Two theorems on provability logics

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Contents

Numerations

In this talk, T and U always denote $\mathcal{L}_A$-theories extending $I\Sigma_1$, where $\mathcal{L}_A$ is the language of first-order arithmetic.

Definition

A formula $\tau(v)$ is a numeration of T
$:\iff$ for any sentence φ ($\varphi \in T \iff \mathrm{PA} \vdash \tau(\ulcorner \varphi \urcorner)$).

Fact

If T is recursively enumerable (r.e.), then there exists a Σ_1 numeration of T.

Provability predicates

For each numeration $\tau(v)$ of T,
we can naturally construct a formula $\mathrm{Pr}_\tau(x)$ saying that

"x is provable from the set of all sentences satisfying $\tau(v)$."

The formula $\mathrm{Pr}_\tau(x)$ is said to be a provability predicate of $\tau(v)$.

Fact

Let $\tau(v)$ be any numeration of T.

- If $T \vdash \varphi$, then $\mathrm{PA} \vdash \mathrm{Pr}_\tau(\ulcorner \varphi \urcorner)$.
- $\mathrm{PA} \vdash \mathrm{Pr}_\tau(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_\tau(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_\tau(\ulcorner \psi \urcorner))$.
- If φ is Σ_1, then $\mathrm{PA} \vdash \varphi \to \mathrm{Pr}_\tau(\ulcorner \varphi \urcorner)$.

Provability Logic is a research area investigating these properties by means of modal logic.

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Arithmetical interpretations

Definition

A mapping f from the set of all propositional variables to the set of $\mathcal{L}_A$-sentences is said to be an arithmetical interpretation.

Let $\tau(v)$ be any numeration of T.

Each arithmetical interpretation f is uniquely extended to the mapping f_τ from the set of all modal formulas to the set of $\mathcal{L}_A$-sentences so that f_τ satisfies the following conditions:

- $f_\tau(\bot)$ is $0 = 1$.
- $f_\tau(A \wedge B)$ is $f_\tau(A) \wedge f_\tau(B)$.
- $\cdots$
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Provability logics

Definition

Let U be any theory and $\tau(v)$ be any numeration of T.

$\mathsf{PL}_\tau(U) := \{A : U \vdash f_\tau(A)$ for all arithmetical interpretations $f\}$.

The set $\mathsf{PL}_\tau(U)$ is said to be the provability logic of $\tau(v)$ relative to U.

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Modal logics **K** and **GL**

Definition

The axioms of the modal logic K are as follows:

- **all tautologies in the language of propositional modal logic,**
- $\Box(p \to q) \to (\Box p \to \Box q)$.

The inference rules of K are modus ponens $\dfrac{A, \; A \to B}{B}$, **necessitation** $\dfrac{A}{\Box A}$ **and substitution.**

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For each set X of modal formulas, let $\mathsf{GL}X$ be the logic whose axioms are all theorems of GL and all elements of X, and whose inference rules are modus ponens and substitution.

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For each $n \in \omega$, let F_n be the modal formula $\Box^{n+1}\bot \to \Box^n\bot$.

Definition

Let α be a subset of ω, and β be a cofinite subset of ω.

- $\mathsf{GL}_\alpha = \mathsf{GL}\{F_n : n \in \alpha\}$
- $\mathsf{GL}^-_\beta = \mathsf{GL}\{\bigvee_{n \notin \beta} \neg F_n\}$
- $\mathsf{D} = \mathsf{GL}\{\Box(\Box p \vee \Box q) \to (\Box p \vee \Box q)\}$
- $\mathsf{S} = \mathsf{GL}\{\Box p \to p\}$
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Major achievements of provability logics

Arithmetical completeness theorems (Solovay, 1976)

Let $\tau(v)$ be any Σ_1 numeration of T.

- If T is Σ_1-sound, then $\mathsf{PL}_\tau(T) = \mathsf{GL}$.
- If T is sound, then $\mathsf{PL}_\tau(\mathsf{TA}) = \mathsf{S}$,

where $\mathsf{TA} = \{\varphi : \mathbb{N} \models \varphi\}$.

Theorem (Artemov - Visser - Japaridze - Beklemishev, 1980–1989)

Let $\tau(v)$ be any Σ_1 numeration of T.

Then $\mathsf{PL}_\tau(U)$ is one of GL_α, D_β, S_β and GL_β^-.

