Rosser-type Henkin sentences and local reflection principles

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2. Rosser-type local reflection principles

Henkin's problem

We fix a theory T which is

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2. **recursively axiomatized; and**
3. **an extension of Peano arithmetic PA.**

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Is every sentence asserting its own T-provability provable in T?

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Fix a standard provability predicate $\mathrm{Pr}_T(x)$ of T.

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For any φ, $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi \Rightarrow T \vdash \varphi$.

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Thus every Henkin sentence of T is provable in T.

Rosser-type Henkin sentences

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Let $\mathrm{Prf}_T(x, y)$ be a Δ_1 formula s.t. $\mathrm{Pr}_T(x) \equiv \exists y \mathrm{Prf}_T(x, y)$.

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A Rosser provability predicate $\mathsf{Pr}_T^R(x)$ of T is defined as $\exists y(\mathsf{Prf}_T(x, y) \land \forall z \leq y \neg \mathsf{Prf}_T(\neg x, z))$.

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A sentence φ is a Rosser-type Henkin sentence of $\mathrm{Pr}_T^R(x)$ $\overset{\text{def.}}{\Leftrightarrow} T \vdash \varphi \leftrightarrow \mathrm{Pr}_T^R(\ulcorner \varphi \urcorner)$.

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Question (Halbach and Visser (2014))

Is every Rosser-type Henkin sentence of $\mathrm{Pr}_T^R(x)$ either provable or refutable in T?

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Is every Rosser-type Henkin sentence of $\mathrm{Pr}_T^R(x)$ either provable or refutable in T?

Answer (K.)

Whether $\mathrm{Pr}_T^R(x)$ has an independent Rosser-type Henkin sentence is dependent on the choice of $\mathrm{Pr}_T^R(x)$.

Rosser predicate with an independent Rosser-type Henkin sentence

Theorem (K.)

For any Σ_1 sentence φ, T.F.A.E.:

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For any Σ_1 sentence φ, T.F.A.E.:

1. **There is a proof predicate $\mathsf{Prf}'_T(x, y)$ s.t.**
   - **PA $\vdash \forall x(\mathsf{Pr}_T(x) \leftrightarrow \mathsf{Pr}'_T(x))$,**
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2. There is a Σ_1 sentence ψ s.t.
   - PA $\vdash \neg\varphi \lor \neg\psi$,
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Since every Σ_1 Rosser sentence of $\mathrm{Pr}^R_T(x)$ satisfies the condition 2 in the statement, we obtained the following corollary.

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Since every Σ_1 Rosser sentence of $\mathrm{Pr}^R_T(x)$ satisfies the condition 2 in the statement, we obtained the following corollary.

Corollary

There is a Rosser provability predicate of T having an independent Rosser-type Henkin sentence.

Rosser predicate without independent Rosser-type Henkin sentences

On the other hand, we obtained the following theorem.

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There is a Rosser provability predicate $\mathrm{Pr}_T^R(x)$ of T s.t. for any sentence φ,
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Corollary

There is a Rosser provability predicate of T having no independent Rosser-type Henkin sentence.

1. **Rosser-type Henkin sentences**
2. **Rosser-type local reflection principles**

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The set $\text{Rfn}^R(T) := \{\text{Pr}_T^R(\ulcorner \varphi \urcorner) \to \varphi : \varphi \text{ is a sentence}\}$ is called the Rosser-type local reflection principle for $\text{Pr}_T^R(x)$.

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- It is known that the second incompleteness theorem does not hold for $\mathsf{Pr}_T^R(x)$.
- Also $\mathsf{Rfn}^R(T)$ may have different properties from $\mathsf{Rfn}(T)$.

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Goryachev's Theorem (1989)

There is a Rosser provability predicate $\mathsf{Pr}_T^R(x)$ of T s.t. $T + \mathsf{Rfn}(T)$ and $T + \mathsf{Rfn}^R(T)$ are equivalent.

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Question

Is there $\mathsf{Pr}_T^R(x)$ s.t. $T + \mathsf{Rfn}_T^R(x)$ is strictly weaker than $T + \mathsf{Rfn}(T)$?

Shavrukov's problem

Shavrukov's problem (1991)

Is there $\mathrm{Prf}_T(x)$ s.t.
for any distinct sentences $\varphi_0, \ldots, \varphi_{n-1}$,
if $T \vdash \bigvee_{i<n-1} \forall y(\mathrm{Prf}_T(\ulcorner \varphi_i \urcorner, y) \to \exists z \leq y \mathrm{Prf}_T(\ulcorner \varphi_{i+1} \urcorner, z))$,
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- Shavrukov pointed out that an affirmative answer to his problem gives a Rosser provability predicate $\mathsf{Pr}_T^R(x)$ s.t. $T + \mathsf{Rfn}^R(T)$ is strictly weaker than $T + \mathsf{Rfn}(T)$.

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Shavrukov's problem is solved affirmatively.

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- $T + \mathsf{Rfn}_{\Pi_1}(T) \nvdash \mathsf{Rfn}_{\Sigma_1}(T)$.

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$T + \mathsf{Rfn}_{\Sigma_1}(T)$ always contains $\mathsf{Rfn}_{\Pi_1}(T)$.

Σ_1 and Π_1 reflection principles

However, the situation for Σ_1 and Π_1 local reflection principles is different.

$T + \mathrm{Rfn}_{\Sigma_1}(T)$ always contains $\mathrm{Rfn}_{\Pi_1}(T)$.

Theorem (K.)

Whether $T + \mathrm{Rfn}^R_{\Sigma_1}(T)$ contains $\mathrm{Rfn}^R_{\Pi_1}(T)$ is dependent on the choice of $\mathrm{Pr}^R_T(x)$.

Open problems

Problem

For $\Gamma \in \{\Sigma_n, \Pi_n : n \geq 1\}$,
is $T + \mathsf{Rfn}^R(T)$ a Γ-conservative extension of $T + \mathsf{Rfn}^R_\Gamma(T)$?

Problem

- Is $T + \mathsf{Rfn}^R_{\Pi_1}(T)$ finitely axiomatizable over T?
- For $\Gamma \in \{\Sigma_n, \Pi_{n+1} : n \geq 1\}$,
  is $T + \mathsf{Rfn}^R_\Gamma(T)$ not finitely axiomatizable over T?

Problem

Study Rosser-type uniform reflection principles.