# Heterodox Models of Peano Arithmetic

**Taishi Kurahashi**
**National Institiute of Technology, Kisarazu College, Japan**

**Makoto Kikuchi**
**Kobe University, Japan**

**15th CLMPS**
**Helsinki**
**August 5, 2015**

**Unfortunately, the abstract in the book is not ours.**
**The correct abstract can be found in the CLMPS web page.**

## Outline

## Provability predicates

- **In this talk, we assume that Peano Arithmetic PA is sound, that is, $\forall\varphi(\mathsf{PA} \vdash \varphi \Rightarrow \mathbb{N} \models \varphi)$.**

## Provability predicates

- **In this talk, we assume that Peano Arithmetic $\mathsf{PA}$ is sound, that is, $\forall\varphi(\mathsf{PA} \vdash \varphi \Rightarrow \mathbb{N} \models \varphi)$.**
- **We fix a $\Sigma_1$ formula $\mathsf{Pr}(x)$ satisfying the following conditions:**

1. $\mathsf{PA} \vdash \varphi \Leftrightarrow \mathsf{PA} \vdash \mathsf{Pr}(\ulcorner \varphi \urcorner)$
2. $\mathsf{PA} \vdash \mathsf{Pr}(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}(\ulcorner \varphi \urcorner) \to \mathsf{Pr}(\ulcorner \psi \urcorner))$
3. $\varphi : \Sigma_1 \Rightarrow \mathsf{PA} \vdash \varphi \to \mathsf{Pr}(\ulcorner \varphi \urcorner)$

**$\mathsf{Pr}(x)$ is called a provability predicate of $\mathsf{PA}$.**

## Provability predicates

- **In this talk, we assume that Peano Arithmetic PA is sound, that is, $\forall\varphi(\mathsf{PA} \vdash \varphi \Rightarrow \mathbb{N} \models \varphi)$.**
- **We fix a $\Sigma_1$ formula $\mathsf{Pr}(x)$ satisfying the following conditions:**

1. $\mathsf{PA} \vdash \varphi \Leftrightarrow \mathsf{PA} \vdash \mathsf{Pr}(\ulcorner \varphi \urcorner)$
2. $\mathsf{PA} \vdash \mathsf{Pr}(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}(\ulcorner \varphi \urcorner) \to \mathsf{Pr}(\ulcorner \psi \urcorner))$
3. $\varphi : \Sigma_1 \Rightarrow \mathsf{PA} \vdash \varphi \to \mathsf{Pr}(\ulcorner \varphi \urcorner)$

**$\mathsf{Pr}(x)$ is called a provability predicate of PA.**

- $\mathsf{Con}_{\mathsf{PA}} :\equiv \neg \mathsf{Pr}(\ulcorner 0 = 1 \urcorner)$.

## Models having a proof of $\mathbf{0} = \mathbf{1}$

- **By the second incompleteness theorem, $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}}$ since $\mathsf{PA}$ is consistent.**

## Models having a proof of $\mathbf{0} = \mathbf{1}$

- **By the second incompleteness theorem, $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}}$ since $\mathsf{PA}$ is consistent.**
- **Then there exists a model $M$ of $\mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.**

## Models having a proof of $\mathbf{0} = \mathbf{1}$

- **By the second incompleteness theorem, $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}}$ since $\mathsf{PA}$ is consistent.**
- **Then there exists a model $M$ of $\mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.**
- **Since $M \models \mathsf{Pr}(\ulcorner 0 = 1 \urcorner)$, $M$ has a non-standard proof of $0 = 1$.**

## Models having a proof of $\mathbf{0} = \mathbf{1}$

- **By the second incompleteness theorem, $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}}$ since $\mathsf{PA}$ is consistent.**
- **Then there exists a model $M$ of $\mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.**
- **Since $M \models \mathsf{Pr}(\ulcorner 0 = 1 \urcorner)$, $M$ has a non-standard proof of $0 = 1$.**

**This fact is well-known, but is not well-understood.**

## Models having a proof of $\mathbf{0} = \mathbf{1}$

- By the second incompleteness theorem, $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}}$ since $\mathsf{PA}$ is consistent.
- Then there exists a model $M$ of $\mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.
- Since $M \models \mathsf{Pr}(\ulcorner 0 = 1 \urcorner)$, $M$ has a non-standard proof of $0 = 1$.

