Syntax and Semantics of Predicate Modal Logic of Provability

Taishi Kurahashi

Kobe University
Research Fellow of the Japan Society for the Promotion of Science (DC2)

Seminar in Logic and Philosophy of Mathematics
March 26, 2013

Contents

1. Propositional modal logic of provability
2. Predicate modal logic of provability
3. Main theorem
4. Predicate provability logics of fragments of PA
5. Further work

Provability predicates

T: r.e. extension of $\mathrm{I}\Sigma_1$

Definition

A formula $\mathrm{Pr}_T(x)$ is called a **provability predicate** of T if for any φ and ψ,

- $\mathrm{Pr}_T(x)$ is Σ_1;
- $T \vdash \varphi \Leftrightarrow \mathrm{I}\Sigma_1 \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$;
- $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$;
- φ: $\Sigma_1 \Rightarrow T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

Provability predicates

T: r.e. extension of $\mathbf{I\Sigma_1}$

Definition

A formula $\mathrm{Pr}_T(x)$ is called a provability predicate of T if for any φ and ψ,

- **$\mathrm{Pr}_T(x)$ is Σ_1;**
- $T \vdash \varphi \Leftrightarrow \mathbf{I\Sigma_1} \vdash \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$;
- $T \vdash \mathrm{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$;
- φ: $\Sigma_1 \Rightarrow T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner \varphi \urcorner)$.

We compare the following three notions on modal formulas:

- **Provability in formal systems of modal logic**
- **Validity on Kripke frames**
- **Validity on arithmetical semantics**

Arithmetical semantics

Let F be the set of all propositional modal formulas.

Arithmetical semantics

Let F be the set of all propositional modal formulas.

Definition

A mapping $*$ from F to all sentences in the language of T is called a *T-interpretation* if it satisfies the following conditions:

- $\bot^* \equiv \ 0 = 1$;
- $(A \to B)^* \equiv (A^* \to B^*)$;
- $\cdots$;
- $(\Box A)^* \equiv \mathrm{Pr}_T(\ulcorner A^* \urcorner)$.

Arithmetical semantics

Let F be the set of all propositional modal formulas.

Definition

A mapping $*$ from F to all sentences in the language of T is called a **T-interpretation**
if it satisfies the following conditions:

- $\bot^* \equiv \ 0 = 1$;
- $(A \to B)^* \equiv (A^* \to B^*)$;
- $\cdots$;
- $(\Box A)^* \equiv \mathrm{Pr}_T(\ulcorner A^* \urcorner)$.

Definition

- A: propositional modal formula.
  A is **T-valid** $\overset{\text{def.}}{\Leftrightarrow}$ $\forall *$: T-interpretation, $T \vdash A^*$.

Arithmetical semantics

Let F be the set of all propositional modal formulas.

Definition

A mapping $*$ from F to all sentences in the language of T is called a **T-interpretation**
if it satisfies the following conditions:

- $\bot^* \equiv \ 0 = 1$;
- $(A \to B)^* \equiv (A^* \to B^*)$;
- $\cdots$;
- $(\Box A)^* \equiv \mathrm{Pr}_T(\ulcorner A^* \urcorner)$.

Definition

- A: propositional modal formula.
  A is **T-valid** $\overset{\text{def.}}{\Leftrightarrow}$ $\forall *$: T-interpretation, $T \vdash A^*$.
- **PL(T)** $:= \{A \mid A$ is T-valid$\}$

Arithmetical semantics

Let F be the set of all propositional modal formulas.

Definition

A mapping $*$ from F to all sentences in the language of T is called a T-interpretation

if it satisfies the following conditions:

- $\bot^* \equiv \ 0 = 1$;
- $(A \to B)^* \equiv (A^* \to B^*)$;
- $\cdots$;
- $(\Box A)^* \equiv \mathrm{Pr}_T(\ulcorner A^* \urcorner)$.

Definition

- A: **propositional modal formula.**
  A **is T-valid** $\overset{\text{def.}}{\Leftrightarrow}$ $\forall *$: T-**interpretation,** $T \vdash A^*$.
- **PL**$(T) := \{A \mid A$ **is** T-**valid**$\}$: **the provability logic of** T.

Propositional modal logic **GL**

Propositional modal logic **GL**

- **Axioms:**
  - **Tautologies;**
  - $\Box(A \to B) \to (\Box A \to \Box B)$;
  - $\Box(\Box A \to A) \to \Box A$.
- **Inference rules:**

modus ponens **from** A **and** $A \to B$ **infer** B;

necessitation **form** A **infer** $\Box A$.

Propositional modal logic **GL**

Propositional modal logic **GL**

- **Axioms:**
  - **Tautologies;**
  - $\Box(A \to B) \to (\Box A \to \Box B)$;
  - $\Box(\Box A \to A) \to \Box A$.
- **Inference rules:**

modus ponens **from** A **and** $A \to B$ **infer** B;

necessitation **form** A **infer** $\Box A$.

Th(GL):= $\{A \mid$ **GL** $\vdash A\}$.

Propositional modal logic **GL**

Propositional modal logic **GL**

- **Axioms:**
  - **Tautologies;**
  - $\Box(A \to B) \to (\Box A \to \Box B)$;
  - $\Box(\Box A \to A) \to \Box A$.
- **Inference rules:**

modus ponens **from** A **and** $A \to B$ **infer** B;

necessitation **form** A **infer** $\Box A$.

Th(GL)$:= \{A \mid$ **GL** $\vdash A\}$.

Note that Th(GL) $\subseteq$ **PL**(T).

Kripke semantics

Definition

Kripke frame is a system $\langle W, \prec \rangle$ where

- **W is a non-empty set of worlds;**
- **$\prec$ is a binary relation on W: accessibility relation.**

Kripke semantics

Definition

Kripke frame is a system $\langle W, \prec \rangle$ where

- W is a non-empty set of **worlds**;
- $\prec$ is a binary relation on W: **accessibility relation.**

Kripke model is a system $\mathcal{M} = \langle W, \prec, \Vdash \rangle$ where

- $\langle W, \prec \rangle$ is a Kripke frame;
- $\Vdash$ is a binary relation on $W \times F$ such that $\forall w \in W$,
  - $w \nVdash \bot$;
  - $w \Vdash A \to B \Leftrightarrow (w \nVdash A$ or $w \Vdash B)$;
  - $\cdots$;
  - $w \Vdash \Box A \Leftrightarrow \forall w' \in W (w \prec w' \Rightarrow w' \Vdash A)$.
  - $w \Vdash \Diamond A \Leftrightarrow \exists w' \in W (w \prec w' \ \& \ w' \Vdash A)$.

GL-frames and Kripke completeness theorem

Definition

A: modal formula, $\mathcal{F}$: Kripke frame, $\mathcal{M}$: Kripke model.

- A is valid in $\mathcal{M}$ $\overset{\text{def.}}{\Leftrightarrow}$ $\forall w \in W$, $w \Vdash A$.
- A is valid in $\mathcal{F}$ $\overset{\text{def.}}{\Leftrightarrow}$ A is valid in $\langle \mathcal{F}, \Vdash \rangle$ for any $\Vdash$.

