Rosser-type formalizations of Yablo's paradox

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	Σ_1-sound (ω-consistent)	consistent
Liar paradox	Gödel (1931)	Rosser (1936)
Yablo's paradox	Priest (1997)	?

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Standard proof predicates

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- for the formula $\mathsf{Pr}_T(x) \equiv \exists y \mathsf{Prf}_T(x, y)$,
  - $\mathsf{PA} \vdash \mathsf{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}_T(\ulcorner \varphi \urcorner) \to \mathsf{Pr}_T(\ulcorner \psi \urcorner))$;

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Gödel constructed a s.p.p. of each such a theory T.

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- $\mathsf{Pr}^R_T(x) \equiv \exists y(\mathsf{Prf}_T(x, y) \land \forall z \leq y \neg \mathsf{Prf}_T(\neg x, z))$
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Theorem (Gödel, 1931)

For any φ satisfying $\mathsf{PA} \vdash \varphi \leftrightarrow \neg \mathsf{Pr}_T(\ulcorner \varphi \urcorner)$,

- T: **consistent** $\Rightarrow T \nvdash \varphi$;
- T: **Σ_1-sound** $\Rightarrow T \nvdash \neg\varphi$.

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$\Leftrightarrow \forall \varphi$: Σ_1 sentence ($T \vdash \varphi \Rightarrow \mathbb{N} \models \varphi$).

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Theorem (Rosser, 1936)

For any ψ satisfying $\mathsf{PA} \vdash \psi \leftrightarrow \neg \mathsf{Pr}^R_T(\ulcorner \psi \urcorner)$,
T: consistent $\Rightarrow T \nvdash \psi$ and $T \nvdash \neg\psi$.

Guaspari and Solovay's results

Theorem (Guaspari and Solovay, 1979)

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- $\mathsf{Pr}_T^R(\ulcorner \varphi \urcorner)$ **means "**φ **has a smaller proof in** T **than any proof of** $\neg\varphi$**".**

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- $\mathsf{Pr}^R_T(\ulcorner \varphi \urcorner)$ **means "φ has a smaller proof in T than any proof of $\neg\varphi$".**
- **They constructed a new s.p.p. with the required conditions by rearrenging proofs of a given s.p.p.**

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Yablo's paradox

Yablo's paradox (Yablo, 1993)

- **Let $Y_0, Y_1, \ldots,$ be an infinite sequence of propositions.**
- **Each Y_i states that "For every $j > i$, Y_j is false".**
- **Then we cannot determine whether Y_i is true or false.**

Gödel's theorems based on Yablo's paradox

Formalization of Yablo's sequence

Let $Y(x)$ be a formula satisfying the following equivalence:

$$\mathsf{PA} \vdash \forall x(Y(x) \leftrightarrow \forall y > x \neg \mathsf{Pr}_T(\ulcorner Y(\dot{y}) \urcorner)),$$

where y is free in $\mathsf{Pr}_T(\ulcorner Y(\dot{y}) \urcorner)$.

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Theorem (Kikuchi and K., 2011; Cieśliński and Urbaniak, 2012)

- **For any** $n \in \mathbb{N}$, $\mathsf{PA} \vdash Y(\bar{n}) \leftrightarrow \mathsf{Con}_T$.

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Theorem (Kikuchi and K., 2011; Cieśliński and Urbaniak, 2012)

- **For any** $n \in \mathbb{N}$, $\mathsf{PA} \vdash Y(\bar{n}) \leftrightarrow \mathsf{Con}_T$.
- **For any** $m, n \in \mathbb{N}$, $\mathsf{PA} \vdash Y(\bar{m}) \leftrightarrow Y(\bar{n})$.

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It is dependent on the choice of a s.p.p.

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Formalizations of Yablo's sequence using Rosser predicates

Rosser-type formalization of Yablo's sequence

Let $Y^R(x)$ be a formula satisfying the following equivalence:

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Question

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Question

T: **consistent** $\Rightarrow \forall n \in \mathbb{N}, \quad T \nvdash \neg Y^R(\bar{n})$?

Answer

No.
If T is consistent but not Σ_1-sound, then the provability of $\neg Y^R(\bar{n})$ is dependent on the choice of a s.p.p.

Main Theorems

Theorem 1

$\boldsymbol{T}$**: consistent**

$\Rightarrow$ **there is a s.p.p. of** $\boldsymbol{T}$ **s.t.** $\forall \boldsymbol{n} \in \mathbb{N}, \quad \boldsymbol{T} \nvdash \neg \boldsymbol{Y}^{\boldsymbol{R}}(\bar{\boldsymbol{n}})$**.**

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Theorem 2

$\boldsymbol{T}$: **not $\boldsymbol{\Sigma_1}$-sound**

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Main Theorems

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T: consistent
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Theorem 2

T: not Σ_1-sound
$\Rightarrow$ there is a s.p.p. of T s.t. $\forall n \in \mathbb{N}$, $T \vdash \neg Y^R(\bar{n})$.

We proved these theorems by using the technique of Gaspari and Solovay.

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- **The provability of $\forall x \forall y (Y^R(x) \leftrightarrow Y^R(y))$ is dependent on the choice of a s.p.p.**

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1. **There is a s.p.p. of T s.t. $\mathsf{PA} \vdash \forall x \forall y (Y^R(x) \leftrightarrow Y^R(y))$.**

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Problem

Is there a s.p.p. of T such that $Y^R(0)$ and $Y^R(\bar{1})$ are not provably equivalent?

Thank you for your attention!