This fact is well-known, but is not well-understood.

- We want to know how to obtain such a proof in $M$.

## Models having a proof of $\mathbf{0} = \mathbf{1}$

- By the second incompleteness theorem, $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}}$ since $\mathsf{PA}$ is consistent.
- Then there exists a model $M$ of $\mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.
- Since $M \models \mathsf{Pr}(\ulcorner 0 = 1 \urcorner)$, $M$ has a non-standard proof of $0 = 1$.

This fact is well-known, but is not well-understood.

- We want to know how to obtain such a proof in $M$.
- For this purpose, we investigate the provability in models of $\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$.

## Theorems in non-standard models

### Definition

Let $M \models \mathsf{PA}$.

$$\mathsf{Thm}(M) := \{\varphi \mid M \models \mathsf{Pr}(\ulcorner \varphi \urcorner)\}.$$

## Theorems in non-standard models

**Definition**

Let $M \models \mathsf{PA}$.

$$\mathsf{Thm}(M) := \{\varphi \mid M \models \mathsf{Pr}(\ulcorner \varphi \urcorner)\}.$$

**Proposition**

1. $\mathsf{Thm}(\mathbb{N}) = \{\varphi \mid \mathsf{PA} \vdash \varphi\}$.

## Theorems in non-standard models

### Definition

Let $M \models \mathsf{PA}$.

$$\mathsf{Thm}(M) := \{\varphi \mid M \models \mathsf{Pr}(\ulcorner \varphi \urcorner)\}.$$

### Proposition

1. $\mathsf{Thm}(\mathbb{N}) = \{\varphi \mid \mathsf{PA} \vdash \varphi\}$.
2. $M \subseteq_e N \Rightarrow \mathsf{Thm}(M) \subseteq \mathsf{Thm}(N)$.

# Theorems in non-standard models

## Definition

Let $M \models \mathsf{PA}$.

$$\mathsf{Thm}(M) := \{\varphi \mid M \models \mathsf{Pr}(\ulcorner \varphi \urcorner)\}.$$

## Proposition

1. $\mathsf{Thm}(\mathbb{N}) = \{\varphi \mid \mathsf{PA} \vdash \varphi\}$.
2. $M \subseteq_e N \Rightarrow \mathsf{Thm}(M) \subseteq \mathsf{Thm}(N)$.
3. $\mathsf{Thm}(\mathbb{N}) \subseteq \mathsf{Thm}(M)$.

## Theorems in non-standard models

### Definition

Let $M \models \mathsf{PA}$.

$$\mathsf{Thm}(M) := \{\varphi \mid M \models \mathsf{Pr}(\ulcorner\varphi\urcorner)\}.$$

### Proposition

1. $\mathsf{Thm}(\mathbb{N}) = \{\varphi \mid \mathsf{PA} \vdash \varphi\}$.
2. $M \subseteq_e N \Rightarrow \mathsf{Thm}(M) \subseteq \mathsf{Thm}(N)$.
3. $\mathsf{Thm}(\mathbb{N}) \subseteq \mathsf{Thm}(M)$.
4. $M \models \mathsf{Con}_{\mathsf{PA}} \Leftrightarrow \exists\varphi$ s.t. $\varphi \notin \mathsf{Thm}(M)$.

## Theorems in non-standard models

### Definition

Let $M \models \mathsf{PA}$.

$$\mathsf{Thm}(M) := \{\varphi \mid M \models \mathsf{Pr}(\ulcorner \varphi \urcorner)\}.$$

### Proposition

1. $\mathsf{Thm}(\mathbb{N}) = \{\varphi \mid \mathsf{PA} \vdash \varphi\}$.
2. $M \subseteq_e N \Rightarrow \mathsf{Thm}(M) \subseteq \mathsf{Thm}(N)$.
3. $\mathsf{Thm}(\mathbb{N}) \subseteq \mathsf{Thm}(M)$.
4. $M \models \mathsf{Con}_{\mathsf{PA}} \Leftrightarrow \exists \varphi$ s.t. $\varphi \notin \mathsf{Thm}(M)$.

### Questions

1. Is there a model $M$ of $\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $\mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(M)$?

