Conditional threshold autoregression (CoTAR)

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AMES 2022
Keio University (Hybrid)
August 8, 2022
A time series often has heterogeneous properties below versus above a certain threshold (threshold effects).

One of the most well-known models in this field is the threshold autoregression (TAR) proposed by Tong (1978).

In TAR, a target series \( y \) follows AR\((p)\) with coefficients being different across regimes, and a regime switch is triggered when a threshold variable \( x \) crosses a constant threshold parameter \( \mu \).

Constant-threshold models like TAR have been extended in many ways so that thresholds are time-varying or state-dependent.
We propose the conditional threshold autoregression (CoTAR), where the threshold $\mu_t$ is specified as an empirical quantile of recent observations of the threshold variable $x_t$.

The proposed conditional threshold $\mu_t$ traces the fluctuation of $x_t$, which can enhance the fit and interpretation of the model.

In CoTAR, the existence of threshold effects can be tested by the wild-bootstrap tests of Hansen (1996).

The estimation and hypothesis testing of CoTAR satisfy desired statistical properties in both large and small samples.

We fit CoTAR to daily new confirmed COVID-19 cases in Japan, finding significant conditional threshold effects.
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CoTAR: Motivation and specification

- Let \( \{y_t\}_1^n \) be a target variable; let \( \{x_t\}_1^n \) be a threshold variable.

- Consider Tong’s (1978) threshold autoregression (TAR):

\[
y_t = \begin{cases} 
\alpha_1 + \sum_{k=1}^{p} \phi_1 y_{t-k} + u_t & \text{if } x_{t-d} < \mu, \\
\alpha_2 + \sum_{k=1}^{p} \phi_2 y_{t-k} + u_t & \text{if } x_{t-d} \geq \mu.
\end{cases}
\]

- \( y \) has different autocorrelation structures below vs. above the **unconditional** threshold \( \mu \).

- “Unconditional” means that \( \mu \) is time-independent and chosen from the entire memory of \( x \):

\[
X_1^n = \{x_1, \ldots, x_n\}.
\]
We propose to replace $\mu$ with a conditional threshold $\mu_t$.

$\mu_t$ is time-dependent and chosen from a local memory of size $m$:

$$x_{t-m+1}^t = \{x_{t-m+1}, \ldots, x_t\}.$$ 

We propose the conditional threshold autoregression (CoTAR):

$$y_t = \begin{cases} 
\alpha_1 + \sum_{k=1}^{p} \phi_{1k} y_{t-k} + u_t & \text{if } x_{t-d} < \mu_{t-d-1}(c), \\
\alpha_2 + \sum_{k=1}^{p} \phi_{2k} y_{t-k} + u_t & \text{if } x_{t-d} \geq \mu_{t-d-1}(c). 
\end{cases}$$

$\mu_t(c)$ is the $mc$-th smallest value (the $100c\%$ point) of $x_{t-m+1}^t$.

$c \in \{1/m, 2/m, \ldots, 1\}$ signifies the relevant percentile.

When $x_t = y_t$, we have the self-exciting CoTAR (SE-CoTAR).
CoTAR: Matrix representation

- Define $\beta_r = (\alpha_r, \phi_{r1}, \ldots, \phi_{rp})^\top$ for $r \in \{1, 2\}$.
- Define $\beta = (\beta_1^\top, \beta_2^\top)^\top$ and $\gamma = (d, c)^\top$.
- Define binary variables which determine the regime:

$$I_{1t}(c) = 1 \{x_t < \mu_{t-1}(c)\}, \quad I_{2t}(c) = 1 \{x_t \geq \mu_{t-1}(c)\}.$$ 

- Stack the regressors:

$$z_{t-1} = (1, y_{t-1}, \ldots, y_{t-p})^\top, \quad Z_{t-1}(\gamma) = \begin{bmatrix} z_{t-1}I_{1,t-d}(c) \\ z_{t-1}I_{2,t-d}(c) \end{bmatrix}.$$

- CoTAR is rewritten in matrix form as:

$$y_t = Z_{t-1}(\gamma)^\top \beta + u_t.$$
Profiling estimation

- To estimate the regression parameter $\beta$ and the nuisance parameter $\gamma$, we adopt a two-step procedure called profiling.

- If $\gamma$ were given, then the least squares estimator for $\beta$ would be analytically available:

$$\hat{\beta}(\gamma) = \left\{ \sum_{t=1}^{n} Z_{t-1}(\gamma) Z_{t-1}(\gamma)^\top \right\}^{-1} \left\{ \sum_{t=1}^{n} Z_{t-1}(\gamma) y_t \right\}.$$

- The profiling estimator for $\gamma$ is given by:

$$\hat{\gamma} = \arg\min_{\gamma \in \Gamma} \sum_{t=1}^{n} \left\{ y_t - Z_{t-1}(\gamma)^\top \hat{\beta}(\gamma) \right\}^2.$$

- The profiling estimator for $\beta$ is given by $\hat{\beta} = \beta(\hat{\gamma})$. 
Asymptotic properties of the profiling estimator depends crucially on whether conditional threshold effects are present or absent.

Conditional threshold effects are **present** if $\beta_1 \neq \beta_2$, in which case $\gamma$ is **identifiable**.

Conditional threshold effects are **absent** if $\beta_1 = \beta_2$, in which case CoTAR reduces to the single-regime AR($p$) and $\gamma$ is **unidentifiable**.