GL-frames and Kripke completeness theorem

Definition

A: modal formula, $\mathcal{F}$: Kripke frame, $\mathcal{M}$: Kripke model.

- A is valid in $\mathcal{M} \overset{\text{def.}}{\Leftrightarrow} \forall w \in W$, $w \Vdash A$.
- A is valid in $\mathcal{F} \overset{\text{def.}}{\Leftrightarrow} A$ is valid in $\langle \mathcal{F}, \Vdash \rangle$ for any $\Vdash$.

Definition

- Kripke frame $\langle W, \prec \rangle$ is a **GL-frame** if $\prec$ is
  1. transitive,
  2. conversely well-founded.

GL-frames and Kripke completeness theorem

Definition

A: modal formula, $\mathcal{F}$: Kripke frame, $\mathcal{M}$: Kripke model.

- A is valid in $\mathcal{M} \overset{\text{def.}}{\Leftrightarrow} \forall w \in W$, $w \Vdash A$.
- A is valid in $\mathcal{F} \overset{\text{def.}}{\Leftrightarrow} A$ is valid in $\langle \mathcal{F}, \Vdash \rangle$ for any $\Vdash$.

Definition

- Kripke frame $\langle W, \prec \rangle$ is a **GL-frame** if $\prec$ is
  1. transitive,
  2. conversely well-founded.
- **Fr(GL)**$:= \{A \mid A$ is valid in any GL-frame $\}$.

GL-frames and Kripke completeness theorem

Definition

A: modal formula, $\mathcal{F}$: Kripke frame, $\mathcal{M}$: Kripke model.

- A is valid in $\mathcal{M}$ $\overset{\text{def.}}{\Leftrightarrow}$ $\forall w \in W$, $w \Vdash A$.
- A is valid in $\mathcal{F}$ $\overset{\text{def.}}{\Leftrightarrow}$ A is valid in $\langle \mathcal{F}, \Vdash \rangle$ for any $\Vdash$.

Definition

- Kripke frame $\langle W, \prec \rangle$ is a **GL-frame** if $\prec$ is
  1. transitive,
  2. conversely well-founded.
- **Fr(GL)**$:= \{A \mid A$ is valid in any GL-frame $\}$.

Theorem (Segerberg, 1971)

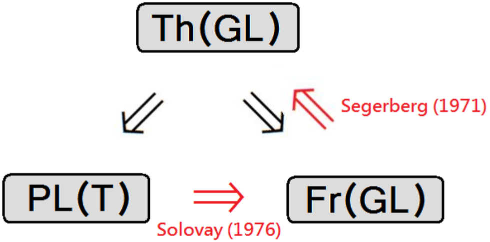
Th(GL) = Fr(GL).

Solovay's arithmetical completeness theorem

T: Σ_1-sound.

Theorem (Solovay, 1976)

$\mathbf{Th}(\mathbf{GL}) = \mathbf{PL}(T)$.

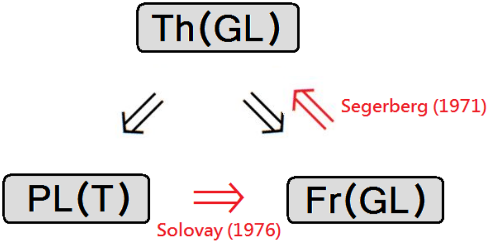


Solovay's arithmetical completeness theorem

T: Σ_1-sound.

Theorem (Solovay, 1976)

$\mathbf{Th}(\mathbf{GL}) = \mathbf{PL}(T)$.



If T is Σ_1-sound r.e. extension of $\mathrm{I}\Sigma_1$, then $\mathbf{Th}(\mathbf{GL}) = \mathbf{Fr}(\mathbf{GL}) = \mathbf{PL}(T)$.

1. Propositional modal logic of provability
2. **Predicate modal logic of provability**
3. Main theorem
4. Predicate provability logics of fragments of PA
5. Further work

Predicate modal logic of provability

- **QGL is a natural extension of GL to predicate modal logic.**

Predicate modal logic of provability

- **QGL is a natural extension of GL to predicate modal logic.**
- **Define Th(QGL), Fr(QGL) and QPL(T) similarly as in the propositional case.**

Predicate modal logic of provability

- **QGL is a natural extension of GL to predicate modal logic.**
- **Define Th(QGL), Fr(QGL) and QPL(T) similarly as in the propositional case.**

- **T-interpretations of predicate modal logic map each k-ary predicate symbol to a k-ary formula in the language of T.**

Predicate modal logic of provability

- **QGL is a natural extension of GL to predicate modal logic.**
- **Define Th(QGL), Fr(QGL) and QPL(T) similarly as in the propositional case.**

- ***T*-interpretations of predicate modal logic map each k-ary predicate symbol to a k-ary formula in the language of T.**
- **Kripke frame for predicate modal logic is a triple $\langle W, \prec, \{D_w\}_{w \in W} \rangle$:**
  - **$\{D_w\}_{w \in W}$ is a family of non-empty sets.**
  - $\forall w, w' \in W,\ w \prec w' \Rightarrow D_w \subseteq D_{w'}$.

Predicate modal logic of provability

- **QGL is a natural extension of GL to predicate modal logic.**
- **Define Th(QGL), Fr(QGL) and QPL(T) similarly as in the propositional case.**

- **T-interpretations of predicate modal logic map each k-ary predicate symbol to a k-ary formula in the language of T.**
- **Kripke frame for predicate modal logic is a triple $\langle W, \prec, \{D_w\}_{w \in W} \rangle$:**
  - **$\{D_w\}_{w \in W}$ is a family of non-empty sets.**
  - **$\forall w, w' \in W,\ w \prec w' \Rightarrow D_w \subseteq D_{w'}$.**
- **Kripke model for predicate modal logic is a 4-tuple $\langle W, \prec, \{D_w\}_{w \in W}, \Vdash \rangle$:**
  $\Vdash$ is a relation between elements w of W and closed formulas with parameters form D_w.

Montagna's theorem

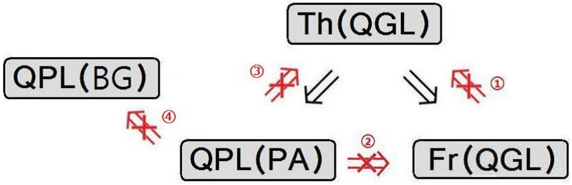
By the definitions, Th(QGL) ⊆ Fr(QGL) ∩ QPL(T).

Montagna's theorem

By the definitions, Th(QGL) $\subseteq$ Fr(QGL) $\cap$ QPL(T).