## Theorems in non-standard models

### Definition

Let $M \models \mathsf{PA}$.

$$\mathsf{Thm}(M) := \{\varphi \mid M \models \mathsf{Pr}(\ulcorner \varphi \urcorner)\}.$$

### Proposition

1. $\mathsf{Thm}(\mathbb{N}) = \{\varphi \mid \mathsf{PA} \vdash \varphi\}$.
2. $M \subseteq_e N \Rightarrow \mathsf{Thm}(M) \subseteq \mathsf{Thm}(N)$.
3. $\mathsf{Thm}(\mathbb{N}) \subseteq \mathsf{Thm}(M)$.
4. $M \models \mathsf{Con}_{\mathsf{PA}} \Leftrightarrow \exists \varphi$ s.t. $\varphi \notin \mathsf{Thm}(M)$.

### Questions

1. Is there a model $M$ of $\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $\mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(M)$?
2. Moreover, is there a model $M$ s.t. $\mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(M)$ and $\mathsf{Thm}(M) \subseteq \mathsf{TA}$? (Where $\mathsf{TA} = \{\varphi \mid \mathbb{N} \models \varphi\}$)

## Terminology used in this talk

### Definition

**Let $M \models \mathsf{PA}$.**

## Terminology used in this talk

### Definition

**Let $M \models$ PA.**

1. **$M$ is heterodox $:\Leftrightarrow$ $\mathsf{Thm}(M) \not\subseteq$ TA.**

## Terminology used in this talk

### Definition

Let $M \models \mathsf{PA}$.

1. $M$ is heterodox $:\Leftrightarrow \mathsf{Thm}(M) \not\subseteq \mathsf{TA}$.
2. $M$ is illusory $:\Leftrightarrow \mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(M)$.

## Terminology used in this talk

### Definition

Let $M \models \mathsf{PA}$.

1. $M$ is heterodox $:\Leftrightarrow \mathsf{Thm}(M) \not\subseteq \mathsf{TA}$.
2. $M$ is illusory $:\Leftrightarrow \mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(M)$.
3. $M$ is insane $:\Leftrightarrow M \models \neg\mathsf{Con}_{\mathsf{PA}}$.

## Terminology used in this talk

**Definition**

Let $M \models \mathsf{PA}$.

1. $M$ is heterodox $:\Leftrightarrow \mathsf{Thm}(M) \not\subseteq \mathsf{TA}$.
2. $M$ is illusory $:\Leftrightarrow \mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(M)$.
3. $M$ is insane $:\Leftrightarrow M \models \neg\mathsf{Con}_{\mathsf{PA}}$.

It is easy to see the following implications.

$$M\text{: insane} \Rightarrow M\text{: heterodox} \Rightarrow M\text{: illusory}$$

## Illusory models

**Proposition**

$\exists M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $M$ is illusory.

# Illusory models

## Proposition

$\exists M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $M$ is illusory.

## Proof.

It is known $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}} \to \neg\mathsf{Pr}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}}\urcorner)$.

# Illusory models

**Proposition**

$\exists M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $M$ is illusory.

**Proof.**

It is known $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}} \to \neg \mathsf{Pr}(\ulcorner \neg \mathsf{Con}_{\mathsf{PA}} \urcorner)$.

Then $\exists M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $M \models \mathsf{Pr}(\ulcorner \neg \mathsf{Con}_{\mathsf{PA}} \urcorner)$.

# Illusory models

**Proposition**

$\exists M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $M$ is illusory.

**Proof.**

It is known $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}} \rightarrow \neg\mathsf{Pr}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}}\urcorner)$.
Then $\exists M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $M \models \mathsf{Pr}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}}\urcorner)$.
Thus $M$ is illusory since $\neg\mathsf{Con}_{\mathsf{PA}} \notin \mathsf{Thm}(\mathbb{N})$ and $\neg\mathsf{Con}_{\mathsf{PA}} \in \mathsf{Thm}(M)$. □

# Illusory models

**Proposition**

$\exists M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $M$ is illusory.

