Conditional threshold autoregression (CoTAR)

Kaiji Motegi\(^1\) John W. Dennis\(^2\) Shigeyuki Hamori\(^1\)

\(^1\)Kobe University
\(^2\)Institute for Defense Analyses

EcoSta Conference 2022
Virtual Session EO269
June 4, 2022
Introduction

- A time series often has heterogeneous properties below versus above a certain threshold (threshold effects).

- One of the most well-known models in this field is the threshold autoregression (TAR) proposed by Tong (1978).

- In TAR, a target series $y$ follows AR($p$) with coefficients being different across regimes, and a regime switch is triggered when a threshold variable $x$ crosses a constant threshold parameter $\mu$.

- Constant-threshold models like TAR have been extended in many ways so that thresholds are time-varying or state-dependent.
We propose the conditional threshold autoregression (CoTAR), where the threshold $\mu_t$ is specified as an empirical quantile of recent observations of the threshold variable $x$.

The proposed conditional threshold $\mu_t$ traces the fluctuation of $x_t$, which can enhance the fit and interpretation of the model.

In CoTAR, the existence of threshold effects can be tested by the wild-bootstrap tests of Hansen (1996).

The estimation and hypothesis testing of CoTAR satisfy desired statistical properties in both large and small samples.

We fit CoTAR to daily new confirmed COVID-19 cases in Japan, finding significant conditional threshold effects.
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Let \( \{y_t\}_1^n \) be a target variable; let \( \{x_t\}_1^n \) be a threshold variable.

Consider Tong’s (1978) threshold autoregression (TAR):

\[
y_t = \begin{cases} 
\alpha_1 + \sum_{k=1}^{p} \phi_1 k y_{t-k} + u_t & \text{if } x_{t-d} < \mu, \\
\alpha_2 + \sum_{k=1}^{p} \phi_2 k y_{t-k} + u_t & \text{if } x_{t-d} \geq \mu.
\end{cases}
\]

\( y \) has different autocorrelation structures below vs. above the unconditional threshold \( \mu \).

“Unconditional” means that \( \mu \) is time-independent and chosen from the entire memory of \( x \):

\[
\mathcal{X}_1^n = \{x_1, \ldots, x_n\}.
\]
CoTAR: Motivation and specification

- We propose to replace $\mu$ with a **conditional** threshold $\mu_t$.
- $\mu_t$ is time-dependent and chosen from a local memory of size $m$:

$$\chi^t_{t-m+1} = \{x_{t-m+1}, \ldots, x_t\}.$$ 

- We propose the conditional threshold autoregression (**CoTAR**):

$$y_t = \begin{cases} 
\alpha_1 + \sum_{k=1}^{p} \phi_1ky_{t-k} + u_t & \text{if } x_{t-d} < \mu_{t-d-1}(c), \\
\alpha_2 + \sum_{k=1}^{p} \phi_2ky_{t-k} + u_t & \text{if } x_{t-d} \geq \mu_{t-d-1}(c). 
\end{cases}$$

- $\mu_t(c)$ is the $mc$-th smallest value (the $100c\%$ point) of $\chi^t_{t-m+1}$.
- $c \in \{1/m, 2/m, \ldots, 1\}$ signifies the relevant percentile.
- When $x_t = y_t$, we have the self-exciting **CoTAR** (**SE-CoTAR**).
Stack the regression parameters for each regime:

$$\beta_r = (\alpha_r, \phi_{r1}, \ldots, \phi_{rp})^\top, \quad r \in \{1, 2\}.$$ 

Define:

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \gamma = \begin{bmatrix} d \\ c \end{bmatrix}.$$ 

Define binary variables which determine the regime:

$$I_{1t}(c) = 1 \left\{ x_t < \mu_{t-1}(c) \right\}, \quad I_{2t}(c) = 1 \left\{ x_t \geq \mu_{t-1}(c) \right\}.$$ 

