_n\text{-DP} \Rightarrow \mathcal{B}(\Sigma_{n-1})$$

Proposition

If T has $\mathcal{B}(\Sigma_{n-1})$-DP and is Σ_n-sound, then T has Σ_n-DP.

Proposition

- There exists a Σ_n-definable Σ_{n-1}-sound theory which has $\mathcal{B}(\Sigma_{n-1})$-DP but does not have Σ_n-DP.
- There exists a Σ_n-definable sound theory which has Σ_n-DP but does not have Π_n-DP.
- There exists a sound theory which has Π_n-DP but does not have $\mathcal{B}(\Sigma_n)$-DP.
- There exists a Σ_2-definable theory which has Π_n-DP but does not have $\mathcal{B}(\Sigma_n)$-DP.

Friedman's theorem

Definition

A theory T has the (numerical) existence property (EP)
$\overset{\text{def.}}{\Longleftrightarrow}$ **for any formula $\varphi(x)$,**
$T \vdash \exists x \varphi(x) \Rightarrow T \vdash \varphi(k)$ **for some k.**

Theorem (Kleene, 1945)

Heyting arithmetic HA has EP.

Theorem (Friedman, 1975)

Let T be a Σ_1-definable consistent intuitionistic number theory containing HA. T.F.A.E.:

1. **T has EP.**
2. **T has DP.**

Partial existence properties

Definition

A theory T has the Γ-existence property (Γ-EP) $\overset{\text{def.}}{\Longleftrightarrow}$ for any Γ formula $\varphi(x)$, $T \vdash \exists x \varphi(x) \Rightarrow T \vdash \varphi(k)$ for some k.

Proposition

T has Γ-EP $\Rightarrow$ T has Γ-DP.

Proof.

Suppose $T \vdash \varphi \lor \psi$ for $\varphi, \psi \in \Gamma$.
Let $\sigma(x)$ be a Γ formula equivalent to $(x = 0 \land \varphi) \lor (x \neq 0 \land \psi)$.
Then $T \vdash \varphi \lor \psi \to \exists x \sigma(x)$ and thus $T \vdash \exists x \sigma(x)$.
By Γ-EP, $T \vdash \sigma(k)$ for some k.
If $k = 0$, then $T \vdash \varphi$.
If $k \neq 0$, then $T \vdash \psi$. □

Γ-completeness

Definition

A theory T is **Γ-complete** (Γ-compl.)
$\stackrel{\text{def.}}{\Longleftrightarrow}$ for any Γ sentence φ ($\mathbb{N} \models \varphi \Rightarrow T \vdash \varphi$).

Fact

Every extension of PA is Σ_1-complete.

Proposition (Kikuchi and Kurahashi, 201?)

If T is consistent, then T.F.A.E.:

1. T is Σ_{n+1}-complete.
2. T is Σ_n-sound and for any Σ_n sentence φ, $T \vdash \varphi$ or $T \vdash \neg\varphi$.
3. T is Σ_n-sound and has $\mathcal{B}(\Sigma_n)$-DP.

Theorem

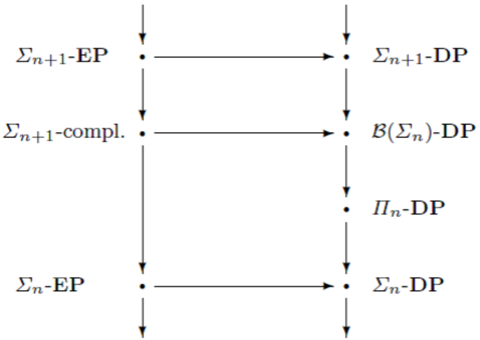
If T is consistent, then T.F.A.E.:

1. **T has Σ_n-EP.**
2. **T has Π_{n-1}-EP.**
3. **T is Σ_n-sound and T has Σ_n-DP.**
4. **T is Σ_n-sound and T is Σ_n-complete.**

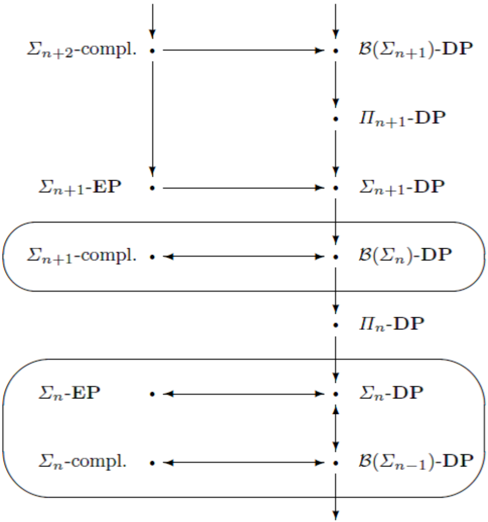
Corollary

If T is Σ_{n+1}-complete and consistent, then T has Σ_n-EP.

Implications for consistent theories



Implications for Σ_n-sound theories



1. **The first incompleteness theorem and DP**
2. **DP and related properties**
3. **$\mathbf{\Sigma_n}$-definable theories**
4. **Unprovability of formalized DP**

Σ_1-definable theories

We have already mentioned that the following theorems hold for Σ_1-definable theories.