**Proof.**

It is known $\mathsf{PA} \nvdash \mathsf{Con}_{\mathsf{PA}} \to \neg \mathsf{Pr}(\ulcorner \neg \mathsf{Con}_{\mathsf{PA}} \urcorner)$.
Then $\exists M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ s.t. $M \models \mathsf{Pr}(\ulcorner \neg \mathsf{Con}_{\mathsf{PA}} \urcorner)$.
Thus $M$ is illusory since $\neg \mathsf{Con}_{\mathsf{PA}} \notin \mathsf{Thm}(\mathbb{N})$ and $\neg \mathsf{Con}_{\mathsf{PA}} \in \mathsf{Thm}(M)$. □

Moreover,

**Theorem**

The cardinality of the set $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$ is $2^{\aleph_0}$.

## Theorem

**If $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ is illusory, then**

### Theorem

**If $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ is illusory, then**

1. **$\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a true $\Pi_1$ sentence; and**

**Theorem**

If $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ is illusory, then

1. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a true $\Pi_1$ sentence; and
2. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a false $\Sigma_1$ sentence.

## Theorem

If $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ is illusory, then

1. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a true $\Pi_1$ sentence; and
2. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a false $\Sigma_1$ sentence.

## Proof.

Let $\varphi$ be any sentence s.t. $\varphi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.

## Theorem

If $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ is illusory, then

1. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a true $\Pi_1$ sentence; and
2. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a false $\Sigma_1$ sentence.

## Proof.

Let $\varphi$ be any sentence s.t. $\varphi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.
Let $\pi$ be a $\Pi_1$ sentence satisfying
$\mathsf{PA} \vdash \pi \leftrightarrow \forall y(\mathsf{Prf}(\ulcorner \varphi \urcorner, y) \rightarrow \exists z \leq y \mathsf{Prf}(\ulcorner \pi \urcorner, z))$.

### Theorem

If $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ is illusory, then

1. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a true $\Pi_1$ sentence; and
2. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a false $\Sigma_1$ sentence.

### Proof.

Let $\varphi$ be any sentence s.t. $\varphi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.
Let $\pi$ be a $\Pi_1$ sentence satisfying
$\mathsf{PA} \vdash \pi \leftrightarrow \forall y(\mathsf{Prf}(\ulcorner \varphi \urcorner, y) \rightarrow \exists z \leq y \mathsf{Prf}(\ulcorner \pi \urcorner, z))$.
Then $\mathbb{N} \models \pi$ and $\pi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.

**Theorem**

If $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ is illusory, then

1. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a true $\Pi_1$ sentence; and
2. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a false $\Sigma_1$ sentence.

**Proof.**

Let $\varphi$ be any sentence s.t. $\varphi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.
Let $\pi$ be a $\Pi_1$ sentence satisfying
$\mathsf{PA} \vdash \pi \leftrightarrow \forall y(\mathsf{Prf}(\ulcorner \varphi \urcorner, y) \to \exists z \leq y \mathsf{Prf}(\ulcorner \pi \urcorner, z))$.
Then $\mathbb{N} \models \pi$ and $\pi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.
Let $\sigma$ be a $\Sigma_1$ sentence $\mathsf{Pr}(\ulcorner \varphi \urcorner)$.

## Theorem

If $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ is illusory, then

1. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a true $\Pi_1$ sentence; and
2. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a false $\Sigma_1$ sentence.

## Proof.

Let $\varphi$ be any sentence s.t. $\varphi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.
Let $\pi$ be a $\Pi_1$ sentence satisfying
$\mathsf{PA} \vdash \pi \leftrightarrow \forall y(\mathsf{Prf}(\ulcorner \varphi \urcorner, y) \to \exists z \leq y \mathsf{Prf}(\ulcorner \pi \urcorner, z))$.
Then $\mathbb{N} \models \pi$ and $\pi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.
Let $\sigma$ be a $\Sigma_1$ sentence $\mathsf{Pr}(\ulcorner \varphi \urcorner)$.
Then $\mathbb{N} \models \neg\sigma$ and $\sigma \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$. □

**Theorem**

If $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ is illusory, then

1. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a true $\Pi_1$ sentence; and
2. $\mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$ contains a false $\Sigma_1$ sentence.

