Yablo's paradox and Rosser's theorem

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	Σ_1-sound (ω-consistent)	consistent
Liar paradox	**Gödel (1931)**	**Rosser (1936)**
Yablo's paradox	**Priest (1997)**	?

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Key words

- Gödel and Rosser's incompleteness theorems
- Yablo's paradox
- Standard proof predicates
- Binumerations

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2. Yablo's paradox
3. Rosser's theorem based on Yablo's paradox

Gödel's theorems

T: primitive recursive theory of arithmetic extending PA

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T: **primitive recursive theory of arithmetic extending PA**

Gödel's incompleteness theorems (1931)

1. **If T is Σ_1-sound, then T is incomplete.**
2. **If T is consistent, then T cannot prove own consistency.**

T is Σ_1-sound
$\Leftrightarrow \forall \varphi (T \vdash \varphi \Rightarrow \mathbb{N} \models \varphi)$.

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Key points

1. **Provability predicates**
2. **An analogy with Liar paradox**

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Definition

A formula $\mathrm{Prf}(x, y)$ is a standard proof predicate (s.p.p.) of T
$\Leftrightarrow$ $\mathrm{Prf}(x, y)$ is a PR formula which represents some proof relation of T, and satisfies the derivability conditions,
i.e., $\mathrm{Prf}(x, y)$ satisfies the following conditions:

- $\mathrm{Prf}(x, y)$ is a PR formula;
- $T \vdash \varphi \Rightarrow \exists p \in \mathbb{N}$ s.t. $\mathrm{PA} \vdash \mathrm{Prf}(\ulcorner \varphi \urcorner, \bar{p})$;
- $T \nvdash \varphi \Rightarrow \forall p \in \mathbb{N}$ s.t. $\mathrm{PA} \vdash \neg \mathrm{Prf}(\ulcorner \varphi \urcorner, \bar{p})$;
- The formalized modus ponens, the formalized Σ_1-completeness.

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- Define $\text{Pr}_T(x)$ to be the formula $\exists y \text{Prf}_\tau(x, y)$.
- $\text{Pr}_T(x)$: A provability predicate of T.

An analogy with Liar paradox

Gödel constructed a sentence π asserting that "π is not provable in T".

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The Fixed Point Lemma

$\forall\varphi(x)$: **formula,** $\exists\psi$: **sentence s.t.**
PA $\vdash \psi \leftrightarrow \varphi(\ulcorner\psi\urcorner)$.

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Definition (Gödel sentences)

- **There exists a sentence π s.t. PA $\vdash \pi \leftrightarrow \neg\mathrm{Pr}_T(\ulcorner\pi\urcorner)$.**

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Liar paradox

Let A be a proposition asserting that "A is false".
Then we cannot determine the truth of A.

Gödel's incompleteness theorems

The first incompleteness theorem

π: **Gödel sentence of T**

- T: **consistent** $\Rightarrow T \nvdash \pi$,
- T: Σ_1-**sound** $\Rightarrow T \nvdash \neg\pi$.

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The second incompleteness theorem

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- $\mathsf{PA} \vdash \pi \leftrightarrow \mathrm{Con}_T$,
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Remark

Gödel's theorems hold for any s.p.p. of T.

Rosser's first incompleteness theorem

Rosser provability predicate

- **Prf(x, y): s.p.p. of T**
- **Pr$_T(x) \equiv \exists y$Prf(x, y)**

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- $\mathsf{Prf}(x, y)$: s.p.p. of T
- $\mathsf{Pr}_T(x) \equiv \exists y \mathsf{Prf}(x, y)$
- $\mathsf{Pr}_T^R(x) \equiv \exists y(\mathsf{Prf}(x, y) \land \forall z < y \neg \mathsf{Prf}(\neg x, z))$

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Theorem (Rosser, 1936)

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Remark

Rosser's theorem holds for any s.p.p. of T.

$\mathsf{Pr}_T(x)$ is dependent on the choice of $\tau(z)$

We defined $\mathsf{Pr}_T(x)$ to be $\exists y \mathsf{Prf}_\tau(x, y)$ for some PR binumeration $\tau(z)$ of T.

