Nonstandard models of arithmetic and QGL

Taishi Kurahashi

Kobe University

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- **Provability predicate of r.e. theory T is a Σ_1 formula $\mathrm{Pr}_T(x)$ which weakly represents the set $\{\ulcorner\varphi\urcorner \mid T \vdash \varphi\}$ in PA, i.e., $\forall\varphi$: sentence, PA $\vdash \mathrm{Pr}_T(\ulcorner\varphi\urcorner) \Leftrightarrow T \vdash \varphi$.**
- **Fix a provability predicate which satisfies the following five conditions:**

The properties of $\mathrm{Pr}_T(x)$

D1 $T \vdash \varphi \Rightarrow$ PA $\vdash \mathrm{Pr}_T(\ulcorner\varphi\urcorner)$

D2 PA $\vdash \mathrm{Pr}_T(\ulcorner\varphi \to \psi\urcorner) \to (\mathrm{Pr}_T(\ulcorner\varphi\urcorner) \to \mathrm{Pr}_T(\ulcorner\psi\urcorner))$

D3 PA $\vdash \mathrm{Pr}_T(\ulcorner\varphi\urcorner) \to \mathrm{Pr}_T(\ulcorner\mathrm{Pr}_T(\ulcorner\varphi\urcorner)\urcorner)$

Löb PA $\vdash \mathrm{Pr}_T(\ulcorner\mathrm{Pr}_T(\ulcorner\varphi\urcorner) \to \varphi\urcorner) \to \mathrm{Pr}_T(\ulcorner\varphi\urcorner)$.

Σ_1-comp. φ: $\Sigma_1 \Rightarrow T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner\varphi\urcorner)$.

Gödel's thesis (1933)

The provability of a formal system can be considered as a modality.

The system **GL** of propositional modal logic

- **Axioms:**
  - **Tautologies;**
  - $\Box(A \to B) \to (\Box A \to \Box B)$;
  - $\Box(\Box A \to A) \to \Box A$.
- **Inference rules:**

  modus ponens **from** A **and** $A \to B$ **infer** B;

  necessitation **form** A **infer** $\Box A$.

Th(GL)$:= \{A \mid \textbf{GL} \vdash A\}$.

Let F be the set of all propositional modal sentences.

Definition

Kripke frame is a system $\langle W, \prec \rangle$ where

- W is a non-empty set of **worlds**;
- $\prec$ is a binary relation on W: **accessibility relation**.

Kripke model is a system $\mathcal{M} = \langle W, \prec, \Vdash \rangle$ where

- $\langle W, \prec \rangle$ is a Kripke frame;
- $\Vdash$ is a binary relation on $W \times F$ such that $\forall w \in W$,
  - $w \nVdash \bot$;
  - $w \Vdash A \to B \Leftrightarrow (w \nVdash A$ or $w \Vdash B)$;
  - $\cdots$;
  - $w \Vdash \Box A \Leftrightarrow \forall w' \in W (w \prec w' \Rightarrow w' \Vdash A)$.
  - $w \Vdash \Diamond A \Leftrightarrow \exists w' \in W (w \prec w' \ \& \ w' \Vdash A)$.

Definition

A: modal sentence, $\mathcal{F}$: Kripke frame, $\mathcal{M}$: Kripke model.

- A is valid in $\mathcal{M} \stackrel{\text{def.}}{\Leftrightarrow} \forall w \in W$, $w \Vdash A$.
- A is valid in $\mathcal{F} \stackrel{\text{def.}}{\Leftrightarrow} A$ is valid in $\langle \mathcal{F}, \Vdash \rangle$ for any $\Vdash$.

Definition

- Kripke frame $\langle W, \prec \rangle$ is a **GL-frame** if $\prec$ is
  1. transitive,
  2. conversely well-founded.
- **Fr(GL)**$:= \{A \mid A$ is valid in any GL-frame $\}$.

Theorem (Segerberg, 1971)

$\mathbf{Th}(\mathbf{GL}) = \mathbf{Fr}(\mathbf{GL})$.

T: r.e. theory.

Definition (arithmetical interpretation)

A mapping $*$ from F to all sentences in the language of T is called a **T-interpretation** if it satisfies the following conditions:

- $\bot^* \equiv \ 0 = 1$;
- $(A \to B)^* \equiv (A^* \to B^*)$;
- $(\Box A)^* \equiv \mathrm{Pr}_T(\ulcorner A^* \urcorner)$.