Theorem (Montagna, 1984)

1. **Fr(QGL) $\not\subseteq$ Th(QGL).**
2. **QPL(PA) $\not\subseteq$ Fr(QGL).**
3. **QPL(PA) $\not\subseteq$ Th(QGL).**
4. **QPL(PA) $\not\subseteq$ QPL(BG) (BG: Bernays-Gödel set theory).**



Vardanyan's theorem on Π^0_2-completeness

Theorem (Vardanyan, 1985)

QPL(PA) is Π^0_2-complete.

- **QPL(PA) is not Σ^0_1.**
- **QPL(PA) cannot be characterized by any recursive extension of QGL.**

Vardanyan's theorem on Π^0_2-completeness

Theorem (Vardanyan, 1985)

QPL(PA) is Π^0_2-complete.

- **QPL(PA) is not Σ^0_1.**
- **QPL(PA) cannot be characterized by any recursive extension of QGL.**

Montagna's conjecture (1984)

$\bigcap\{\mathbf{QPL}(T) \mid T : \ \Sigma_1\text{-sound r.e. extension of PA}\} = \mathbf{Th}(\mathbf{QGL})$

Vardanyan's theorem on Π^0_2-completeness

Theorem (Vardanyan, 1985)

QPL(PA) is Π^0_2-complete.

- **QPL(PA) is not Σ^0_1.**
- **QPL(PA) cannot be characterized by any recursive extension of QGL.**

Montagna's conjecture (1984)

$\bigcap\{\mathbf{QPL}(T) \mid T : \Sigma_1\text{-sound r.e. extension of PA}\} = \mathbf{Th}(\mathbf{QGL})$

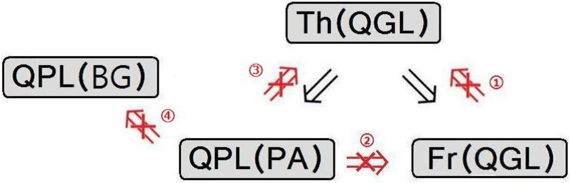
In a similar way as in the proof of Vardanyan's theorem,

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of PA}\}$ **is** Π^0_2**-hard.**

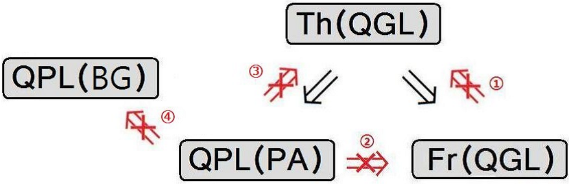
Corollary

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of PA}\} \not\subseteq \mathbf{Th}(\mathbf{QGL})$.

How about other inclusions?

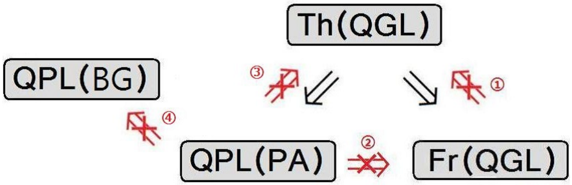


How about other inclusions?



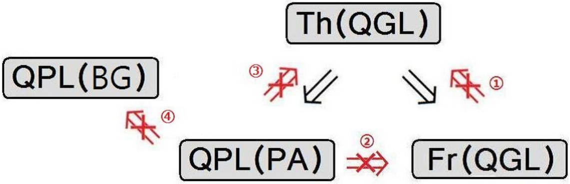
- **Fr(QGL)** $\subseteq$ **QPL**(T)**?**

How about other inclusions?



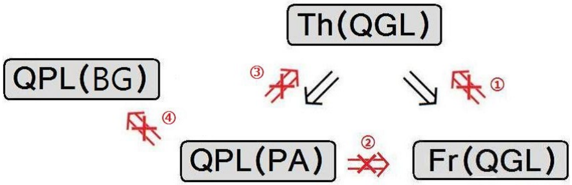
- $\mathbf{Fr}(\mathbf{QGL}) \subseteq \mathbf{QPL}(T)$?
- $\bigcap\{\mathbf{QPL}(T) \mid T : \textbf{r.e. extension of PA}\} \subseteq \mathbf{Fr}(\mathbf{QGL})$?

How about other inclusions?



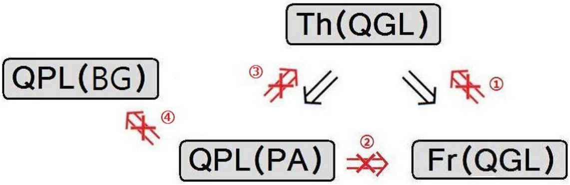
- $\mathbf{Fr}(\mathbf{QGL}) \subseteq \mathbf{QPL}(T)$?
- $\bigcap\{\mathbf{QPL}(T) \mid T : \textbf{r.e. extension of PA}\} \subseteq \mathbf{Fr}(\mathbf{QGL})$?
- $\bigcap\{\mathbf{QPL}(T) \mid T : \textbf{r.e. extension of PA}\} \cap \mathbf{Fr}(\mathbf{QGL}) \subseteq \mathbf{Th}(\mathbf{QGL})$?

How about other inclusions?



- **Fr**(**QGL**) $\subseteq$ **QPL**(T)?
- $\bigcap\{$**QPL**$(T) \mid T$: **r.e. extension of PA**$\} \subseteq$ **Fr**(**QGL**)?
- $\bigcap\{$**QPL**$(T) \mid T$: **r.e. extension of PA**$\} \cap$ **Fr**(**QGL**) $\subseteq$ **Th**(**QGL**)?
- $\exists i, j \in \omega (i \neq j)$ **s.t. QPL**$(\mathrm{I}\Sigma_i) =$ **QPL**$(\mathrm{I}\Sigma_j)$?

How about other inclusions?



- Fr(QGL)$\not\subseteq$QPL(T).
- $\bigcap${QPL(T) | T : r.e. extension of PA}$\not\subseteq$Fr(QGL).
- $\bigcap${QPL(T) | T : r.e. extension of PA} $\cap$ Fr(QGL) $\subseteq$ Th(QGL)?
- $\exists i, j \in \omega (i \neq j)$ s.t. QPL($\mathrm{I}\Sigma_i$) $=$ QPL($\mathrm{I}\Sigma_j$)?

Prop

Fr(QGL) $\not\subseteq$ **QPL**(T) **for any** Σ_2**-sound r.e. extension** T **of** $\mathrm{I}\Sigma_1$**.**

Prop

Fr(QGL) $\not\subseteq$ QPL(T) for any Σ_2-sound r.e. extension T of $\mathrm{I}\Sigma_1$.

- **Montagna proved $A \equiv \forall x \exists y \Box(p(x) \to \Diamond p(y)) \to \forall x \Box \neg p(x)$ witnesses the non-inclusion Fr(QGL) $\not\subseteq$ Th(QGL).**

Prop

$\mathsf{Fr}(\mathsf{QGL}) \not\subseteq \mathsf{QPL}(T)$ for any Σ_2-sound r.e. extension T of $\mathrm{I}\Sigma_1$.