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(Feferman)

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$\Rightarrow \exists\tau(z)$: PR binumeration of T s.t. $T \nvdash \neg\mathsf{Con}_\tau$.

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(Orey)

T: not Σ_1-sound
$\Rightarrow \exists\tau(z)$: PR binumeration of T s.t. $T \vdash \neg\mathsf{Con}_\tau$.

$\mathsf{Pr}_T^R(x)$ is dependent on the choice of a s.p.p.

Theorem (Guaspari and Solovay, 1979)

1. **There is a s.p.p. s.t. not all of whose Rosser sentences are provably equivalent.**

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- **$\mathsf{Pr}_T^R(\ulcorner\varphi\urcorner)$ means "φ has a smaller proof in T than any proof of $\neg\varphi$".**
- **They constructed a new s.p.p. with the required conditions by rearrenging proofs of a given s.p.p.**

Yablo's paradox

Yablo, 1993

Let $Y_0, Y_1, \ldots, Y_n, \ldots$ be an infinite sequence of propositions s.t. for each $i \in \mathbb{N}$,

$$Y_i \Leftrightarrow \forall j > i (Y_j \text{ is false}).$$

Then we cannot determine the truth of each Y_i.

Gödel's theorems based on Yablo's paradox

Priest, 1997

$\mathsf{PA} \vdash Y(x) \leftrightarrow \forall y > x \neg \mathsf{Pr}_T(\ulcorner Y(\dot{y}) \urcorner)$.

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Theorem (Kikuchi and K., 2011; Cieśliński and Urbaniak, 2012)

- $\mathsf{PA} \vdash Y(\bar{n}) \leftrightarrow \mathrm{Con}_T$.

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It is dependent on the choice of a s.p.p.!

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Formalizations of Yablo's sequence using Rosser predicates

Priest

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$\mathsf{PA} \vdash Y(x) \leftrightarrow \forall y > x \neg \mathsf{Pr}_T(\ulcorner Y(\dot{y}) \urcorner).$

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Problem

T: consistent $\Rightarrow \forall n \in \mathbb{N},\ \ T \nvdash \neg Y^R(\bar{n})$?

Answer

No. If T is consistent but not Σ_1-sound, then the provability of $\neg Y^R(\bar{n})$ is dependent on the choice of a s.p.p.

Result 1

T: consistent

$\Rightarrow$ there is a s.p.p. of T s.t. $\forall n \in \mathbb{N}, \quad T \nvdash \neg Y^R(\bar{n})$.

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Result 2

T: not Σ_1-sound
$\Rightarrow$ there is a s.p.p. of T s.t. $\forall n \in \mathbb{N}, \quad T \vdash \neg Y^R(\bar{n})$.

Theorem 1

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$\exists$**Prf**(x, y): **s.p.p. of** T **s.t.**

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- **PA** $\vdash$ **Pr**$_\tau(x) \leftrightarrow \exists y$**Prf**$(x, y)$;

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$\exists$Prf(x, y): s.p.p. of T s.t.

- **PA $\vdash$ Pr$_\tau(x) \leftrightarrow \exists yPrf(x, y)$;**
- **PA $\vdash Y^R(0) \leftrightarrow \pi$ for a Rosser sentence π of Prf(x, y).**

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- **$T \nvdash \neg\pi$ by Rosser's theorem**

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- $T \nvdash \neg\pi$ by Rosser's theorem
- $T \nvdash \neg Y^R(0)$

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- T: consistent
- $T \nvdash \neg\pi$ by Rosser's theorem
- $T \nvdash \neg Y^R(0)$
- $T \nvdash \neg Y^R(\bar{n})$ since $\vdash Y^R(0) \rightarrow Y^R(\bar{n})$.

Theorem 2

$\forall \tau(z)$: PR binumeration of T
$\exists$Prf(x, y): s.p.p. of T s.t.

