Simple method for generation of 1-D and 2-D random permeability field by using stochastic fractal model

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Abstract A simple method for generation of 1-D and 2-D fractal permeability fields is presented. The method is based on theoretical consideration of spatial variations of permeability. The method is characterized by the repetition of the same procedure similar to the process of generating other fractal figures. Then, the statistical properties (e.g., probability density function, spectral density, autocorrelation function) of random fractal field generated by this method are examined. Although this model is quite simple, it can reproduce quite correctly the features that were obtained from field observations.

Key words fractal; permeability; power spectrum; Autocorrelation Function; Integral Scale

INTRODUCTION

A great deal of research has been made on geostatistical models. Random-field generation is one of important topics. The purpose of this paper is to present the theoretical investigation of the spatial distribution of permeability in a stratum that is considered to be geologically homogeneous and proposes a methodology for numerical generation of more realistic non-uniform ground. Initially 1-D model will be considered, and expanded to 2-D model. Then, the statistical properties (e.g., probability density function, spectral density, autocorrelation function) of random fractal field generated by this method will be examined.

PROCEDURE

1-D model

In the case of 1-D model, the generation procedure is described as follows. Firstly, 1-dimensional sample of which the average permeability is $K^{(0)}$, is divided into two parts as shown in Figure 1.

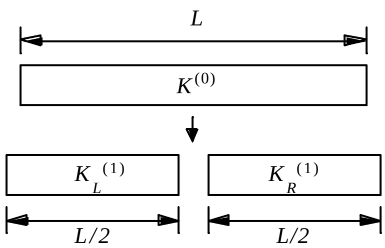

Fig. 1 Division of sample

Permeability of divided samples are defined as $K_L^{(1)}$ and $K_R^{(1)}$. Since the soil is non-uniform, the relationship of $K^{(0)}$, $K_L^{(1)}$ and $K_R^{(1)}$ is expressed as following inequality:

$$K_L^{(1)} < K^{(0)} < K_R^{(1)} \ \ or \ \ K_R^{(1)} < K^{(0)} < K_L^{(1)} \tag{1}$$

$K^{(0)}$ is also defined by using $K_L^{(1)}$ and $K_R^{(1)}$ as,

$$K^{(0)} = \frac{2K_L^{(1)}K_R^{(1)}}{K_L^{(1)} + K_R^{(1)}} \tag{2}$$

Then, the random variable a that means spatial fluctuation of permeability is introduced and defined as,

$$a = K_L^{(1)} / K^{(0)} \tag{3}$$

By substituting a in (2), $K_R^{(1)}$ is given by

$$K_R^{(1)} = \frac{aK^{(0)}}{2a - 1} \tag{4}$$

Therefore, constraint of a is given by

$$0.5 < a \tag{5}$$

Moreover, $K_m^{(1)}$, $K_n^{(1)}$, a_m and a_n are defined as,

$$K_m^{(1)} = \max(K_L^{(1)}, K_R^{(1)}),\ K_n^{(1)} = \min(K_L^{(1)}, K_R^{(1)}),\ a_m = K_m^{(1)}/K^{(0)},\ a_n = K_n^{(1)}/K^{(0)} \tag{6}$$

By using (4), (5) and (6), constraints of a_m and a_n are given by

$$1 < a_m,\ 0.5 < a_n = \frac{a_m}{2a_m - 1} < 1 \tag{7}$$

And, assuming the probability density function of a_m with exponential function as,

$$p(a_m) = q_{am}\exp\{-q_{am}(a_m\text{-}1)\} \ \ (1 < a_m) \tag{8}$$

where $p(a_m)$ is the probability density of a_m, q_{am} is the parameter of characterizing degree of spatial fluctuation of permeability. Thus, the sample is divided into 2^N parts by iterating the procedure mentioned above N times. The correlation in this model can be controlled by increasing or decreasing q_{am} as iteration increases. Such as,

$$q_{am}{}^{(i+1)} = eq_{am}{}^{(i)} \qquad (1 = \mathrm{i} < N,\ \ 0 < e) \tag{9}$$

where e is a constant value.

2-D model

In the case of 2-D model, the generation procedure is similar to 1-D model. Figure 2 shows the definition of divided region.