Stack the regressors:

$$z_{t-1} = (1, y_{t-1}, \ldots, y_{t-p})^\top, \quad Z_{t-1}(\gamma) = \begin{bmatrix} z_{t-1}I_{1,t-d}(c) \\ z_{t-1}I_{2,t-d}(c) \end{bmatrix}.$$ 

CoTAR is rewritten in matrix form as:

$$y_t = Z_{t-1}(\gamma)^\top \beta + u_t.$$
Profiling estimation

- To estimate the regression parameter $\beta$ and the nuisance parameter $\gamma$, we adopt a two-step procedure called profiling.
- If $\gamma$ were given, then the least squares estimator for $\beta$ would be analytically available:

$$
\hat{\beta}(\gamma) = \left\{ \sum_{t=1}^{n} \mathbf{Z}_{t-1}(\gamma) \mathbf{Z}_{t-1}(\gamma)^\top \right\}^{-1} \left\{ \sum_{t=1}^{n} \mathbf{Z}_{t-1}(\gamma) y_t \right\}.
$$

- The profiling estimator for $\gamma$ is given by:

$$
\hat{\gamma} = \arg \min_{\gamma \in \Gamma} \sum_{t=1}^{n} \left\{ y_t - \mathbf{Z}_{t-1}(\gamma)^\top \hat{\beta}(\gamma) \right\}^2.
$$

- The profiling estimator for $\beta$ is given by $\hat{\beta} = \beta(\hat{\gamma})$. 
Profiling estimation

- Asymptotic properties of the profiling estimator depends crucially on whether conditional threshold effects are present or absent.

- Conditional threshold effects are **present** if $\beta_1 \neq \beta_2$, in which case $\gamma$ is **identifiable**.

- Conditional threshold effects are **absent** if $\beta_1 = \beta_2$, in which case CoTAR reduces to the single-regime AR($p$) and $\gamma$ is **unidentifiable**.

- Define the no-threshold-effect hypothesis:

\[ H_0^*: \beta_1 = \beta_2 \quad \text{vs.} \quad H_1^*: \beta_1 \neq \beta_2. \]
Theorem 1 (Profiling estimator)

Under standard regularity conditions, the following are true:

1. \( \sqrt{n}\{\hat{\beta}(\gamma) - \beta_0\} \Rightarrow \mathcal{N}\{0, V(\gamma)\} \) for each fixed \( \gamma \in \Gamma \).

2. \( \hat{\beta}(\gamma) \xrightarrow{p} \beta_0 \) uniformly over \( \gamma \in \Gamma \).

3. Under \( H_1^* \), \( \hat{\gamma} - \gamma_0 = O_p(n^{-1}) \) and \( \sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} \mathcal{N}\{0, V(\gamma_0)\} \).

- See the full paper for the regularity conditions, the construction of \( V(\gamma) \), and the proof of Theorem 1.
- Under \( H_0^* \), the asymptotic distribution of \( \hat{\beta} \) is non-standard.
Testing the no-threshold-effect hypothesis

- Testing $H_0^*$ requires the **wild bootstrap** of Hansen (1996), as $\gamma$ is unidentified and $\hat{\beta}$ is not asymptotically normal under $H_0^*$.

- Formulate the no-threshold-effect hypothesis $H_0^*$ as a linear parametric restriction:

\[
H_0^* : R^* \beta = 0 \quad \text{vs.} \quad H_1^* : R^* \beta \neq 0.
\]

where $R^* = (I_{p+1}, -I_{p+1})$.

- The Wald test statistic conditional on $\gamma$ is given by:

\[
W_n^*(\gamma) = n\hat{\beta}(\gamma)^\top (R^*)^\top \left\{ R^* \hat{V}_n(\gamma) (R^*)^\top \right\}^{-1} R^* \hat{\beta}(\gamma).
\]

- See the full paper for the construction of $\hat{V}_n(\gamma)$. 
Testing the no-threshold-effect hypothesis

- Incorporate all possible values of $\gamma$ as in:

$$\sup \mathcal{W}_n^* = \sup_{\gamma \in \Gamma} \mathcal{W}_n^*(\gamma).$$

- Let $g(\mathcal{W}_n^*)$ be either $\sup \mathcal{W}_n^*$, $\text{ave} \mathcal{W}_n^*$, or $\exp \mathcal{W}_n^*$.