- Montagna proved $A \equiv \forall x \exists y \Box(p(x) \to \Diamond p(y)) \to \forall x \Box \neg p(x)$ witnesses the non-inclusion $\mathsf{Fr}(\mathsf{QGL}) \not\subseteq \mathsf{Th}(\mathsf{QGL})$.
- This sentence also witnesses $\mathsf{Fr}(\mathsf{QGL}) \not\subseteq \mathsf{QPL}(T)$.

Prop

Fr(QGL) $\not\subseteq$ QPL(T) for any Σ_2-sound r.e. extension T of $\mathrm{I}\Sigma_1$.

- Montagna proved $A \equiv \forall x \exists y \Box(p(x) \to \Diamond p(y)) \to \forall x \Box \neg p(x)$ witnesses the non-inclusion Fr(QGL) $\not\subseteq$ Th(QGL).
- This sentence also witnesses Fr(QGL) $\not\subseteq$ QPL(T).

Theorem (Solovay-Somoryński-Friedman)

$\exists \varphi(x)$: Π_1 formula s.t.
$\forall n \in \omega$,

1. $T + \varphi(\bar{n})$ is consistent, and
2. $\mathrm{I}\Sigma_1 \vdash \varphi(\bar{n}) \to \mathrm{Con}_{T+\varphi(\overline{n+1})}$.

Prop

$\mathrm{Fr}(\mathsf{QGL}) \not\subseteq \mathsf{QPL}(T)$ for any Σ_2-sound r.e. extension T of $\mathrm{I}\Sigma_1$.

- Montagna proved $A \equiv \forall x \exists y \Box (p(x) \to \Diamond p(y)) \to \forall x \Box \neg p(x)$ witnesses the non-inclusion $\mathrm{Fr}(\mathsf{QGL}) \not\subseteq \mathrm{Th}(\mathsf{QGL})$.
- This sentence also witnesses $\mathrm{Fr}(\mathsf{QGL}) \not\subseteq \mathsf{QPL}(T)$.

Theorem (Solovay-Somoryński-Friedman)

$\exists \varphi(x)$: Π_1 formula s.t.
$\forall n \in \omega$,

1. $T + \varphi(\bar{n})$ is consistent, and
2. $\mathrm{I}\Sigma_1 \vdash \varphi(\bar{n}) \to \mathrm{Con}_{T+\varphi(\overline{n+1})}$.

Let $*$ be a T-interpretation s.t. $(p(x))^* \equiv \varphi(x)$, then $\mathbb{N} \models \neg A^*$.

Prop

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_1\} \not\subseteq \mathbf{Fr}(\mathbf{QGL})$.

Prop

$\bigcap\{$**QPL**$(T) \mid T$: **r.e. extension of** $\mathrm{I}\Sigma_1\} \not\subseteq$ **Fr(QGL).**

$P_+, P_\times, P_S, P_0, P_<, P_=$: **new predicate symbols corresponding to** $+, \times, S, 0, <, =$ **respectively.**

Prop

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_1\} \not\subseteq \mathbf{Fr}(\mathbf{QGL})$.

$P_+, P_\times, P_S, P_0, P_<, P_=$: new predicate symbols corresponding to $+, \times, S, 0, <, =$ respectively.

- For each $\mathcal{L}_A$-formula φ, let $[\![\varphi]\!]$ be one of the relational formula obtained by rewriting φ by using only these new symbols.

Prop

$\bigcap\{\mathbf{QPL}(T) \mid T$: r.e. extension of $\mathrm{I}\Sigma_1\} \not\subseteq \mathbf{Fr}(\mathbf{QGL})$.

$P_+, P_\times, P_S, P_0, P_<, P_=$: new predicate symbols corresponding to $+, \times, S, 0, <, =$ respectively.

- For each $\mathcal{L}_A$-formula φ, let $[\![\varphi]\!]$ be one of the relational formula obtained by rewriting φ by using only these new symbols.

$\mathbf{D} \equiv \neg\Box\bot \wedge \bigwedge\{P_\circ \to \Box P_\circ, \neg P_\circ \to \Box\neg P_\circ \mid \circ \in \mathcal{L}_A\}$,

Prop

$\bigcap\{\mathsf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_1\} \not\subseteq \mathsf{Fr}(\mathsf{QGL})$.

$P_+, P_\times, P_S, P_0, P_<, P_=$: new predicate symbols corresponding to $+, \times, S, 0, <, =$ respectively.

- For each $\mathcal{L}_A$-formula φ, let $[\![\varphi]\!]$ be one of the relational formula obtained by rewriting φ by using only these new symbols.

$\mathsf{D} \equiv \neg\Box\bot \wedge \bigwedge\{P_\circ \to \Box P_\circ, \neg P_\circ \to \Box\neg P_\circ \mid \circ \in \mathcal{L}_A\}$,

Lemma (Artemov)

$\forall *$: T-interpretation, $\forall\varphi$: $\mathcal{L}_A$-sentence,
$\mathrm{I}\Sigma_1 \vdash \mathsf{D}^* \wedge [\![\bigwedge \mathrm{I}\Delta_0(exp)]\!]^* \to (\varphi \leftrightarrow [\![\varphi]\!]^*)$.

Prop

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathbf{I\Sigma_1}\} \not\subseteq \mathsf{Fr}(\mathsf{QGL})$.

$P_+, P_\times, P_S, P_0, P_<, P_=$: new predicate symbols corresponding to $+, \times, S, 0, <, =$ respectively.

- For each $\mathcal{L}_A$-formula φ, let $[\![\varphi]\!]$ be one of the relational formula obtained by rewriting φ by using only these new symbols.

$\mathsf{D} \equiv \neg\Box\bot \wedge \bigwedge\{P_\circ \to \Box P_\circ, \neg P_\circ \to \Box\neg P_\circ \mid \circ \in \mathcal{L}_A\}$,

Lemma (Artemov)

$\forall *$: T-interpretation, $\forall\varphi$: $\mathcal{L}_A$-sentence,
$\mathbf{I\Sigma_1} \vdash \mathsf{D}^* \wedge [\![\bigwedge \mathbf{I\Delta_0}(exp)]\!]^* \to (\varphi \leftrightarrow [\![\varphi]\!]^*)$.

Let $A \equiv \mathsf{D} \wedge [\![\bigwedge \mathbf{I\Delta_0}(exp)]\!] \to [\![\bigwedge \mathbf{I\Sigma_1}]\!]$.

Prop

$\bigcap\{\mathsf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_1\} \nsubseteq \mathsf{Fr}(\mathsf{QGL})$.

$P_+, P_\times, P_S, P_0, P_<, P_=$: new predicate symbols corresponding to $+, \times, S, 0, <, =$ respectively.

- For each $\mathcal{L}_A$-formula φ, let $[\![\varphi]\!]$ be one of the relational formula obtained by rewriting φ by using only these new symbols.