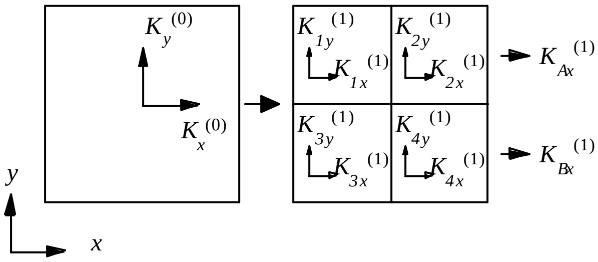

Fig. 2 Division of region

For x-direction, $K_{Ax}^{(1)}$, $K_{Bx}^{(1)}$ and $K_x^{(0)}$ are defined by using $K_{1x}^{(1)}...K_{4x}^{(1)}$ as,

$$K_{Ax}^{(1)} = \frac{2K_{1x}^{(1)}K_{2x}^{(1)}}{K_{1x}^{(1)} + K_{2x}^{(1)}},\ K_{Bx}^{(1)} = \frac{2K_{3x}^{(1)}K_{4x}^{(1)}}{K_{3x}^{(1)} + K_{4x}^{(1)}},\ K_x^{(0)} = \frac{K_{Ax}^{(1)} + K_{Bx}^{(1)}}{2} \tag{10}$$

Then, the random variables a_x, b_x and g_x that mean spatial fluctuation of permeability for x-direction are introduced, and defined as,

$$a_x = K_{1x}^{(1)}/K_x^{(0)},\ b_x = K_{4x}^{(1)}/K_x^{(0)},\ g_x = K_{Ax}^{(1)}/K_x^{(0)} \tag{11}$$

By substituting a_x, b_x and g_x in (10), $K_{2x}^{(1)}$ and $K_{3x}^{(1)}$ are given by

$$K_{2x}^{(1)} = \frac{a_x g_x}{2a_x - g_x} K_x^{(0)},\ K_{3x}^{(1)} = \frac{(2-g_x)b_x}{2b_x + g_x - 2} K_x^{(0)} \tag{12}$$

Therefore, constraints of a_x, b_x and g_x are given by

$$\frac{g_x}{2} < a_x,\ \frac{2-g_x}{2} < b_x,\ 0 < g_x < 2 \tag{13}$$

The random variables a_x', b_x' and g_x' are also introduced, and defined as

$$a_x' = \frac{a_x}{g_x},\ b_x' = \frac{b_x}{2-g_x},\ g_x' = \frac{1}{g_x} \tag{14}$$

By using (13) and (14), constraints of a_x', b_x' and g_x' are given by

$$0.5 < a_x',\ \ 0.5 < b_x',\ \ 0.5 < g_x' \tag{15}$$

Moreover, a_{mx}' b_{mx}' and g_{mx}' are defined same as for the 1-D model:

$$1 < a_{mx}',\ 1 < b_{mx}',\ 1 < g_{mx}' \tag{16}$$

We assume that the probability density functions of a_{mx}', b_{mx}' and g_{mx}' with exponential function are same as 1-D model. Thus, the region is divided into 4^N parts by iterating this procedure N times.

RESULTS AND DISCUSSION

Frequency distribution and variance

In this section the discussion shall mainly focus on 2-D model. Figure 3 shows the distribution of log permeability, $Y(x,y)=\log(K(x,y))$, generated by proposed method. The parameters employed are $K_x^{(0)}=1.0$, $q_{axm}^{(1)}=q_{bxm}^{(1)}=1.25$, $q_{gxm}^{(1)}=2.50$ and $e=1.0$.