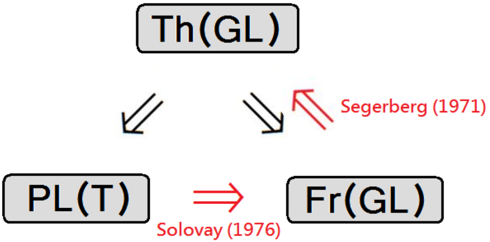
Definition

- A: propositional modal sentence.
  A is **T-valid** $\overset{\text{def.}}{\Leftrightarrow}$ $\forall *$: T-interpretation, $T \vdash A^*$.
- **PL(T)** $:= \{A \mid A$ is T-valid$\}$:**the provability logic** of T.

T: Σ_1-sound r.e. extension of PA.

Theorem (Solovay, 1976)

$\mathsf{Th}(\mathsf{GL}) = \mathsf{PL}(T)$.



$\mathsf{Th}(\mathsf{GL}) = \mathsf{Fr}(\mathsf{GL}) = \mathsf{PL}(T)$ for any Σ_1-sound r.e. extension T of PA.

1. Propositional provability logic
2. Predicate provability logic
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- **Assume that the language of predicate modal logic has no function and constant symbols.**
- **Arithmetical interpretations of predicate modal logic assign a k-ary formula in the language of T to each k-ary predicate symbol.**
- **Kripke frame for predicate modal logic is a system $\langle W, \prec, \{D_w\}_{w \in W} \rangle$ where $\{D_w\}_{w \in W}$ is a sequence of non-empty sets s.t. $w \prec w' \Rightarrow D_w \subseteq D_{w'}$.**
- **Kripke model for predicate modal logic is a system $\langle W, \prec, \{D_w\}_{w \in W}, \Vdash \rangle$ where $\Vdash$ is a binary relation between elements w of W and closed formulas with parameters form D_w.**

- **QGL is a natural extension of GL to predicate modal logic.**
- **Define Th(QGL), Fr(QGL) and PL(T) similarly to the propositional case.**

Theorem

$\mathcal{F}$: Kripke frame. TFAE:

1. **All axioms of QGL are valid in $\mathcal{F}$;**
2. **$\mathcal{F}$ is transitive and conversely well-founded.**

So Th(QGL) $\subseteq$ Fr(QGL).

Theorem

For a predicate modal formula A, TFAE:

1. **QGL $\vdash A$;**
2. **A is valid in any transitive Kripke model where $\Box(\Box B \to B) \to \Box B$ is valid for any predicate modal formula B.**

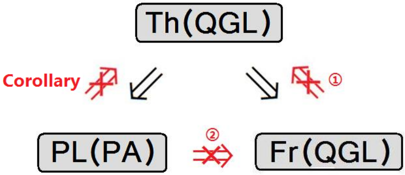
Th(QGL) can be characterized by a class of Kripke models.

Theorem (Montagna, 1984)

1. $\mathbf{Th(QGL)} \subsetneq \mathbf{Fr(QGL)}$.
2. $\mathbf{PL(PA)} \not\subseteq \mathbf{Fr(QGL)}$.

Corollary

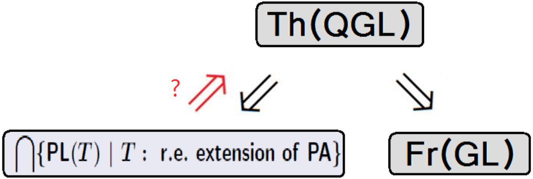
$\mathbf{Th(QGL)} \subsetneq \mathbf{PL(PA)}$.



- **Montagna pointed out that**
  $\exists T, T'$: Σ_1-sound r.e. extensions of PA s.t. $\mathsf{PL}(T) \neq \mathsf{PL}(T')$.

Montagna's conjecture (1984)

$$\bigcap\{\mathsf{PL}(T) \mid T: \text{ r.e. extension of PA}\} = \mathsf{Th}(\mathsf{QGL})?$$

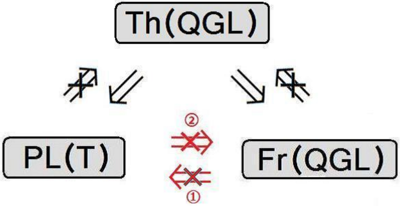


The relationships between Th(QGL), Fr(QGL) and PL(T) have not been understood completely.

Theorem 1 (T.K.)

For any Σ_1-sound r.e. extension T of $\mathbf{I\Sigma_1}$,

$$\mathsf{Fr}(\mathsf{QGL}) \not\subseteq \mathsf{PL}(T).$$



Theorem 2 (T.K.)

$\bigcap\{\mathsf{PL}(T) \mid T : \text{r.e. } \mathcal{L}_A\text{-theory extending } \mathbf{I\Sigma_1}\} \not\subseteq \mathsf{Fr}(\mathsf{QGL})$.