$\mathsf{D} \equiv \neg\Box\bot \wedge \bigwedge\{P_\circ \to \Box P_\circ, \neg P_\circ \to \Box\neg P_\circ \mid \circ \in \mathcal{L}_A\}$,

Lemma (Artemov)

$\forall *$: T-interpretation, $\forall\varphi$: $\mathcal{L}_A$-sentence,
$\mathrm{I}\Sigma_1 \vdash \mathsf{D}^* \wedge [\![\bigwedge \mathrm{I}\Delta_0(exp)]\!]^* \to (\varphi \leftrightarrow [\![\varphi]\!]^*)$.

Let $A \equiv \mathsf{D} \wedge [\![\bigwedge \mathrm{I}\Delta_0(exp)]\!] \to [\![\bigwedge \mathrm{I}\Sigma_1]\!]$.

- Since $T \vdash \bigwedge \mathrm{I}\Sigma_1$, $A \in \mathsf{QPL}(T)$ by Artemov's lemma.
- From the fact that $\mathrm{I}\Delta_0(exp) \nvdash \bigwedge \mathrm{I}\Sigma_1$, there is a Kripke model where A is not valid.

1. Propositional modal logic of provability
2. Predicate modal logic of provability
3. Main theorem
4. Predicate provability logics of fragments of PA
5. Further work

Main theorem

Theorem (T.K.)

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathbf{Fr}(\mathbf{QGL}) \nsubseteq \mathbf{Th}(\mathbf{QGL})$.

Main theorem

Theorem (T.K.)

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathbf{Fr}(\mathbf{QGL}) \not\subseteq \mathbf{Th}(\mathbf{QGL})$.

Actually, we proved

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathbf{Fr}(\mathbf{QGL}) \not\subseteq \mathbf{QPL}(\mathrm{I}\Sigma_1)$.

Main theorem

Theorem (T.K.)

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathbf{Fr}(\mathbf{QGL}) \not\subseteq \mathbf{Th}(\mathbf{QGL})$.

Actually, we proved

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathbf{Fr}(\mathbf{QGL}) \not\subseteq \mathbf{QPL}(\mathrm{I}\Sigma_1)$.

- $\forall x \exists y \Box(p(x) \to \Diamond p(y)) \to \forall x \Box \neg p(x)$ **cannot be a witness of the non-inclusion.**

Main theorem

Theorem (T.K.)

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathbf{Fr}(\mathbf{QGL}) \nsubseteq \mathbf{Th}(\mathbf{QGL})$.

Actually, we proved

$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathbf{Fr}(\mathbf{QGL}) \nsubseteq \mathbf{QPL}(\mathrm{I}\Sigma_1)$.

- $\forall x \exists y \Box(p(x) \to \Diamond p(y)) \to \forall x \Box \neg p(x)$ **cannot be a witness of the non-inclusion.**
- **We introduce another method of constructing a witness of each non-inclusion $\mathbf{Fr}(\mathbf{QGL}) \nsubseteq \mathbf{QPL}(T)$.**

Sufficient conditions

We describe our method of constructing a witness of $\mathbf{Fr(QGL)} \not\subseteq \mathbf{QPL(PA)}$.

Sufficient conditions

We describe our method of constructing a witness of Fr(QGL) $\not\subseteq$ QPL(PA).
It suffices to find a predicate modal sentence A s.t.

(i)' $\neg A \in$ **Fr(QGL) and**
(ii)' $\neg A \notin$ **QPL(PA).**

Sufficient conditions

We describe our method of constructing a witness of Fr(QGL) $\not\subseteq$ QPL(PA).
It suffices to find a predicate modal sentence A s.t.

(i)' $\neg A \in$ Fr(QGL) and
(ii)' $\neg A \notin$ QPL(PA).

These conditions are equivalent to the following conditions respectively:

Conditions

(i) $\forall \mathcal{M} = \langle W, \prec, \{D_w\}_{w \in W}, \Vdash \rangle$: transitive Kripke model,
if $\exists w \in W$ s.t. $w \Vdash A$,
then $\prec$ is not conversely well-founded.

(ii) $\exists \mathcal{M} \models$ PA $\exists *$: PA-interpretation s.t. $\mathcal{M} \models A^*$.

Parameterized iterated consistency assertions

Definition (iterated consistency assertions)

$\mathbf{Con}^0 :\equiv (0 = 0);$
$\mathbf{Con}^{n+1} :\equiv \mathbf{Con}(\mathbf{PA} + \mathbf{Con}^n).$

Parameterized iterated consistency assertions

Definition (iterated consistency assertions)

$\mathsf{Con}^0 :\equiv (0 = 0)$;
$\mathsf{Con}^{n+1} :\equiv \mathsf{Con}(\mathsf{PA} + \mathsf{Con}^n)$.

Definition (parameterized iterated consistency assertions)

Let $\mathsf{Con}_{\mathsf{PA}}(x)$ be one of the $\mathcal{L}_A$-formula $\varphi(x)$ which satisfies

$$\mathsf{PA} \vdash \forall x(\varphi(x) \leftrightarrow [\mathsf{Con}(\mathsf{PA} + \varphi(\dot{x} \dot{-} 1)) \lor x = 0]).$$

Parameterized iterated consistency assertions

Definition (iterated consistency assertions)

$\mathsf{Con}^0 :\equiv (0 = 0)$;
$\mathsf{Con}^{n+1} :\equiv \mathsf{Con}(\mathsf{PA} + \mathsf{Con}^n)$.

Definition (parameterized iterated consistency assertions)

Let $\mathsf{Con}_{\mathsf{PA}}(x)$ be one of the $\mathcal{L}_A$-formula $\varphi(x)$ which satisfies

$$\mathsf{PA} \vdash \forall x(\varphi(x) \leftrightarrow [\mathsf{Con}(\mathsf{PA} + \varphi(\dot{x} \dot{-} 1)) \vee x = 0]).$$

$\forall n \in \omega$, $\mathsf{PA} \vdash \mathsf{Con}_{\mathsf{PA}}(\bar{n}) \leftrightarrow \mathsf{Con}^n$.

Main idea

A main idea of the construction is based on the following proposition.

Proposition

1. $\mathsf{PA} \vdash \forall x(\mathsf{Con}_{\mathsf{PA}}(x+1) \to \mathsf{Con}(\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}(\dot{x})))$.
2. $\mathbb{N} \models \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$.

Main idea

A main idea of the construction is based on the following proposition.

Proposition

1. $\mathsf{PA} \vdash \forall x(\mathsf{Con}_{\mathsf{PA}}(x+1) \to \mathsf{Con}(\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}(\dot{x})))$.
2. $\mathbb{N} \models \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$.

- $B :\equiv \forall x p(x) \land \Box \forall x(p(x+1) \to \Diamond p(x))$.

Main idea

A main idea of the construction is based on the following proposition.

Proposition

1. $\mathbf{PA} \vdash \forall x(\mathsf{Con}_{\mathbf{PA}}(x+1) \to \mathsf{Con}(\mathbf{PA} + \mathsf{Con}_{\mathbf{PA}}(\dot{x})))$.
2. $\mathbb{N} \models \mathsf{Con}(\mathbf{PA} + \forall x \mathsf{Con}_{\mathbf{PA}}(x))$.