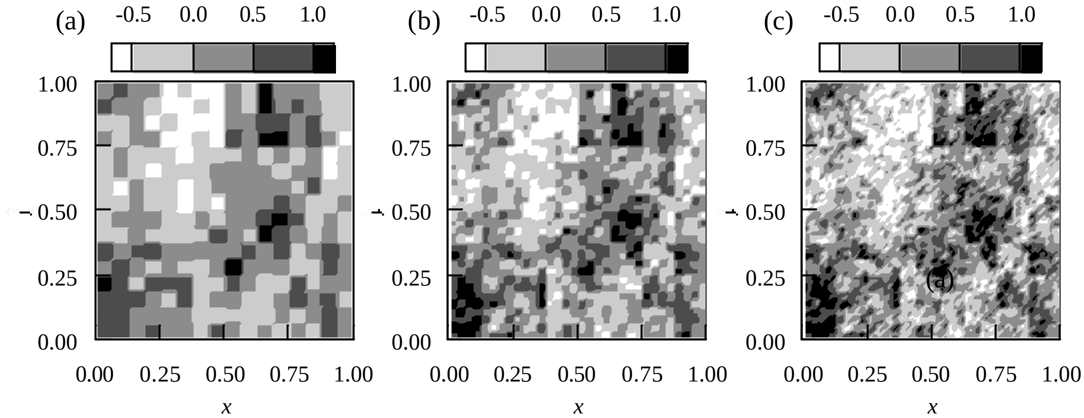

Fig.3 Distribution of log permeability $Y(x,y)$; (a) N=4; (b) N=5; (c) N=6

Figure 4 shows frequency distribution of permeability. It is clear that the permeability is lognormaly distributed although the frequency distribution is not specified in advance. The same result applies to 1-D model, but there is space only for 2-D model. However, these results are applied only to e˜1.0. Therefore, it seems reasonable to suppose that e should be nearly equal to 1.0.

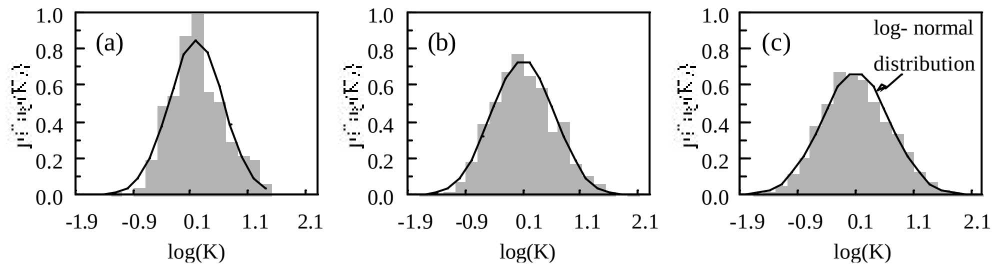

Fig.4 Frequency distribution; (a) N=4; (b) N=5; (c) N=6

For the moment let us look closely at the variance. Figure 5 shows the relation of variance and iteration number N. The variance is in proportion to the number of iteration of generating procedure.

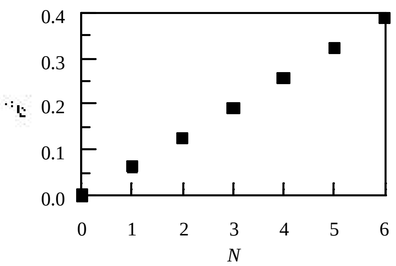

Fig.5 Ensemble mean of Variances *vs* iteration N (e =1.0; 100 realizations)

Spectral density

The ensemble mean of spectral density for e=1.0 and e=1.5 with their regression lines are plotted in Figure 6. From this figure, spectral densities would be assumed as

$$S(|\boldsymbol{f}|) \propto |\boldsymbol{f}|^{-z} \tag{17}$$

where $\boldsymbol{f}$ is a wavenumber vector, and z is a constant associated with e.

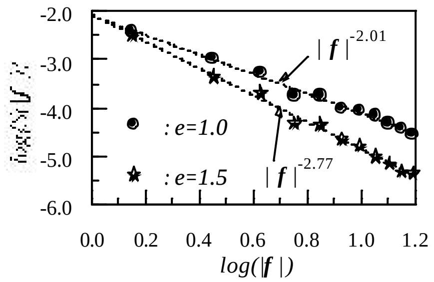

Fig.6 The ensemble mean of spectral density for e=1.0 and e=1.5 (100 realizations)

Autocorrelation function and Integral Scale

The autocorrelation function is written as,

$$r(\boldsymbol{r}) = 1 - \frac{1}{2s_Y^2}\left\langle \{Y(\boldsymbol{p}+\boldsymbol{r}) - Y(\boldsymbol{p})\}^2 \right\rangle \tag{18}$$

where $r(\boldsymbol{r})$ is autocorrelation function, $\boldsymbol{r}$ is displacement vector, $\boldsymbol{p}$ is location vector, and < > indicates the ensemble mean. Figure 7 shows the ensemble mean of autocorrelation function for e=1.0. It is distributed linearly on the piece logarithm paper. From this result, when the autocorrelation function is approximated with exponential function, the integral scale appears to become nearly equal to 0.1 times of support scale. Moreover, this result is supported by field data presented by Gelhar (1993) and Di Federico & Neuman(1997). On the other hand, this figure exhibits the nugget effect. Although the nugget effect can be attributed to measurement error (de Marsily, 1986), this result shows that it can also come from spatial structure.