The first incompleteness theorem (Gödel, 1931; Rosser, 1936)

If T is Σ_1-definable and consistent, then there exists a Σ_1 sentence φ such that $T \nvdash \varphi$ and $T \nvdash \neg\varphi$.

Theorem (Macintyre and Simmons, 1975)

If T is Σ_1-definable and consistent, then T does not have Π_1-DP.

Theorem (Guaspari, 1979)

Let T be a Σ_1-definable consistent theory. T.F.A.E.:

1. **T is Σ_1-sound.**
2. **T has Σ_1-DP.**

Can we generalize these results?

Question

First, we generalize Gödel-Rosser incompleteness theorem.
However the following statement is false.

If T is Σ_2-definable and consistent, then T is incomplete.

Because

Fact

There exists a complete consistent theory that is Σ_2-definable.

A generalization

We can generalize the Gödel-Rosser theorem as follows.

Proposition

T **is consistent** $\iff$ T **is** Σ_0**-sound.**

The first incompleteness theorem (rephrased)

If T **is** Σ_1**-definable and** Σ_0**-sound, then there exists a** Σ_1 **sentence** φ **such that** $T \nvdash \varphi$ **and** $T \nvdash \neg\varphi$**.**

Theorem (Kikuchi and Kurahashi, 201?)

If T **is** Σ_n**-definable and** Σ_{n-1}**-sound, then there exists a** Σ_n **sentence** φ **such that** $T \nvdash \varphi$ **and** $T \nvdash \neg\varphi$**.**

Corollary

If T **is** Σ_n**-definable and** Σ_{n-1}**-sound,**
then T **does not have** $\mathcal{B}(\Sigma_n)$**-DP.**

A generalization of Macintyre and Simmons' theorem

Moreover, the following strengthening of a generalization of the Gödel-Rosser theorem holds, that is a generalization of Macintyre and Simmons' theorem.

Theorem

If T is Σ_n-definable and Σ_{n-1}-sound, then T does not have Π_n-DP.

Guaspari's and Friedman's theorems

We proved a generalized version of Guaspari's theorem, that is also a counterpart to Friedman's theorem.

Theorem

If T is Σ_n-definable, Σ_{n-1}-sound and has Σ_n-DP, then T has Σ_n-EP.

Corollary

If T is Σ_n-definable and Σ_n-complete, T.F.A.E.:

1. **T is Σ_n-sound.**
2. **T has Σ_n-EP.**
3. **T has Σ_n-DP.**

This is best possible.

Proposition

There exists a Σ_{n+1}-definable Σ_{n-1}-sound theory which has DP but does not have Σ_n-EP.

1. The first incompleteness theorem and DP
2. DP and related properties
3. Σ_n-definable theories
4. Unprovability of formalized DP

Myhill's theorem

Myhill proved the following unprovability result.

Theorem (Myhill, 1973)

Let T be a Σ_1-definable consistent intuitionistic number theory containing HA. If T has DP, then $T \nvdash$ "T has DP".

We investigate Myhill's theorem in our framework.

Provability predicates

- For each Σ_n-definition $\sigma(x)$ of T,
  we can construct a Σ_n formula $\mathrm{Prf}_\sigma(x, y)$ saying that

  "y is a proof of x in the theory defined by σ"
- Let $\mathrm{Pr}_\sigma(x)$ be the Σ_1 formula $\exists y \mathrm{Prf}_\sigma(x, y)$.

$$\mathrm{DP}_\sigma \equiv \forall x \forall y(\mathrm{Sent}(x) \land \mathrm{Sent}(y) \land \mathrm{Pr}_\sigma(x \lor y) \to \mathrm{Pr}_\sigma(x) \lor \mathrm{Pr}_\sigma(y))$$

$$\mathrm{DP}_\sigma(\Sigma_n) \equiv \forall x \forall y(\Sigma_n(x) \land \Sigma_n(y) \land \mathrm{Pr}_\sigma(x \lor y) \to \mathrm{Pr}_\sigma(x) \lor \mathrm{Pr}_\sigma(y))$$

Theorem

Let T be a Σ_n-definable Σ_n-complete theory. T.F.A.E.:

1. T has Σ_n-DP.
2. $T \nvdash \mathrm{DP}_\sigma(\Sigma_n)$ for all Σ_n definitions $\sigma(x)$ of T.
3. $T \nvdash \mathrm{DP}_\sigma$ for all Σ_n definitions $\sigma(x)$ of T.

Corollary

Let T be a Σ_n-definable theory. If T has Σ_n-EP, then $T \nvdash \mathrm{DP}_\sigma(\Sigma_n)$ for all Σ_n definitions $\sigma(x)$ of T.

The assumption of Σ_n-completeness in the statement of Theorem cannot be removed.

Proposition

There exists a Σ_2-definable theory having DP such that $T \vdash \mathrm{DP}_\sigma$ for some Σ_2 definition $\sigma(x)$ of T.

Thank you!