Define the no-threshold-effect hypothesis:

$$H_0^* : \beta_1 = \beta_2 \quad \text{vs.} \quad H_1^* : \beta_1 \neq \beta_2.$$
Profiling estimation

Theorem 1 (Profiling estimator)

Under standard regularity conditions, the following are true:

1. \( \sqrt{n}\{\hat{\beta}(\gamma) - \beta_0\} \Rightarrow N\{0, V(\gamma)\} \) for each fixed \( \gamma \in \Gamma \).
2. \( \hat{\beta}(\gamma) \overset{p}{\to} \beta_0 \) uniformly over \( \gamma \in \Gamma \).
3. Under \( H^*_1 \), \( \hat{\gamma} - \gamma_0 = O_p(n^{-1}) \) and
   \( \sqrt{n}(\hat{\beta} - \beta_0) \overset{d}{\to} N\{0, V(\gamma_0)\} \).

- See the full paper for the regularity conditions, the construction of \( V(\gamma) \), and the proof of Theorem 1.
- Under \( H^*_0 \), the asymptotic distribution of \( \hat{\beta} \) is non-standard.
Testing the no-threshold-effect hypothesis

- Testing $H_0^*$ requires the **wild bootstrap** of Hansen (1996), as $\gamma$ is unidentified and $\hat{\beta}$ is not asymptotically normal under $H_0^*$.

- Formulate the no-threshold-effect hypothesis $H_0^*$ as a linear parametric restriction:

  $H_0^*: \mathbf{R}^* \beta = 0 \quad \text{vs.} \quad H_1^*: \mathbf{R}^* \beta \neq 0.$

  where $\mathbf{R}^* = (I_{p+1}, -I_{p+1})$.

- The Wald test statistic conditional on $\gamma$ is given by:

  $\mathcal{W}_n^*(\gamma) = n\hat{\beta}(\gamma)^\top (\mathbf{R}^*)^\top \left\{ \mathbf{R}^* \hat{V}_n(\gamma)(\mathbf{R}^*) \right\}^{-1} \mathbf{R}^* \hat{\beta}(\gamma)$.

- See the full paper for the construction of $\hat{V}_n(\gamma)$. 
Testing the no-threshold-effect hypothesis

- Incorporate all possible values of $\gamma$ as in:

$$\sup W_n^* = \sup_{\gamma \in \Gamma} W_n^*(\gamma).$$

- Let $g(W_n^*)$ be either $\sup W_n^*$, $\text{ave} W_n^*$, or $\exp W_n^*$.

- Let $\{g\{W_n^{*(b)}\}\}_{b=1}^B$ be the set of wild-bootstrap test statistics. (See the full paper for the bootstrap procedure.)

- The bootstrap p-value is defined as:

$$\hat{p}_n^B(H_0^*) = \frac{1}{B} \sum_{b=1}^B 1 \left[ g \left\{ W_n^{*(b)} \right\} \geq g(W_n^*) \right].$$

- Reject $H_0^*$ if $\hat{p}_n^B(H_0^*) < a$, where $a \in (0, 1)$ is the nominal size.
Testing the no-threshold-effect hypothesis

Theorem 2 (Bootstrap test for $H_0^\ast$)

Under standard regularity conditions, the following are true:

1. Under $H_0^\ast$, $\hat{p}_n^B(H_0^\ast)$ is asymptotically uniform on $[0, 1]$.

2. Under $H_1^\ast$, $\hat{p}_n^B(H_0^\ast) \xrightarrow{p} 0$ as $n \to \infty$ and $B \to \infty$.

- See the full paper for the regularity conditions and the proof.

- The bootstrap test for $H_0^\ast$ is asymptotically valid; the test has size approaching the nominal size $a$ under $H_0^\ast$, and power approaching 1 under $H_1^\ast$. 
Empirical application: Set-up

- We analyze the number of daily new confirmed cases per million people in Japan, denoted as \( \{w_t\}_{t=1}^n \).

- Sample period: April 4, 2020 – June 23, 2021 (\( n = 446 \) days).

- We fit the SE-CoTAR model with \( p = 3 \) and \( m = 14 \) to
  \[
y_t = \Delta \ln w_t = \ln w_t - \ln w_{t-1}:
  \]
  \[
y_t = \begin{cases} 
  \alpha_1 + \sum_{k=1}^3 \phi_1 k y_{t-k} + u_t & \text{if } y_{t-d} < \mu_{t-d-1}(c), \\
  \alpha_2 + \sum_{k=1}^3 \phi_2 k y_{t-k} + u_t & \text{if } y_{t-d} \geq \mu_{t-d-1}(c).
  \end{cases}
  \]

- Regime 1 is a **deceleration** phase where the change in new confirmed cases is small relative to the local memory.

- Regime 2 is an **acceleration** phase where the change is relatively large.
Several waves of the pandemic are observed in the log series.

The log-difference series is the target of the SE-CoTAR model.

The log-difference series exhibits rather complex fluctuations with persistent swings and temporary noise being combined, which suggests the presence of nonlinear effects.
The estimated conditional threshold $\mu_t(\hat{c})$ well traces the persistent swing of $y_t$, highlighting the virtue of CoTAR.

The p-value of the bootstrap test for $H_0^*$ is 0.002, indicating the presence of conditional threshold effects.

Hence, we conclude that the deceleration and acceleration phases are significantly different from each other.
Conclusion

- We have proposed the conditional threshold autoregression (CoTAR), where the threshold is specified as an empirical quantile of the local memory of a threshold variable $x$.

- The resulting conditional threshold traces the fluctuation of $x$, which can enhance the fit and interpretation of the model.

- The parameters of CoTAR can be estimated via profiling.

- The bootstrap test for the no-threshold-effect hypothesis $H_0^*$ is asymptotically valid.

- We fitted SE-CoTAR to the daily new confirmed COVID-19 cases of Japan, finding significant conditional threshold effects.