**Proof.**

Let $\varphi$ be any sentence s.t. $\varphi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.
Let $\pi$ be a $\Pi_1$ sentence satisfying
$\mathsf{PA} \vdash \pi \leftrightarrow \forall y(\mathsf{Prf}(\ulcorner \varphi \urcorner, y) \to \exists z \leq y \mathsf{Prf}(\ulcorner \pi \urcorner, z))$.
Then $\mathbb{N} \models \pi$ and $\pi \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$.
Let $\sigma$ be a $\Sigma_1$ sentence $\mathsf{Pr}(\ulcorner \varphi \urcorner)$.
Then $\mathbb{N} \models \neg\sigma$ and $\sigma \in \mathsf{Thm}(M) \setminus \mathsf{Thm}(\mathbb{N})$. □

**Corollary**

$M$: illusory $\Leftrightarrow$ $M$: heterodox.

## Completeness

**We have shown that for any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$, $\mathsf{Thm}(M) \neq \mathsf{TA}$.**

## Completeness

**We have shown that for any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$, $\mathsf{Thm}(M) \neq \mathsf{TA}$.**

**Proposition**

**For any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$, $\mathsf{Thm}(M)$ is not complete.**

## Completeness

**We have shown that for any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$, $\mathsf{Thm}(M) \neq \mathsf{TA}$.**

### Proposition

**For any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$, $\mathsf{Thm}(M)$ is not complete.**

### Proof.

**Let $\varphi$ be a Rosser sentence of $\mathsf{PA}$.**

## Completeness

We have shown that for any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$, $\mathsf{Thm}(M) \neq \mathsf{TA}$.

**Proposition**

For any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$, $\mathsf{Thm}(M)$ is not complete.

**Proof.**

Let $\varphi$ be a Rosser sentence of $\mathsf{PA}$.

Then

$\mathsf{PA} \vdash \mathsf{Con}_{\mathsf{PA}} \to \neg\mathsf{Pr}(\ulcorner \varphi \urcorner) \land \neg\mathsf{Pr}(\ulcorner \neg\varphi \urcorner)$.

## Completeness

We have shown that for any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$, $\mathsf{Thm}(M) \neq \mathsf{TA}$.

**Proposition**

For any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$, $\mathsf{Thm}(M)$ is not complete.

**Proof.**

Let $\varphi$ be a Rosser sentence of $\mathsf{PA}$.

Then

$\mathsf{PA} \vdash \mathsf{Con}_{\mathsf{PA}} \to \neg\mathsf{Pr}(\ulcorner\varphi\urcorner) \wedge \neg\mathsf{Pr}(\ulcorner\neg\varphi\urcorner)$.

Thus for any $M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$,

$\varphi, \neg\varphi \notin \mathsf{Thm}(M)$. □

# Maximality

**There is no complete theory in $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$.**

# Maximality

**There is no complete theory in $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$.**
**However, this family can have a maximal element w.r.t. $\subseteq$.**

## Maximality

**There is no complete theory in $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$.**
**However, this family can have a maximal element w.r.t. $\subseteq$.**
**In fact,**

**Theorem**

**The family $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$ contains $2^{\aleph_0}$ maximal elements.**

## Maximality

**There is no complete theory in $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$.**
**However, this family can have a maximal element w.r.t. $\subseteq$.**
**In fact,**

### Theorem

**The family $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$ contains $2^{\aleph_0}$ maximal elements.**

**We obtained the following results concerning maximal elements.**

# Maximality

**There is no complete theory in $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$.**
**However, this family can have a maximal element w.r.t. $\subseteq$.**
**In fact,**

**Theorem**

**The family $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$ contains $2^{\aleph_0}$ maximal elements.**

**We obtained the following results concerning maximal elements.**
$\mathsf{Con}^2_{\mathsf{PA}} :\equiv \mathsf{Con}_{\mathsf{PA}+\mathsf{Con}_{\mathsf{PA}}}$

## Maximality

There is no complete theory in $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$.
However, this family can have a maximal element w.r.t. $\subseteq$.
In fact,

**Theorem**

The family $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$ contains $2^{\aleph_0}$ maximal elements.

We obtained the following results concerning maximal elements.
$\mathsf{Con}^2_{\mathsf{PA}} :\equiv \mathsf{Con}_{\mathsf{PA}+\mathsf{Con}_{\mathsf{PA}}}$

**Proposition**

Every $M \models \mathsf{PA} + \mathsf{Con}^2_{\mathsf{PA}}$ is not maximal.

