On Kripke frames and arithmetical interpretations for QGL

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- **Provability predicate** of r.e. theory T is a Σ_1 formula $\mathrm{Pr}_T(x)$ which weakly represents the set $\{\ulcorner\varphi\urcorner \mid T \vdash \varphi\}$ in PA, i.e., $\forall\varphi$: sentence, PA $\vdash \mathrm{Pr}_T(\ulcorner\varphi\urcorner) \Leftrightarrow T \vdash \varphi$.

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- **Fix a provability predicate which satisfies the following five conditions:**

The properties of $\mathrm{Pr}_T(x)$

D1 $T \vdash \varphi \Rightarrow$ PA $\vdash \mathrm{Pr}_T(\ulcorner\varphi\urcorner)$

D2 PA $\vdash \mathrm{Pr}_T(\ulcorner\varphi \to \psi\urcorner) \to (\mathrm{Pr}_T(\ulcorner\varphi\urcorner) \to \mathrm{Pr}_T(\ulcorner\psi\urcorner))$

D3 PA $\vdash \mathrm{Pr}_T(\ulcorner\varphi\urcorner) \to \mathrm{Pr}_T(\ulcorner\mathrm{Pr}_T(\ulcorner\varphi\urcorner)\urcorner)$

Löb PA $\vdash \mathrm{Pr}_T(\ulcorner\mathrm{Pr}_T(\ulcorner\varphi\urcorner) \to \varphi\urcorner) \to \mathrm{Pr}_T(\ulcorner\varphi\urcorner)$.

Σ_1-comp. φ: $\Sigma_1 \Rightarrow T \vdash \varphi \to \mathrm{Pr}_T(\ulcorner\varphi\urcorner)$.

Gödel's thesis (1933)

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The system **GL** of propositional modal logic

- **Axioms:**
  - **Tautologies;**
  - $\Box(A \to B) \to (\Box A \to \Box B)$;
  - $\Box(\Box A \to A) \to \Box A$.
- **Inference rules:**

  modus ponens **from** A **and** $A \to B$ **infer** B;

  necessitation **form** A **infer** $\Box A$.

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Th(GL)$:= \{A \mid$ **GL** $\vdash A\}$.

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Kripke frame is a system $\langle W, \prec \rangle$ where

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Kripke model is a system $\mathcal{M} = \langle W, \prec, \Vdash \rangle$ where

- $\langle W, \prec \rangle$ is a Kripke frame;
- $\Vdash$ is a binary relation on $W \times F$ such that $\forall w \in W$,
  - $w \nVdash \bot$;
  - $w \Vdash A \to B \Leftrightarrow (w \nVdash A$ or $w \Vdash B)$;
  - $\cdots$;
  - $w \Vdash \Box A \Leftrightarrow \forall w' \in W (w \prec w' \Rightarrow w' \Vdash A)$.
  - $w \Vdash \Diamond A \Leftrightarrow \exists w' \in W (w \prec w' \ \& \ w' \Vdash A)$.

Definition

A: modal sentence, $\mathcal{F}$: Kripke frame, $\mathcal{M}$: Kripke model.

- **A is valid in $\mathcal{M}$ $\overset{\text{def.}}{\Leftrightarrow}$ $\forall w \in W$, $w \Vdash A$.**
- **A is valid in $\mathcal{F}$ $\overset{\text{def.}}{\Leftrightarrow}$ A is valid in $\langle \mathcal{F}, \Vdash \rangle$ for any $\Vdash$.**

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- Kripke frame $\langle W, \prec \rangle$ is a **GL-frame** if $\prec$ is
  1. transitive,
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- **Fr(GL)**:= $\{A \mid A$ is valid in any GL-frame $\}$.

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Theorem (Segerberg, 1971)

Th(GL) $=$ Fr(GL).