- $B :\equiv \forall x p(x) \land \Box \forall x(p(x+1) \to \Diamond p(x))$.
- $*$: **PA-interpretation s.t.** $(p(x))^* \equiv \mathsf{Con}_{\mathbf{PA}}(x)$.

Main idea

A main idea of the construction is based on the following proposition.

Proposition

1. $\mathsf{PA} \vdash \forall x(\mathsf{Con}_{\mathsf{PA}}(x+1) \to \mathsf{Con}(\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}(\dot{x})))$.
2. $\mathbb{N} \models \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$.

- $B :\equiv \forall x p(x) \land \Box \forall x(p(x+1) \to \Diamond p(x))$.
- $*$: PA-interpretation s.t. $(p(x))^* \equiv \mathsf{Con}_{\mathsf{PA}}(x)$.
- Then $\exists \mathcal{M} \models \mathsf{PA}$ s.t. $\mathcal{M} \models B^*$.

Main idea

A main idea of the construction is based on the following proposition.

Proposition

1. $\mathsf{PA} \vdash \forall x(\mathsf{Con}_{\mathsf{PA}}(x+1) \to \mathsf{Con}(\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}(\dot{x})))$.
2. $\mathbb{N} \models \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$.

- $B :\equiv \forall x p(x) \land \Box \forall x(p(x+1) \to \Diamond p(x))$.
- $*$: PA-interpretation s.t. $(p(x))^* \equiv \mathsf{Con}_{\mathsf{PA}}(x)$.
- Then $\exists \mathcal{M} \models \mathsf{PA}$ s.t. $\mathcal{M} \models B^*$.
- We construct an infinite increasing sequence of worlds from this sentence by starting from a non-standard element of a non-standard model of arithmetic.

$$B \equiv \forall x \forall y (P_S(y,x) \wedge p(x) \rightarrow \Diamond p(y)).$$

Let A be the conjunction of the following six sentences:

1. $\forall x p(x)$
2. B
3. $\Box B$
4. $\forall x \forall y (P_S(x,y) \rightarrow \Box P_S(x,y))$
5. $[\![\bigwedge \mathsf{Q}]\!]$
6. $[\![\neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))]\!]$

where Q is Robinson's arithmetic.

$$B \equiv \forall x \forall y (P_S(y,x) \wedge p(x) \to \Diamond p(y)).$$

Let A be the conjunction of the following six sentences:

1. $\forall x p(x)$
2. B
3. $\Box B$
4. $\forall x \forall y (P_S(x,y) \to \Box P_S(x,y))$
5. $[\![\bigwedge \mathsf{Q}]\!]$
6. $[\![\neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))]\!]$

where Q is Robinson's arithmetic.

Then A satisfies (i) and (ii).

(i) $\forall \mathcal{M} = \langle W, \prec, \{D_w\}_{w \in W}, \Vdash \rangle$: transitive Kripke model, if $\exists w \in W$ s.t. $w \Vdash A$, then $\prec$ is not conversely well-founded.

(ii) $\exists \mathcal{M} \models \mathsf{PA}$ $\exists *$: PA-interpretation s.t. $\mathcal{M} \models A^*$.

$$B \equiv \forall x \forall y (P_S(y,x) \wedge p(x) \to \Diamond p(y))$$

$$A \equiv \forall x p(x) \wedge B \wedge \Box B \wedge \forall x \forall y (P_S(x,y) \to \Box P_S(x,y))$$

$$\wedge [\![\bigwedge \mathsf{Q}]\!] \wedge [\![\neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))]\!]$$

$B \equiv \forall x \forall y (P_S(y, x) \wedge p(x) \rightarrow \Diamond p(y))$
$A \equiv \forall x p(x) \wedge B \wedge \Box B \wedge \forall x \forall y (P_S(x, y) \rightarrow \Box P_S(x, y))$
$\wedge [\![\bigwedge \mathsf{Q}]\!] \wedge [\![\neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))]\!]$

(i)

- **Assume that $\mathcal{M} = \langle W, \prec, \{D_w\}_{w \in W}, \Vdash \rangle$ is a transitive Kripke model and $w_0 \in W$ satisfies A.**
- **w_0 is a model of Q.**
- **Since $\mathbb{N} \models \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$, w_0 must be non-standard.**

$\Rightarrow \prec$ is not conversely well-founded.

$B \equiv \forall x \forall y (P_S(y,x) \wedge p(x) \to \Diamond p(y))$
$A \equiv \forall x p(x) \wedge B \wedge \Box B \wedge \forall x \forall y (P_S(x,y) \to \Box P_S(x,y))$
$\wedge [\![\bigwedge \mathsf{Q}]\!] \wedge [\![\neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))]\!]$

(i)

- Assume that $\mathcal{M} = \langle W, \prec, \{D_w\}_{w \in W}, \Vdash \rangle$ is a transitive Kripke model and $w_0 \in W$ satisfies A.
- w_0 is a model of Q.
- Since $\mathbb{N} \models \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$, w_0 must be non-standard.

$\Rightarrow \prec$ is not conversely well-founded.

(ii)

- $\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x)$: consistent.
- $\exists \mathcal{M} \models \mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x) + \neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$.
- $*$: natural PA-interpretation s.t. $(p(x))^* \equiv \mathsf{Con}_{\mathsf{PA}}(x)$.

$\Rightarrow \mathcal{M} \models A^*$.

Witness of our main theorem

$$\bigcap\{\mathbf{QPL}(T) \mid T : \text{r.e. extension of } \mathbf{I\Sigma_2}\} \cap \mathbf{Fr}(\mathbf{QGL}) \not\subseteq \mathbf{QPL}(\mathbf{I\Sigma_1}).$$

$$B \equiv \forall x \forall y (P_S(y, x) \land p(x) \to \Diamond p(y)),$$
$$C \equiv \forall x p(x) \land B \land \Box B \land [\![\neg \mathsf{Con}(\mathbf{I\Sigma_1} + \forall x \mathsf{Con}_{\mathbf{I\Sigma_1}}(x))]\!],$$
$$\mathsf{D} \equiv \neg\Box\bot \land \bigwedge\{P_\circ \to \Box P_\circ, \neg P_\circ \to \Box\neg P_\circ \mid \circ \in \mathcal{L}_A\},$$
$$A' \equiv [\![\mathbf{I\Sigma_1}]\!] \land \mathsf{D} \land C \to [\![\mathbf{I\Sigma_2}]\!].$$

Witness of our main theorem

$\bigcap\{\mathsf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathsf{Fr}(\mathsf{QGL}) \nsubseteq \mathsf{QPL}(\mathrm{I}\Sigma_1).$

$B \equiv \forall x \forall y (P_S(y,x) \land p(x) \to \Diamond p(y))$,
$C \equiv \forall x p(x) \land B \land \Box B \land [\![\neg \mathsf{Con}(\mathrm{I}\Sigma_1 + \forall x \mathsf{Con}_{\mathrm{I}\Sigma_1}(x))]\!]$,
$\mathsf{D} \equiv \neg\Box\bot \land \bigwedge\{P_\circ \to \Box P_\circ, \neg P_\circ \to \Box\neg P_\circ \mid \circ \in \mathcal{L}_A\}$,
$A' \equiv [\![\mathrm{I}\Sigma_1]\!] \land \mathsf{D} \land C \to [\![\mathrm{I}\Sigma_2]\!]$.