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Outline of proof

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PA $\vdash$ Pr$_T^R(\ulcorner\pi\urcorner) \leftrightarrow \exists y > 0Pr_T^R(\ulcorner Y^R(\dot{y})\urcorner)$.

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PA $\vdash \mathsf{Pr}_T^R(\ulcorner \pi \urcorner) \leftrightarrow \exists y > 0 \mathsf{Pr}_T^R(\ulcorner Y^R(\dot{y}) \urcorner)$.

π has a smaller proof than any proof of $\neg \pi$
$\Leftrightarrow \exists z > 0$ s.t. $Y^R(z)$ has a smaller proof than any proof of $\neg Y^R(z)$.

Define Prf(x, y) by copying Prf$_\tau(x, y)$ until a proof of one of π, $\neg\pi$ and $Y^R(a)$ appears.

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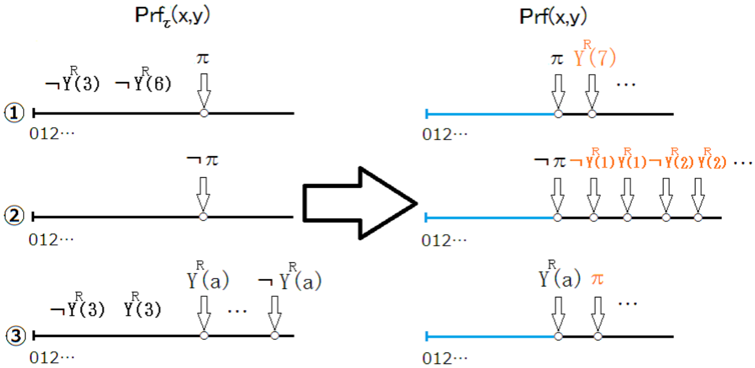

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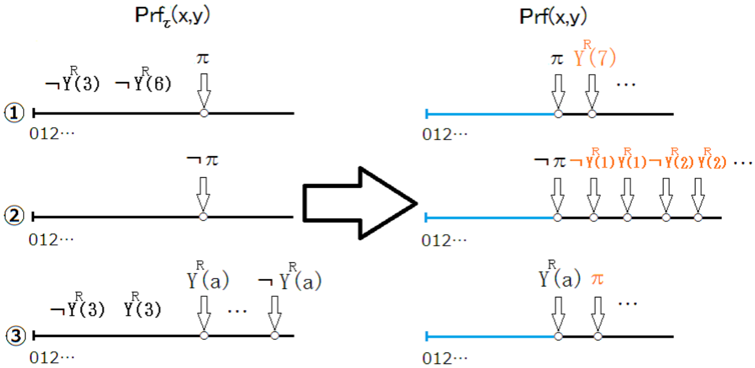

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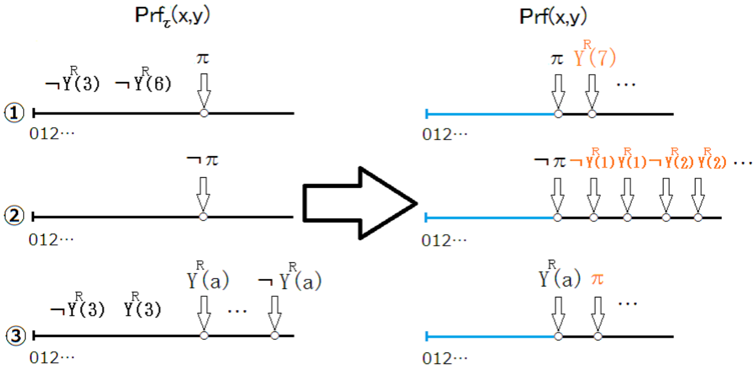

- **π has a smaller proof than any proof of $\neg\pi$ $\Leftrightarrow \exists z > 0$ s.t. $Y^R(z)$ has a smaller proof than any proof of $\neg Y^R(z)$.**
- **We can construct Prf(x, y) so that the theorems of Prf(x, y) coincide with that of Prf$_\tau(x, y)$.**

Thank you for your attention!