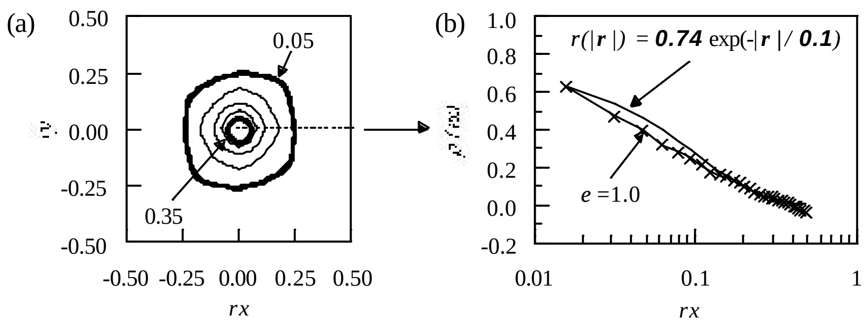

Fig.7 The ensemble mean of autocorrelation function; (a) planimetric; (b) cross section of rx>0, ry=0 ($K_x^{(0)}$=1.0, $q_{axm}^{(1)}$=$q_{bxm}^{(1)}$=1.25, $q_{gxm}^{(1)}$=2.50, e=1.0; 100 realizations)

Figure 8 shows comparison in the different support scale of distributions of log permeability generated by FFT using (17) for z=2.0, and ensemble mean of autocorrelation functions are shown in Figure 9. These results show that the integral scale is completely in proportion to the scale and the difference between FFT and iteration procedure is nugget effect.

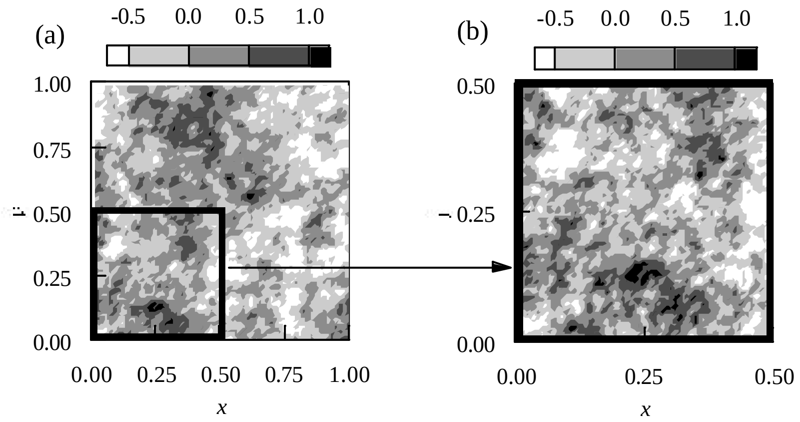

Fig.8 Distribution of log permeability (by FFT, z=2.0); (a)1×1; (b) 0.5×0.5

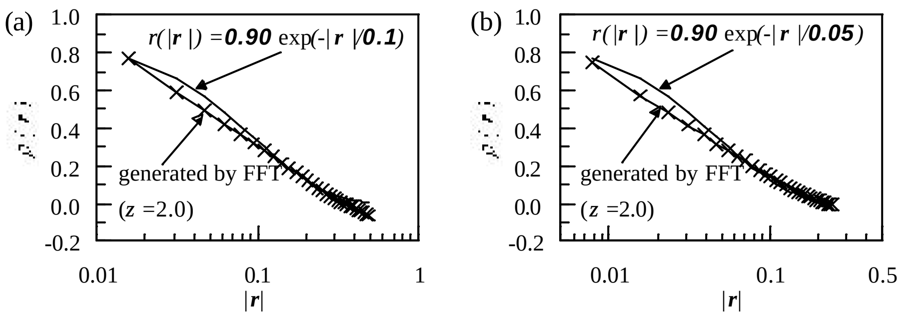

Fig.9 Ensemble mean of autocorrelation Function (100 realizations); (a) 1×1; (b) 0.5×0.5

CONCLUSIONS

1. The hydraulic conductivity of the generated fields are always lognormally distributed although the frequency distribution is not specified in advance.
2. The variance is in proportion to the number of iteration of generating procedure.
3. The spectral density of a spatial variation of the permeability resulted in $\boldsymbol{f}^{-z}$ type.
4. The integral scale is nearly 0.1 times of support scale when the ensemble mean of the autocorrelation function obtained from $\boldsymbol{f}^{-2}$ type models is approximated with exponential model.

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