## Maximality

**There is no complete theory in $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$.**
**However, this family can have a maximal element w.r.t. $\subseteq$.**
**In fact,**

**Theorem**

**The family $\{\mathsf{Thm}(M) \mid M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}\}$ contains $2^{\aleph_0}$ maximal elements.**

**We obtained the following results concerning maximal elements.**
$\mathsf{Con}^2_{\mathsf{PA}} :\equiv \mathsf{Con}_{\mathsf{PA}+\mathsf{Con}_{\mathsf{PA}}}$

**Proposition**

**Every $M \models \mathsf{PA} + \mathsf{Con}^2_{\mathsf{PA}}$ is not maximal.**

**Theorem**

**$\exists M \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}} + \neg\mathsf{Con}^2_{\mathsf{PA}}$ s.t. $M$ is not maximal.**

1. **Background**
2. **Theorems in models of $\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$**
3. **Models having a proof of $\mathbf{0} = \mathbf{1}$**

## Models increasing their theorems gradually

**It is easy to prove the following proposition by using the arithmetized completeness theorem.**

### Proposition

$\exists K \models \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}, \quad \exists M, N \subseteq_e K$ s.t.

1. $M$ and $N$ are non-standard models of $\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$;
2. $\mathsf{Thm}(M) = \mathsf{Thm}(\mathbb{N})$; and
3. $\mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(N)$.

## Models increasing their theorems gradually

**It is easy to prove the following proposition by using the arithmetized completeness theorem.**

### Proposition

$\exists K \models \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}, \quad \exists M, N \subseteq_e K$ s.t.

1. $M$ and $N$ are non-standard models of $\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$;
2. $\mathsf{Thm}(M) = \mathsf{Thm}(\mathbb{N})$; and
3. $\mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(N)$.

**Such a model $M$ increases its theorems gradually.**

## Models increasing their theorems gradually

**It is easy to prove the following proposition by using the arithmetized completeness theorem.**

**Proposition**

$\exists K \models \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}, \quad \exists M, N \subseteq_e K$ s.t.

1. $M$ and $N$ are non-standard models of $\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$;
2. $\mathsf{Thm}(M) = \mathsf{Thm}(\mathbb{N})$; and
3. $\mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(N)$.

**Such a model $M$ increases its theorems gradually.**
**Next, we consider two special insane models.**

## Models proving $\mathbf{0} = \mathbf{1}$ suddenly

**First, we proved the existence of an insane model which proves $\mathbf{0} = \mathbf{1}$ suddenly.**

## Models proving $\mathbf{0} = \mathbf{1}$ suddenly

**First, we proved the existence of an insane model which proves $\mathbf{0} = \mathbf{1}$ suddenly.**

**Theorem**

$\exists N \models \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$ s.t.
$\forall I \subseteq_e N (I \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}} \Rightarrow \mathsf{Thm}(I) = \mathsf{Thm}(\mathbb{N}))$.

## Models proving $0 = 1$ suddenly

**First, we proved the existence of an insane model which proves $0 = 1$ suddenly.**

**Theorem**

$\exists N \models \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$ s.t.
$\forall I \subseteq_e N(I \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}} \Rightarrow \mathsf{Thm}(I) = \mathsf{Thm}(\mathbb{N}))$.

**We proved this theorem by using the following theorem by Krajíček and Pudlák (1989).**

**Theorem(Krajíček and Pudlák (1989))**

$\forall M$: non-standard model of $\mathsf{PA}$, $\forall a$: non-standard element of $M$
$\exists N \models \mathsf{PA}$ s.t. $M \restriction a \simeq N \restriction a$ and $N \models \exists y < 2^{2^a} \mathsf{Prf}(\ulcorner 0 = 1 \urcorner, y)$.

## Models which are illusory by nature

**Secondly, we proved the existence of an insane model which is illusory by nature.**

## Models which are illusory by nature

**Secondly, we proved the existence of an insane model which is illusory by nature.**

**Theorem**

$\exists M \models \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$ s.t.
$\forall N \subseteq_e M (N\text{: non-standard model of } \mathsf{PA} \Rightarrow \mathsf{Thm}(\mathbb{N}) \subsetneq \mathsf{Thm}(N))$.

**We give our proof of this theorem.**

**We give our proof of this theorem.**

Proof.

**Let $T = \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.**

**We give our proof of this theorem.**

**Proof.**

Let $T = \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.