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For each fixed recursively axiomatized consistent extension U of PA, which modal logic can be of the form $\mathsf{PL}_\tau(U)$ for some numeration $\tau(v)$?

We investigated this question in the following two particular cases.

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Jeroslow's theorem

First, we restrict our considerations to Σ_1 numerations $\tau(v)$.
We paid attention to a theorem due to Jeroslow.

Theorem (Jeroslow, 1971)

For any r.e. consistent extension U of PA, there exists a Σ_1 numeration $\tau(v)$ of some extension T of $I\Sigma_1$ s.t. $U = T + \mathsf{Con}_\tau$.

Corollary

For any r.e. consistent extension U of PA, there exists a Σ_1 numeration $\tau(v)$ of some extension T of $I\Sigma_1$ s.t. $\mathsf{PL}_\tau(U) = \mathsf{GL}_\alpha$ where $\alpha = \{0\}$.

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The first theorem

Theorem 1

Let U be any r.e. consistent extension of PA.
If L is one of the logics GL_α, D_β, S_β and GL_β^- where $\alpha \subseteq \omega$ is r.e. and $\beta \subseteq \omega$ is cofinite.
Then there exists a Σ_1 numeration $\tau(v)$ of some extension of $I\Sigma_1$ s.t. $\mathsf{PL}_\tau(U) = \mathsf{L}$.

Kurahashi, T., Provability logics relative to a fixed extension of Peano Arithmetic, submitted.

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Remarks on Theorem 1

Remark 1

Theorem 1 is proved by applying Jeroslow's method.
For example, we found a Σ_1 numeration $\tau(v)$ of some T s.t. $U = T + \mathsf{RFN}_{\Sigma_1}(\tau)$, and then $\mathsf{PL}_\tau(U) = \mathsf{D}$.

Remark 2

We found a Σ_1 numeration $\tau(v)$ of $I\Sigma_1$ s.t. $\mathsf{PL}_\tau(\mathrm{PA}) = \mathsf{S}_\beta$ for each cofinite β.
This can be compared with Kreisel and Lévy's theorem stating that $\mathrm{PA} = I\Sigma_1 + \mathsf{RFN}(\sigma)$ for a natural Σ_1 numeration $\sigma(v)$ of $I\Sigma_1$. Then $\mathsf{PL}_\sigma(\mathrm{PA}) = \mathsf{S}$.

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Sacchetti's logics

We restrict our considerations to numerations $\tau(v)$ of U.

- **Sacchetti introduced the logics $\mathsf{K} + \Box(\Box^n p \to p) \to \Box p$ $(n \geq 2)$ and proved that the fixed-point theorem holds for these logics.**
- **There is no Σ_1 numeration $\tau(v)$ of U s.t. $\mathsf{PL}_\tau(U)$ is $\mathsf{K} + \Box(\Box^n p \to p) \to \Box p$ for $n \geq 2$.**

Question (Sacchetti, 2001)

Is there a nonstandard provability predicate of PA whose provability logic is $\mathsf{K} + \Box(\Box^n p \to p) \to \Box p$ for $n \geq 2$?

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The second theorem

We settled Sacchetti's question affirmatively.

Theorem 2

Let U be any recursively axiomatized consistent extension of PA. If L is one of the logics K and $\mathsf{K} + \Box(\Box^n p \to p) \to \Box p$ $(n \geq 2)$. Then there exists a Σ_2 numeration $\tau(v)$ of U s.t. $\mathsf{PL}_\tau(U) = \mathsf{L}$.

- **Kurahashi, T., Arithmetical completeness theorem for modal logic K,** *Studia Logica*, **to appear.**
- **Kurahashi, T., Arithmetical soundness and completeness by Σ_2 numerations, submitted.**

An open problem

Proposition

For every numeration $\tau(v)$ of U, $\mathsf{PL}_\tau(U)$ is a normal modal logic, that is, $\mathsf{PL}_\tau(U)$ contains K and closed under the inference rules of K.

Open Problem

Which normal modal logic can be of the form $\mathsf{PL}_\tau(U)$ for some numeration $\tau(v)$ of U? How about $\mathsf{KD} = \mathsf{K} + \neg\Box\bot$?

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There is no numeration $\tau(v)$ of U whose provability logic $\mathsf{PL}_\tau(U)$ is one of the following normal modal logics:

- $\mathsf{T} = \mathsf{K} + \Box p \to p$ **(well known)**
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