Yablo's paradox and witness comparison

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	Σ_1-sound (ω-consistent)	**consistent**
Liar paradox	**Gödel (1931)**	**Rosser (1936)**
Yablo's paradox	**Priest (1997)**	?

Contents

1. Gödel and Rosser's theorems
2. Gödel's theorems based on Yablo's paradox
3. Rosser-type formalizations
4. Proof of Theorem 1

1. **Gödel and Rosser's theorems**
2. **Gödel's theorems based on Yablo's paradox**
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4. **Proof of Theorem 1**

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Gödel constructed a s.p.p. of each such a theory T.

Gödel and Rosser's incompleteness theorems

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Theorem (Gödel, 1931)

For any sentence φ satisfying $\mathsf{PA} \vdash \varphi \leftrightarrow \neg \mathsf{Pr}_T(\ulcorner \varphi \urcorner)$,

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T is Σ_1-sound
$\Leftrightarrow \forall \varphi$: Σ_1 sentence ($T \vdash \varphi \Rightarrow \mathbb{N} \models \varphi$).

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Theorem (Rosser, 1936)

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Guaspari and Solovay's results

Theorem (Guaspari and Solovay, 1979)

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They constructed required standard proof predicates by rearrenging proofs of a given s.p.p.

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Yablo's paradox

Yablo's paradox (Yablo, 1993)

- **Let $Y_0, Y_1, \ldots,$ be an infinite sequence of propositions.**
- **Each Y_i states that "For every $j > i$, Y_j is false".**
- **Then we cannot determine whether Y_i is true or false.**

Gödel's theorems based on Yablo's paradox

Formalization of Yablo's sequence

Let $Y(x)$ be a formula satisfying the following equivalence:

$$\mathsf{PA} \vdash \forall x(Y(x) \leftrightarrow \forall y > x \neg \mathsf{Pr}_T(\ulcorner Y(\dot{y}) \urcorner)),$$

where y is free in $\mathsf{Pr}_T(\ulcorner Y(\dot{y}) \urcorner)$.

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- T: **consistent** $\Rightarrow \forall n \in \mathbb{N}, \quad T \nvdash Y(\bar{n})$.
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Theorem (Kikuchi and K., 2011; Cieśliński and Urbaniak, 2012)

- **For any** $n \in \mathbb{N}$, $\mathsf{PA} \vdash Y(\bar{n}) \leftrightarrow \mathsf{Con}_T$.

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- **For any** $m, n \in \mathbb{N}$, $\mathsf{PA} \vdash Y(\bar{m}) \leftrightarrow Y(\bar{n})$.

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	Σ_1-sound	consistent
Liar paradox	$\mathsf{Pr}_T(x)$	$\mathsf{Pr}_T^R(x)$
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It is dependent on the choice of a s.p.p.

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Question

T: **consistent** $\Rightarrow \forall n \in \mathbb{N},\quad T \nvdash \neg Y^R(\bar{n})$?

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Question

T: **consistent** $\Rightarrow \forall n \in \mathbb{N},\ \ T \nvdash \neg Y^R(\bar{n})$?

Answer

No.
If T is consistent but not Σ_1-sound, then the provability of $\neg Y^R(\bar{n})$ is dependent on the choice of a s.p.p.

Main Theorems

Theorem 1

T: consistent

$\Rightarrow$ there is a s.p.p. of T s.t. $\forall n \in \mathbb{N}$, $T \nvdash \neg Y^R(\bar{n})$.

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Theorem 1

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Theorem 2

T: not Σ_1-sound

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We proved these theorems by using the technique of Gaspari and Solovay.

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- $\mathsf{PA} \vdash \forall x(\mathsf{Pr}_T(x) \leftrightarrow \mathsf{Pr}'_T(x))$;
- $\mathsf{PA} \vdash Y^R(0) \leftrightarrow \pi$ for a Rosser sentence π of $\mathsf{Prf}_T(x, y)$;

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- T: consistent $\Rightarrow T \nvdash \neg Y^R(\bar{n})$ for all $n \in \omega$.

$T \nvdash \neg\pi$ by Rosser's theorem, and thus $T \nvdash \neg Y^R(0)$.
Since $\mathsf{PA} \vdash Y^R(0) \to Y^R(\bar{n})$, $T \nvdash \neg Y^R(\bar{n})$.

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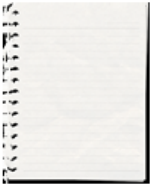
- **By using the fixed point lemma, we can use a Rosser-type Yablo formula $Y^R(x)$ of $\mathsf{Prf}_T(x, y)$ and a Rosser sentence π of $\mathsf{Prf}_T(x, y)$ in the definition of f.**

Proof of Theorem 1

- Our s.p.p. $\mathsf{Prf}_T(x, y)$ is defined as a formula representing a recursive function f enumerating the set of all theorems of T.
- By using the fixed point lemma, we can use
  a Rosser-type Yablo formula $Y^R(x)$ of $\mathsf{Prf}_T(x, y)$ and
  a Rosser sentence π of $\mathsf{Prf}_T(x, y)$
  in the definition of f.
- We prepare a bell and a list, and we define f recursively by using these materials.



Proof of Theorem 1

Until the bell rings

- **Define the values of f by copying $\mathsf{Prf}'_T(x, y)$, i.e.**

$$f(n) := \begin{cases} \varphi & \text{if } n \text{ is a proof of some formula } \varphi; \\ 0 = 0 & \text{o.w.} \end{cases}$$

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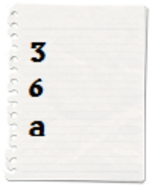
- **If n is a proof of $\neg Y(\bar{a})$ for some $a > 0$, put a into the list.**



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Until the bell rings

- **If n is a proof of one of**
  1. **$Y(\bar{a})$ for some $a > 0$ which is not in the list,**
  2. **π,**
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  then ring the bell!

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After the bell rings at the stage n

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1. **If n is a proof of $Y(\bar{a})$, let**

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After $n+3$, f outputs all formulas.

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and define $f(n+1) = Y^R(\bar{c})$.
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3. **If n is a proof of $\neg\pi$, for a suitable recursive enumeration $\varphi_0, \varphi_1, \ldots$ of all formulas, f enumerates**

$$\neg Y^R(\bar{1}), \varphi_0, \neg Y^R(\bar{2}), \varphi_1, \ldots$$

after $n+1$.

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$\mathsf{PA} \vdash$ "the bell rings" $\leftrightarrow \neg \mathsf{Con}'_T$.

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Lemma 2

$\mathsf{PA} \vdash \exists y > 0 \mathsf{Pr}^R_T(\ulcorner Y^R(\dot{y}) \urcorner) \leftrightarrow \mathsf{Pr}^R_T(\ulcorner \pi \urcorner)$.

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Lemma 2

$\mathbf{PA} \vdash \exists y > 0 \mathsf{Pr}^R_T(\ulcorner Y^R(\dot{y}) \urcorner) \leftrightarrow \mathsf{Pr}^R_T(\ulcorner \pi \urcorner)$.

Then $\mathbf{PA} \vdash Y^R(0) \leftrightarrow \pi$.

Thank you for your attention!