We can take a model $M$ of $T$ omitting the type

$\{\forall y \leq x \neg\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \rightarrow \varphi\urcorner, y) \mid T \nvdash \varphi\} \cup \{x \geq \bar{n} \mid n \in \omega\}$.

**We give our proof of this theorem.**

**Proof.**

Let $T = \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.

We can take a model $M$ of $T$ omitting the type

$\{\forall y \leq x \neg \mathsf{Prf}(\ulcorner \neg\mathsf{Con}_{\mathsf{PA}} \to \varphi \urcorner, y) \mid T \nvdash \varphi\} \cup \{x \geq \bar{n} \mid n \in \omega\}$.

Let $N \subseteq_e M$ be a non-standard model of $\mathsf{PA}$.

**We give our proof of this theorem.**

**Proof.**

Let $T = \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.

We can take a model $M$ of $T$ omitting the type

$\{\forall y \leq x \neg\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \varphi\urcorner, y) \mid T \nvdash \varphi\} \cup \{x \geq \bar{n} \mid n \in \omega\}$.

Let $N \subseteq_e M$ be a non-standard model of $\mathsf{PA}$.

Let $a$ be any non-standard element of $N$.

We give our proof of this theorem.

**Proof.**

Let $T = \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.

We can take a model $M$ of $T$ omitting the type

$\{\forall y \leq x \neg \mathsf{Prf}(\ulcorner \neg \mathsf{Con}_{\mathsf{PA}} \to \varphi \urcorner, y) \mid T \nvdash \varphi\} \cup \{x \geq \bar{n} \mid n \in \omega\}$.

Let $N \subseteq_e M$ be a non-standard model of $\mathsf{PA}$.

Let $a$ be any non-standard element of $N$.

Since $M$ omits the type above, there is a formula $\varphi(x)$ in the type s.t. $M \models \neg\varphi(a)$.

**We give our proof of this theorem.**

**Proof.**

Let $T = \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.

We can take a model $M$ of $T$ omitting the type $\{\forall y \leq x \neg\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \varphi\urcorner, y) \mid T \nvdash \varphi\} \cup \{x \geq \bar{n} \mid n \in \omega\}$.

Let $N \subseteq_e M$ be a non-standard model of $\mathsf{PA}$.

Let $a$ be any non-standard element of $N$.

Since $M$ omits the type above, there is a formula $\varphi(x)$ in the type s.t. $M \models \neg\varphi(a)$.

Because $a$ is non-standard, $M \models a \geq \bar{n}$ for all $n \in \omega$.

We give our proof of this theorem.

**Proof.**

Let $T = \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.
We can take a model $M$ of $T$ omitting the type
$\{\forall y \leq x \neg\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \varphi\urcorner, y) \mid T \nvdash \varphi\} \cup \{x \geq \bar{n} \mid n \in \omega\}$.
Let $N \subseteq_e M$ be a non-standard model of PA.
Let $a$ be any non-standard element of $N$.
Since $M$ omits the type above, there is a formula $\varphi(x)$ in the type s.t. $M \models \neg\varphi(a)$.
Because $a$ is non-standard, $M \models a \geq \bar{n}$ for all $n \in \omega$.
Hence there is $\psi$ s.t. $T \nvdash \psi$ and $M \models \exists y \leq a \mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \psi\urcorner, y)$.

We give our proof of this theorem.

**Proof.**

Let $T = \mathsf{PA} + \neg\mathrm{Con}_{\mathsf{PA}}$.
We can take a model $M$ of $T$ omitting the type
$\{\forall y \leq x \neg\mathrm{Prf}(\ulcorner\neg\mathrm{Con}_{\mathsf{PA}} \to \varphi\urcorner, y) \mid T \nvdash \varphi\} \cup \{x \geq \bar{n} \mid n \in \omega\}$.
Let $N \subseteq_e M$ be a non-standard model of $\mathsf{PA}$.
Let $a$ be any non-standard element of $N$.
Since $M$ omits the type above, there is a formula $\varphi(x)$ in the type s.t. $M \models \neg\varphi(a)$.
Because $a$ is non-standard, $M \models a \geq \bar{n}$ for all $n \in \omega$.
Hence there is $\psi$ s.t. $T \nvdash \psi$ and $M \models \exists y \leq a \mathrm{Prf}(\ulcorner\neg\mathrm{Con}_{\mathsf{PA}} \to \psi\urcorner, y)$.
Since this statement is $\Pi_1$, $N \models \exists y \leq a \mathrm{Prf}(\ulcorner\neg\mathrm{Con}_{\mathsf{PA}} \to \psi\urcorner, y)$
and thus $N \models \mathrm{Pr}(\ulcorner\neg\mathrm{Con}_{\mathsf{PA}} \to \psi\urcorner)$.