Then A' satisfies (i) and (ii).

Witness of our main theorem

$$\bigcap\{\mathsf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathsf{Fr}(\mathsf{QGL}) \nsubseteq \mathsf{QPL}(\mathrm{I}\Sigma_1).$$

$B \equiv \forall x \forall y (P_S(y, x) \land p(x) \to \Diamond p(y))$,
$C \equiv \forall x p(x) \land B \land \Box B \land [\![\neg \mathsf{Con}(\mathrm{I}\Sigma_1 + \forall x \mathsf{Con}_{\mathrm{I}\Sigma_1}(x))]\!]$,
$\mathsf{D} \equiv \neg \Box \bot \land \bigwedge \{P_\circ \to \Box P_\circ, \neg P_\circ \to \Box \neg P_\circ \mid \circ \in \mathcal{L}_A\}$,
$A' \equiv [\![\mathrm{I}\Sigma_1]\!] \land \mathsf{D} \land C \to [\![\mathrm{I}\Sigma_2]\!]$.

Then A' satisfies (i) and (ii).

- $\neg A'$ witnesses $\mathsf{Fr}(\mathsf{QGL}) \nsubseteq \mathsf{QPL}(\mathrm{I}\Sigma_1)$.

Witness of our main theorem

$\bigcap\{\mathsf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\} \cap \mathsf{Fr}(\mathsf{QGL}) \not\subseteq \mathsf{QPL}(\mathrm{I}\Sigma_1)$.

$B \equiv \forall x \forall y (P_S(y, x) \land p(x) \to \Diamond p(y))$,
$C \equiv \forall x p(x) \land B \land \Box B \land [\![\neg \mathsf{Con}(\mathrm{I}\Sigma_1 + \forall x \mathsf{Con}_{\mathrm{I}\Sigma_1}(x))]\!]$,
$\mathsf{D} \equiv \neg \Box \bot \land \bigwedge \{P_\circ \to \Box P_\circ, \neg P_\circ \to \Box \neg P_\circ \mid \circ \in \mathcal{L}_A\}$,
$A' \equiv [\![\mathrm{I}\Sigma_1]\!] \land \mathsf{D} \land C \to [\![\mathrm{I}\Sigma_2]\!]$.

Then A' satisfies (i) and (ii).

- $\neg A'$ witnesses $\mathsf{Fr}(\mathsf{QGL}) \not\subseteq \mathsf{QPL}(\mathrm{I}\Sigma_1)$.
- $\neg A' \in \bigcap\{\mathsf{QPL}(T) \mid T : \text{r.e. extension of } \mathrm{I}\Sigma_2\}$ by Artemov's lemma.

Binumerations

A formula $\alpha(x)$ is called a **binumeration** of T if for any sentence φ,

$$\varphi \in T \quad \Rightarrow \quad T \vdash \alpha(\ulcorner \varphi \urcorner);$$
$$\varphi \notin T \quad \Rightarrow \quad T \vdash \neg\alpha(\ulcorner \varphi \urcorner).$$

Binumerations

A formula $\alpha(x)$ is called a binumeration of T if for any sentence φ,

$$\varphi \in T \quad \Rightarrow \quad T \vdash \alpha(\ulcorner \varphi \urcorner);$$
$$\varphi \notin T \quad \Rightarrow \quad T \vdash \neg\alpha(\ulcorner \varphi \urcorner).$$

Each recursive theory T has a Σ_1 binumeration.

Binumerations

A formula $\alpha(x)$ is called a binumeration of T if for any sentence φ,

$$\varphi \in T \quad \Rightarrow \quad T \vdash \alpha(\ulcorner \varphi \urcorner);$$
$$\varphi \notin T \quad \Rightarrow \quad T \vdash \neg\alpha(\ulcorner \varphi \urcorner).$$

Each recursive theory T has a Σ_1 binumeration.

- For each Σ_1 binumeration $\alpha(x)$ of T,
  let $\mathrm{Pr}_\alpha(x)$ be the provability predicate of T asserting that "x is provabile from the set of all sentences satisfying $\alpha(z)$".

Binumerations

A formula $\alpha(x)$ is called a **binumeration** of T if for any sentence φ,

$$\varphi \in T \quad \Rightarrow \quad T \vdash \alpha(\ulcorner \varphi \urcorner);$$
$$\varphi \notin T \quad \Rightarrow \quad T \vdash \neg\alpha(\ulcorner \varphi \urcorner).$$

Each recursive theory T has a Σ_1 binumeration.

- For each Σ_1 binumeration $\alpha(x)$ of T, let $\mathrm{Pr}_\alpha(x)$ be the provability predicate of T asserting that "x is provabile from the set of all sentences satisfying $\alpha(z)$".
- Let QPL(α) be the provability logic of T which is defined by using the provability predicate $\mathrm{Pr}_\alpha(x)$.

Provability logics and binumerations

Theorem (Artemov, 1986)

T: Σ_1-sound recursive extension of **PA**.
$\forall\alpha(x)$: Σ_1 binumeration of T,
$\exists\beta(x)$: Σ_1 binumeration of T s.t.

$$\mathbf{QPL}(\alpha) \nsubseteq \mathbf{QPL}(\beta).$$

Provability logics and binumerations

Theorem (Artemov, 1986)

T: Σ_1-sound recursive extension of PA.
$\forall\alpha(x)$: Σ_1 binumeration of T,
$\exists\beta(x)$: Σ_1 binumeration of T s.t.

$$\mathbf{QPL}(\alpha) \not\subseteq \mathbf{QPL}(\beta).$$

- Our main theorem holds for any Σ_1 binumeration.

Provability logics and binumerations

Theorem (Artemov, 1986)

T: Σ_1-sound recursive extension of **PA**.
$\forall\alpha(x)$: Σ_1 binumeration of T,
$\exists\beta(x)$: Σ_1 binumeration of T s.t.

$$\mathbf{QPL}(\alpha) \not\subseteq \mathbf{QPL}(\beta).$$

- Our main theorem holds for any Σ_1 binumeration.
- Montagna proved $\mathbf{QPL}(\mathbf{PA}) \not\subseteq \mathbf{QPL}(\mathbf{BG})$ essentially from the follwoing facts:
  - **BG** is finitely axiomatizable,
  - $\mathbf{PA} \vdash \mathrm{Con}_{\mathbf{BG}} \to \mathrm{Con}_{\mathbf{PA}+\mathrm{Con}_{\mathbf{PA}}}$.