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Definition (arithmetical interpretation)

A mapping $*$ from F to all sentences in the language of T is called a *T-interpretation* if it satisfies the following conditions:

- p^* is a sentence in the language of T for any propositional variable p;
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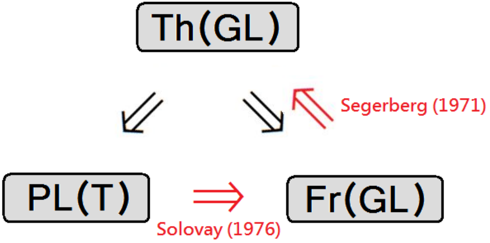
Definition

- A: propositional modal sentence.
  A is T-**valid** $\overset{\text{def.}}{\Leftrightarrow}$ $\forall *$: T-interpretation, $T \vdash A^*$.
- $\mathbf{PL}(T) := \{A \mid A \text{ is } T\text{-valid}\}$:**the provability logic** of T.

T: Σ_1-sound r.e. extension of PA.

Theorem (Solovay, 1976)

$\mathbf{Th}(\mathbf{GL}) = \mathbf{PL}(T)$.



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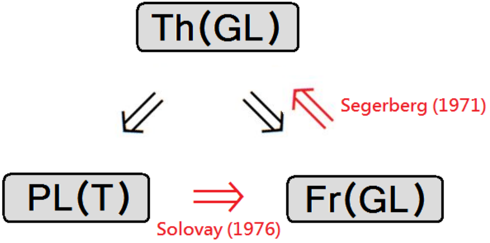

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for any Σ_1-sound r.e. extension T of PA.

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- Kripke frame for predicate modal logic is a triple $\langle W, \prec, \{D_w\}_{w \in W} \rangle$:
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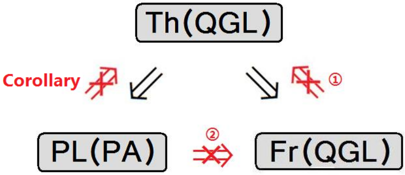
- Th(QGL) $\subseteq$ Fr(QGL) $\cap$ PL(T).
- Th(QGL) is characterized by a class of Kripke **models**.

Theorem (Montagna, 1984)

1. $\mathbf{Fr}(\mathbf{QGL}) \not\subseteq \mathbf{Th}(\mathbf{QGL})$.
2. $\mathbf{PL}(\mathbf{PA}) \not\subseteq \mathbf{Fr}(\mathbf{QGL})$.

Corollary

$\mathbf{PL}(\mathbf{PA}) \not\subseteq \mathbf{Th}(\mathbf{QGL})$.



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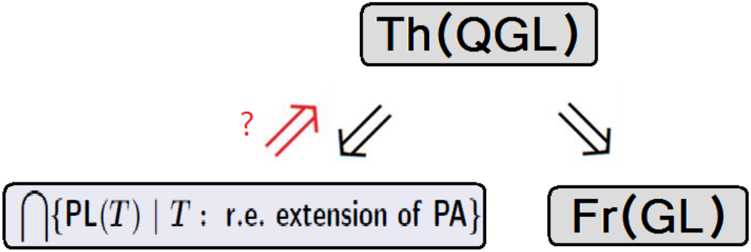
Montagna's conjecture (1984)

$$\bigcap\{\mathbf{PL}(T) \mid T : \text{ r.e. extension of PA}\} = \mathbf{Th}(\mathbf{QGL})?$$



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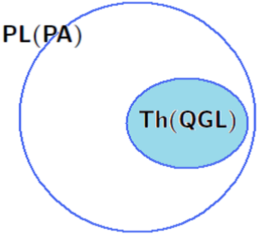

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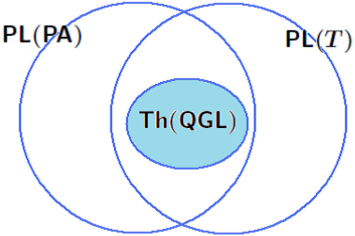

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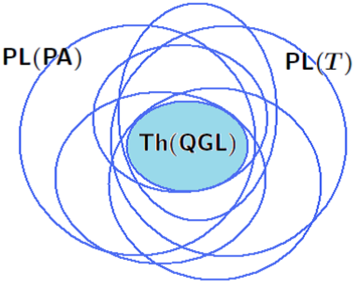

What is an r.e. extension of PA?