$\mathcal{L}_A = \{+, \times, S, 0, <\}$.

The following corollary shows that Montagna's conjecture does not hold for a restricted case.

Corollary of Theorem 2

$\bigcap\{\mathsf{PL}(T) \mid T : \text{r.e. } \mathcal{L}_A\text{-theory extending } \mathsf{I\Sigma}_1\} \nsubseteq \mathsf{Th}(\mathsf{QGL})$.

If we weaken the theory in the condition of the conjunction to $\mathsf{I\Sigma}_2$, then we obtain a stronger version.

Theorem 3 (T.K.)

$\bigcap\{\mathsf{PL}(T) \mid T : \text{r.e. } \mathcal{L}_A\text{-theory extending } \mathsf{I\Sigma}_2\} \cap \mathsf{Fr}(\mathsf{QGL}) \nsubseteq \mathsf{Th}(\mathsf{QGL})$.

An outline of the proof of Theorem 1.

It suffices to find a predicate modal sentence A s.t.

(i)' $\neg A \in$ Fr(QGL) and
(ii)' $\neg A \notin$ PL(PA).

These conditions are equivalent to the following conditions:

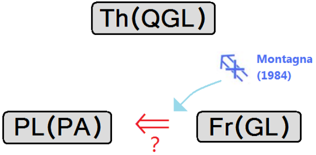
Conditions

(i) $\mathcal{M} = \langle W, \prec, \{D_w\}_{w \in W}, \Vdash \rangle$: **transitive Kripke model.**
$\exists w \in W$ **s.t.** $w \Vdash A$
$\Rightarrow \prec$ **is not conversely well-founded.**

(ii) $\exists \mathcal{M} \models$ **PA** $\exists *$: **PA-interpretation s.t.** $\mathcal{M} \models A^*$.

$p(x)$: a unary predicate symbol.

- Montagna proved that
  $C \equiv \exists x \Diamond p(x) \land \forall x \exists y \Box (p(x) \to \Diamond p(y))$
  satisfies the condition (i).
  So $\neg C \in Fr(QGL)$.



- $C^* \equiv \exists x Con(PA + p^*(x)) \land \forall x \exists y Pr_{PA}(\ulcorner p^*(\dot{x}) \to Con(PA + p^*(\dot{y}))\urcorner)$

(ii) $\exists \mathcal{M} \models PA$ $\exists *$: PA-interpretation s.t. $\mathcal{M} \models C^*$.
However, we do not know the existence of such $\mathcal{M}$ and $*$.

- **We shall modify the predicate modal sentence C so that (ii) holds while (i) is kept.**

Definition (iterated consistency assertions)

$\mathrm{Con}^0 :\equiv (0 = 0)$;
$\mathrm{Con}^{n+1} :\equiv \mathrm{Con}(\mathrm{PA} + \mathrm{Con}^n)$.

Definition (parameterized iterated consistency assertions)

Let $\mathrm{Con}_{\mathrm{PA}}(x)$ be one of the $\mathcal{L}_A$-formula $\varphi(x)$ which satisfies

$$\mathrm{PA} \vdash \forall x(\varphi(x) \leftrightarrow [\mathrm{Con}(\mathrm{PA} + \varphi(\dot{x} \dot{-} 1)) \lor x = 0]).$$

$\forall n \in \omega$, $\mathrm{PA} \vdash \mathrm{Con}_{\mathrm{PA}}(\bar{n}) \leftrightarrow \mathrm{Con}^n$.

$\exists x \Diamond p(x) \land \forall x \exists y \Box (p(x) \rightarrow \Diamond p(y))$
$\Rightarrow \forall x p(x) \land \Box \forall x (p(x+1) \rightarrow \Diamond p(x))$.

The main idea of the proof of Theorem 1

Proposition

1. $\mathsf{PA} \vdash \forall x(\mathsf{Con}_{\mathsf{PA}}(x+1) \to \mathsf{Con}(\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}}(\dot{x})))$.
2. $\mathbb{N} \models \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$.

If $p^*(x) \equiv \mathsf{Con}_{\mathsf{PA}}(x)$, then
$\exists \mathcal{M} \models \mathsf{PA}$
$\mathcal{M} \models (\forall x p(x) \wedge \Box \forall x (p(x+1) \to \Diamond p(x)))^*$.

$\Box \forall x (p(x+1) \to \Diamond p(x))$ asserts the existence of an infinite sequence of worlds starting from any nonstandard element of a nonstandard model of arithmetic.

$X = \{E(x,y), S(x,y), A(x,y,z), M(x,y,z), Z(x), L(x,y)\}$: set of predicate symbols.