Provability logics and binumerations

Theorem (Artemov, 1986)

T: Σ_1-sound recursive extension of PA.
$\forall\alpha(x)$: Σ_1 binumeration of T,
$\exists\beta(x)$: Σ_1 binumeration of T s.t.

$$\mathsf{QPL}(\alpha) \nsubseteq \mathsf{QPL}(\beta).$$

- Our main theorem holds for any Σ_1 binumeration.
- Montagna proved $\mathsf{QPL}(\mathsf{PA}) \nsubseteq \mathsf{QPL}(\mathsf{BG})$ essentially from the follwoing facts:
  - BG is finitely axiomatizable,
  - $\mathsf{PA} \vdash \mathsf{Con}_{\mathsf{BG}} \rightarrow \mathsf{Con}_{\mathsf{PA}+\mathsf{Con}_{\mathsf{PA}}}$.
- These conditions also hold for the theories $\mathrm{I}\Sigma_{i+1}$ and $\mathrm{I}\Sigma_i$.

Provability logics and binumerations

Theorem (Artemov, 1986)

T: Σ_1-sound recursive extension of PA.
$\forall\alpha(x)$: Σ_1 binumeration of T,
$\exists\beta(x)$: Σ_1 binumeration of T s.t.

$$\mathbf{QPL}(\alpha) \nsubseteq \mathbf{QPL}(\beta).$$

- Our main theorem holds for any Σ_1 binumeration.
- Montagna proved $\mathbf{QPL}(\mathbf{PA}) \nsubseteq \mathbf{QPL}(\mathbf{BG})$ essentially from the follwoing facts:
  - BG is finitely axiomatizable,
  - $\mathbf{PA} \vdash \mathrm{Con}_{\mathbf{BG}} \to \mathrm{Con}_{\mathbf{PA}+\mathrm{Con}_{\mathbf{PA}}}$.
- These conditions also hold for the theories $\mathrm{I}\Sigma_{i+1}$ and $\mathrm{I}\Sigma_i$.
- The second condition is dependent on the choice of binumerations of PA and BG.

Predicate provability logics of fragments of **PA**

T_i: **a finite axiomatization of** $\mathrm{I}\Sigma_i$ $(i > 0)$**.**

Predicate provability logics of fragments of **PA**

T_i: **a finite axiomatization of** $\mathrm{I}\Sigma_i$ **(**$i > 0$**).**
$[T_i](x) :\equiv \bigvee\{x = \ulcorner\varphi\urcorner | \varphi \in T_i\}$.

Predicate provability logics of fragments of **PA**

T_i: a finite axiomatization of $\mathrm{I}\Sigma_i$ $(i > 0)$.
$[T_i](x) :\equiv \bigvee\{x = \ulcorner\varphi\urcorner | \varphi \in T_i\}$.

Theorem (T.K.)

For any i, j: natural numbers $(0 < i < j)$,
$\mathbf{QPL}([T_i]) \not\subseteq \bigcup\{\mathbf{QPL}(\beta) \mid \beta(x)$: Σ_1 binumeration of some r.a. of $\mathrm{I}\Sigma_j\}$,
$\mathbf{QPL}([T_i]) \not\supseteq \bigcap\{\mathbf{QPL}(\beta) \mid \beta(x)$: Σ_1 binumeration of some r.a. of $\mathrm{I}\Sigma_j\}$.

where "r.a." is an abbreviation for "recursive axiomatization".

1. **Propositional modal logic of provability**
2. **Predicate modal logic of provability**
3. **Main theorem**
4. **Predicate provability logics of fragments of PA**
5. **Further work**

Montagna's problem

Montagna's problem (extended)

$\bigcap\{$**QPL**$(T) \mid T$: **r.e. theory where PA is relatively interpretable**$\}$
$=$ **Th**$($**QGL**$)$ **?**

Montagna's problem

Montagna's problem (extended)

$\bigcap\{$**QPL**$(T) \mid T$: **r.e. theory where PA is relatively interpretable**$\}$
$=$ **Th**$($**QGL**$)$ **?**

- **Montagna proved QPL(PA)** $\not\subseteq$ **QPL(BG).**

Montagna's problem

Montagna's problem (extended)

$\bigcap\{\mathsf{QPL}(T) \mid T$: r.e. theory where PA is relatively interpretable$\}$
$= \mathsf{Th}(\mathsf{QGL})$?

- Montagna proved $\mathsf{QPL}(\mathsf{PA}) \not\subseteq \mathsf{QPL}(\mathsf{BG})$.
- $\mathsf{QPL}(\mathsf{BG})$ can be defined in many ways.
  - The choice of a binumeration of BG.
  - The definition of $\mathsf{Pr}_{\mathsf{BG}}(x)$.

Characterization of $\mathbf{QPL}(\alpha) = \mathbf{QPL}(\beta)$

Let $\alpha(x), \beta(x)$ be Σ_1 binumerations of T.

Characterization of $\mathbf{QPL}(\alpha) = \mathbf{QPL}(\beta)$

Let $\alpha(x), \beta(x)$ be Σ_1 binumerations of T.

Theorem (Visser and de Jonge, 2006)

$\exists A$: predicate modal sentence s.t. T.F.A.E.:

- $T + \mathbf{Con}_\alpha \vdash \varphi$;
- $A \to [\![\varphi]\!] \in \mathbf{QPL}(\alpha)$.

Characterization of $\mathbf{QPL}(\alpha) = \mathbf{QPL}(\beta)$

Let $\alpha(x), \beta(x)$ be Σ_1 binumerations of T.

Theorem (Visser and de Jonge, 2006)

$\exists A$: predicate modal sentence s.t. T.F.A.E.:

- $T + \mathbf{Con}_\alpha \vdash \varphi$;
- $A \to [\![\varphi]\!] \in \mathbf{QPL}(\alpha)$.

Corollary

If $\mathbf{QPL}(\alpha) = \mathbf{QPL}(\beta)$, then $T \vdash \mathbf{Con}_\alpha \leftrightarrow \mathbf{Con}_\beta$.

Characterization of $\mathbf{QPL}(\alpha) = \mathbf{QPL}(\beta)$

Let $\alpha(x), \beta(x)$ be Σ_1 binumerations of T.

Theorem (Visser and de Jonge, 2006)

$\exists A$: predicate modal sentence s.t. T.F.A.E.:

- $T + \mathbf{Con}_\alpha \vdash \varphi$;
- $A \to [\![\varphi]\!] \in \mathbf{QPL}(\alpha)$.

Corollary

If $\mathbf{QPL}(\alpha) = \mathbf{QPL}(\beta)$, then $T \vdash \mathbf{Con}_\alpha \leftrightarrow \mathbf{Con}_\beta$.

Conjecture

$\mathbf{QPL}(\alpha) = \mathbf{QPL}(\beta)$
if and only if $T \vdash \mathbf{Pr}_\alpha(\ulcorner \varphi \urcorner) \leftrightarrow \mathbf{Pr}_\beta(\ulcorner \varphi \urcorner)$ for any formula φ.

Thank you!