We give our proof of this theorem.

**Proof.**

Let $T = \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.
We can take a model $M$ of $T$ omitting the type
$\{\forall y \leq x \neg\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \varphi\urcorner, y) \mid T \nvdash \varphi\} \cup \{x \geq \bar{n} \mid n \in \omega\}$.
Let $N \subseteq_e M$ be a non-standard model of PA.
Let $a$ be any non-standard element of $N$.
Since $M$ omits the type above, there is a formula $\varphi(x)$ in the type s.t. $M \models \neg\varphi(a)$.
Because $a$ is non-standard, $M \models a \geq \bar{n}$ for all $n \in \omega$.
Hence there is $\psi$ s.t. $T \nvdash \psi$ and $M \models \exists y \leq a\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \psi\urcorner, y)$.
Since this statement is $\Pi_1$, $N \models \exists y \leq a\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \psi\urcorner, y)$ and thus $N \models \mathsf{Pr}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \psi\urcorner)$.
Therefore $\neg\mathsf{Con}_{\mathsf{PA}} \to \psi \in \mathsf{Thm}(N)$ and $\mathsf{PA} \nvdash \neg\mathsf{Con}_{\mathsf{PA}} \to \psi$.

We give our proof of this theorem.

**Proof.**

Let $T = \mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}}$.
We can take a model $M$ of $T$ omitting the type
$\{\forall y \leq x \neg\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \varphi\urcorner, y) \mid T \nvdash \varphi\} \cup \{x \geq \bar{n} \mid n \in \omega\}$.
Let $N \subseteq_e M$ be a non-standard model of $\mathsf{PA}$.
Let $a$ be any non-standard element of $N$.
Since $M$ omits the type above, there is a formula $\varphi(x)$ in the type s.t. $M \models \neg\varphi(a)$.
Because $a$ is non-standard, $M \models a \geq \bar{n}$ for all $n \in \omega$.
Hence there is $\psi$ s.t. $T \nvdash \psi$ and $M \models \exists y \leq a\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \psi\urcorner, y)$.
Since this statement is $\Pi_1$, $N \models \exists y \leq a\mathsf{Prf}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \psi\urcorner, y)$ and thus $N \models \mathsf{Pr}(\ulcorner\neg\mathsf{Con}_{\mathsf{PA}} \to \psi\urcorner)$.
Therefore $\neg\mathsf{Con}_{\mathsf{PA}} \to \psi \in \mathsf{Thm}(N)$ and $\mathsf{PA} \nvdash \neg\mathsf{Con}_{\mathsf{PA}} \to \psi$.
This means $N$ is illusory. □

# Problems

## Problems

1. If $\mathsf{Thm}(M) \subseteq \mathsf{Thm}(N)$, then is there an end-extension $K$ of $M$ s.t. $\mathsf{Thm}(N) = \mathsf{Thm}(K)$?
2. Does every non-standard model which is not maximal have a maximal end-extension?
3. Does every non-standard model $M$ which is not maximal have an end-extension $N$ s.t. $N \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ and $\mathsf{Thm}(M) \subsetneq \mathsf{Thm}(N)$?

## Problems

### Problems

1. If $\mathsf{Thm}(M) \subseteq \mathsf{Thm}(N)$, then is there an end-extension $K$ of $M$ s.t. $\mathsf{Thm}(N) = \mathsf{Thm}(K)$?
2. Does every non-standard model which is not maximal have a maximal end-extension?
3. Does every non-standard model $M$ which is not maximal have an end-extension $N$ s.t. $N \models \mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}$ and $\mathsf{Thm}(M) \subsetneq \mathsf{Thm}(N)$?

The results presented in this talk will appear in

Makoto Kikuchi and Taishi Kurahashi, "Illusory models of Peano arithmetic", *Journal of Symbolic Logic*.