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By the same argument,

$\bigcap\{$**PL**$(T) \mid T$: r.e. $\mathcal{L}_A$-theory extending PA$\}$ is not Σ^0_1.

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$\bigcap\{\mathsf{PL}(T) \mid T$: r.e. $\mathcal{L}_A$-theory extending PA$\} \not\subseteq$ Th(QGL).

The relationships between Th(QGL), Fr(QGL) and PL(T) have not been understood completely.

1. Propositional provability logic
2. Predicate provability logic
3. **Main theorems**
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Theorem 1(T.K.)

For any Σ_1-sound r.e. extension T of $\mathsf{I}\Sigma_1$,

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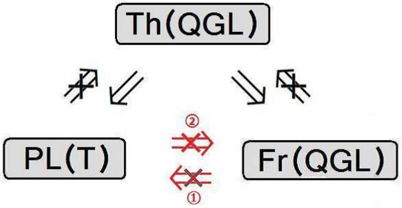

Theorem 2(T.K.)

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We concretely constructed a counter example of the inclusion.

Corollary to Theorem 2

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Theorem 3(T.K.)

$\bigcap\{\mathbf{PL}(T) \mid T : \text{r.e. } \mathcal{L}_A\text{-theory extending } \mathbf{I\Sigma_2}\} \cap \mathbf{Fr}(\mathbf{QGL}) \not\subseteq \mathbf{Th}(\mathbf{QGL})$.

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$$\land \llbracket \bigwedge \mathsf{Q} \rrbracket \land \llbracket \neg \mathsf{Con}(\mathsf{PA} + \forall x \mathsf{Con}_{\mathsf{PA}}(x)) \rrbracket$$

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- $\exists *$: T-interpretation $\exists \mathcal{M}$: model of T s.t. $\mathcal{M} \models A^*$.

Then $\neg A \in \mathsf{Fr}(\mathsf{QGL})$, $\neg A \notin \mathsf{PL}(T)$.

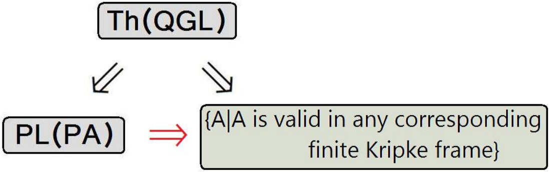
Theorem (Artemov and Dzhaparidze, 1990)

A: predicate modal sentence.

A is PA-valid

$\Rightarrow$ A is valid in any finite transitive and conversely well founded Kripke frame.

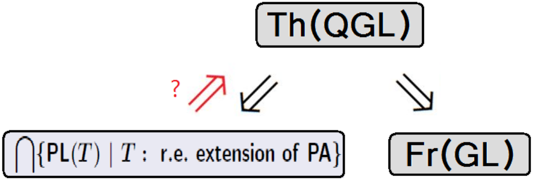
(A frame is finite $\Leftrightarrow$ whose universe and domains are all finite)



Problem

- **Is Montagna's conjecture true?**

$$\bigcap\{\mathbf{PL}(T) \mid T : \text{ r.e. extension of } \mathbf{PA}\} = \mathbf{Th}(\mathbf{QGL})?$$



$$\bigcap\{\mathbf{PL}(T) \mid T : \text{ r.e. theory where } \mathbf{PA} \text{ is relatively interpretable}\}$$
$$= \mathbf{Th}(\mathbf{QGL}) \ ?$$

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