Definition

PA-interpretation $*$ is natural
$\overset{\text{def.}}{\Leftrightarrow}$ $E^*(x,y) \equiv$ "$x = y$", $S^*(x,y) \equiv$ "$S(x) = y$" and so on.

- For each $\mathcal{L}_A$-formula φ, let $[\![\varphi]\!]$ be one of the relational formulas written by using the symbols in X, which is equivalent to φ in the sense of the natural interpretations.

- For any $\mathcal{L}_A$-formula φ and any natural interpretation $*$,

$$\mathsf{PA} \vdash \varphi \leftrightarrow [\![\varphi]\!]^*.$$

$$B \equiv \forall x \forall y (S(y, x) \land p(x) \to \Diamond p(y)).$$

Let A be the conjunction of the following six sentences:

1. $\forall x p(x)$
2. B
3. $\Box B$
4. $\forall x \forall y (S(x, y) \to \Box S(x, y))$
5. $[\![\bigwedge \mathsf{Q}]\!]$
6. $[\![\neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))]\!]$

where $\bigwedge$ Q is a conjunction of all axioms of Robinson's arithmetic Q.

Then A satisfies (i) and (ii).

(i) $\mathcal{M} = \langle W, \prec, \{D_w\}_{w \in W}, \Vdash \rangle$: transitive Kripke model.
$\exists w \in W$ s.t. $w \Vdash A$
$\Rightarrow \prec$ is not conversely well-founded.

(ii) $\exists \mathcal{M} \models \mathsf{PA}\ \exists\ *$: PA-interpretation s.t. $\mathcal{M} \models A^*$.

$$B \equiv \forall x \forall y (S(y,x) \wedge p(x) \to \Diamond p(y))$$
$$A \equiv \forall x p(x) \wedge B \wedge \Box B \wedge \forall x \forall y (S(x,y) \to \Box S(x,y)) \wedge$$
$$[\![\bigwedge \mathsf{Q}]\!] \wedge [\![\neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))]\!]$$

(i)

- Assume that $\mathcal{M} = \langle W, \prec, \{D_w\}_{w \in W}, \Vdash \rangle$ is a transitive Kripke model and $w_0 \in W$ satisfies A.
- w_0 is a model of Q
- Since $\mathbb{N} \models \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$, w_0 must be non-standard.

$\Rightarrow \prec$ is not conversely well-founded.

(ii)

- $\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x)$: consistent.
- $\exists \mathcal{M} \models \mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x) + \neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x))$.
- $*$: natural PA-interpretation s.t. $p^*(x) \equiv \mathsf{Con}_{\mathsf{PA}}(x)$.

$\Rightarrow \mathcal{M} \models A^*$.

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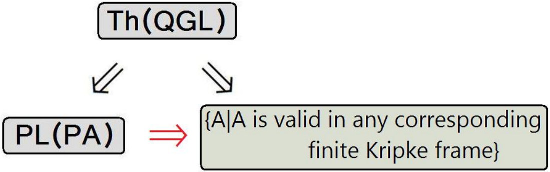
Theorem (Artemov and Dzhaparidze, 1990)

A: predicate modal sentence.

A is PA-valid

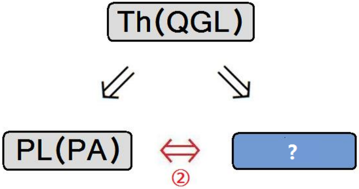
$\Rightarrow$ A is valid in any finite transitive and conversely well founded Kripke frame.

(A frame is finite $\Leftrightarrow$ whose universe and domains are all finite)



Problem 1

- Is there a class of Kripke models which characterizes PL(T)?



Problem 2

- Is Montagna's conjecture true?

References

- K. Segerberg, *An essay in classical modal logic*, Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet, 1971.
- R. Solovay, *Provability interpretations of modal logic*, Israel J. Math. 25 (1976), no. 3-4, 287–304.
- F. Montagna. *The predicate modal logic of provability*. Notre Dame Journal of Formal Logic 25 (1984), 179–189.
- C. Smoryński. *Self-Reference and Modal Logic*. Springer, New York, 1985.
- S. Artemov; G. Dzhaparidze. *Finite Kripke models and predicate logics of provability*, J. Symbolic Logic 55 (1990), no. 3, 1090–1098.
- G. Boolos. *The logic of provability*. Cambridge University Press, Cambridge, 1993.
- G. Dzhaparidze and D. de Jongh. *The Logic of Provability*. Handbook of Proof Theory. North holland, 1998.
- S. Artemov and Lev D. Beklemishev. *Provability logic*. Handbook of Philosophical Logic. Springer, 2005.
- T. Kurahashi. *Semantical Analysis of Predicate Modal Logic of Provability*. Master's Thesis, Kobe University, 2011.