- Let $\{g\{\mathcal{W}_n^{*(b)}\}\}_{b=1}^B$ be the set of wild-bootstrap test statistics. (See the full paper for the bootstrap procedure.)

- The bootstrap p-value is defined as:

$$\hat{p}_n^B(H_0^*) = \frac{1}{B} \sum_{b=1}^B 1 \left[ g \left\{ \mathcal{W}_n^{*(b)} \right\} \geq g(\mathcal{W}_n^*) \right].$$

- Reject $H_0^*$ if $\hat{p}_n^B(H_0^*) < a$, where $a \in (0, 1)$ is the nominal size.
Testing the no-threshold-effect hypothesis

Theorem 2 (Bootstrap test for $H_0^*$)

Under standard regularity conditions, the following are true:

1. Under $H_0^*$, $\hat{p}_n^B(H_0^*)$ is asymptotically uniform on $[0, 1]$.
2. Under $H_1^*$, $\hat{p}_n^B(H_0^*) \xrightarrow{p} 0$ as $n \to \infty$ and $B \to \infty$.

See the full paper for the regularity conditions and the proof.

The bootstrap test for $H_0^*$ is asymptotically valid; the test has size approaching the nominal size $\alpha$ under $H_0^*$, and power approaching 1 under $H_1^*$. 
Empirical application: Set-up

- We analyze the number of daily new confirmed cases per million people in Japan, denoted as \( \{w_t\}_{t=1}^n \).
- Sample period: April 4, 2020 – June 23, 2021 (\( n = 446 \) days).
- We fit the SE-CoTAR model with \( p = 3 \) and \( m = 14 \) to
  \[
  y_t = \Delta \ln w_t = \ln w_t - \ln w_{t-1} \text{ (i.e., the log-difference of the number of daily new confirmed cases per million people):}
  \]
  \[
  y_t = \begin{cases} 
  \alpha_1 + \sum_{k=1}^{3} \phi_{1k} y_{t-k} + u_t & \text{if } y_{t-d} < \mu_{t-d-1}(c), \\
  \alpha_2 + \sum_{k=1}^{3} \phi_{2k} y_{t-k} + u_t & \text{if } y_{t-d} \geq \mu_{t-d-1}(c).
  \end{cases}
  \]
- Regime 1 represents a \textbf{deceleration} phase where the change in new confirmed cases is small relative to the local memory.
- Regime 2 represents an \textbf{acceleration} phase where the change is relatively large.
Several waves of the pandemic are observed in the log series.
The log-difference series is the target of the SE-CoTAR model.
The log-difference series exhibits rather complex fluctuations with persistent swings and temporary noise being combined, which suggests the presence of nonlinear effects.
The estimated conditional threshold $\mu_t(\hat{c})$ well traces the persistent swing of $y_t$, highlighting the virtue of CoTAR.

The p-value of the bootstrap test for $H_0^*$ is 0.002, indicating the presence of conditional threshold effects.

Hence, we conclude that the deceleration and acceleration phases are significantly different from each other.
We have proposed the conditional threshold autoregression (CoTAR), where the threshold is specified as an empirical quantile of the local memory of a threshold variable $x$.

The resulting conditional threshold traces the fluctuation of $x$, which can enhance the fit and interpretation of the model.

The parameters of CoTAR can be estimated via profiling.

The bootstrap test for the no-threshold-effect hypothesis $H_0^*$ is asymptotically valid.

We fitted SE-CoTAR to the daily new confirmed COVID-19 cases of Japan, finding significant conditional threshold effects.