Supplemental material for "*Asymptotic properties of spurious regression and random walks with generalized drifts*"

John W. Dennis[*] – Institute for Defense Analyses

Kaiji Motegi[†] – Kobe University

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1 Linear time trend in the regression model

In the main paper, the regression model is specified as

$$y_t = \alpha + \beta x_t + u_t. \tag{1}$$

Based on model (1), the asymptotic properties of the estimated slope parameter $\hat{\beta}$ and the squared t-statistic $\hat{t}^2_\beta$ are derived in the main paper. It is of interest to investigate how the asymptotic properties change when the linear time trend is added to the model:

$$y_t = \alpha + \beta x_t + \gamma \times t + u_t. \tag{2}$$

A motivation of the inclusion of the time trend t is as follows. Suppose that a data generating process (DGP) for y is a random walk with generalized drift:

$$y_t = d_{yn} + y_{t-1} + \epsilon_{yt}. \tag{3}$$

[*]Institute for Defense Analyses (IDA). Research results and conclusions expressed are those of the authors and do not necessarily reflect the views of IDA. No funding was provided by IDA or an affiliated organization for this paper. E-mail: jay.dennis@alumni.unc.edu

[†]*Corresponding author*. Graduate School of Economics, Kobe University. Address: 2-1 Rokkodai-cho, Nada, Kobe, Hyogo 657-8501 Japan. E-mail: motegi@econ.kobe-u.ac.jp

Equation (3) can be rewritten as

$$y_t = d_{yn} \times t + y_0 + \sum_{\tau=1}^{t} \epsilon_{y\tau}, \tag{4}$$

where y_0 is a non-stochastic initial value. One might think that the time trend t in model (2) would resolve or mitigate the spurious regression, since it would capture the deterministic trend $d_{yn} \times t$ though not the stochastic trend $\sum_{\tau=1}^{t} \epsilon_{y\tau}$. It is therefore of interest to study the consequence of using the extended model (2).

It is beyond the scope of the present paper to formally derive the asymptotic behavior of $\hat{\beta}$ and $\hat{t}_{\beta}^2$ under model (2). Clearly, required algebra would be more tedious than the main scenario since model (2) is more general than model (1). To draw some conjecture on the unknown asymptotic properties, Monte Carlo simulations are performed in this section. We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP:

$$y_t = d_{yn} + y_{t-1} + \epsilon_{yt}, \qquad x_t = d_{xn} + x_{t-1} + \epsilon_{xt}.$$

To make the simulation compact, the drift terms are specified as follows:

Y1 (zero drift in y): $d_{yn} = 0$. **X1 (zero drift in x):** $d_{xn} = 0$.

Y2 (local drift in y): $d_{yn} = n^{-1/2}$. **X2 (local drift in x):** $d_{xn} = n^{-1/2}$.

Y3 (constant drift in y): $d_{yn} = 1$. **X3 (constant drift in x):** $d_{xn} = 1$.

These are special cases of the weak, semi-strong, and strong drifts elaborated in the main paper. The zero drift is the limit case of the weak drift; the local drift here is identical to the semi-strong drift with specific loadings $d_y = d_x = 1$; the constant drift is the boundary case of the strong drift. We write Case Y1X1, for example, to express the combination of Cases Y1 and X1. The error terms are drawn from the bivariate standard normal distributions:

$$\begin{bmatrix} \epsilon_{yt} \\ \epsilon_{xt} \end{bmatrix} \overset{i.i.d.}{\sim} \mathcal{N}(\mathbf{0}, \boldsymbol{I}_2).$$

For each Monte Carlo sample, we fit the extended model (2) and compute the least squares estimator $\hat{\beta}$ and the squared t-statistic $\hat{t}_{\beta}^2$. Then, the empirical distributions of scaled $\hat{\beta}$ and $\hat{t}_{\beta}^2$ are plotted. The scaling factors (i.e., conjecture on the rate of convergence or divergence) are subjectively chosen so that the empirical distributions appear to be stable across the sample sizes $n \in \{100, 500, 1000\}$.

The resulting empirical distributions are shown in Figures 1-9. Changing the regression model from (1) to (2) alters the asymptotic properties of $\hat{\beta}$ and $\hat{t}_{\beta}^2$ substantially. For all figures, the empirical distributions of $\hat{\beta}$ itself seem stable across the sample sizes, which suggests that $\hat{\beta} = O_p(1)$ for all cases. The spurious regression is still present under the extended model (2) in the sense that $\hat{\beta}$ does not converge in probability to 0. A plausible reason for the spurious regression, judging from (4), is that the stochastic trend $\sum_{\tau=1}^{t} \epsilon_{y\tau}$ in y is left uncaptured under model (2). For all cases, the empirical distribution of $\hat{\beta}$ is unimodal and symmetric around 0.

Scaling factors which stabilize $\hat{t}_{\beta}^2$ vary across cases (Figures 1-9). Our conjecture is that $\hat{t}_{\beta}^2 = O_p(n^2)$ if x has a nonzero constant drift (Case X3) and $\hat{t}_{\beta}^2 = O_p(n)$ otherwise (Cases X1 and X2). In all cases, the squared t-statistic diverges and hence the probability of making Type I Error (i.e., rejecting the correct null hypothesis $H_0 : \beta = 0$) approaches 1 asymptotically. This suggests that using model (2) is not a solution to the spurious regression; indeed, including a time trend appears to even exacerbate the symptoms of spurious regression in some cases (i.e., Cases Y1X3 and Y2X3) by accelerating the rate of divergence of $\hat{t}_{\beta}^2$ from n to n^2. Uncovering the reason for this phenomenon is left for future work. For all cases, the empirical distribution of scaled $\hat{t}_{\beta}^2$ is a positively skewed distribution whose peak is located at 0. Finally, the conjecture on the rate of convergence or divergence of $\hat{\beta}$ and $\hat{t}_{\beta}^2$ is summarized in Table 1.

2 Quadratic time trend in the regression model

In the previous section, we studied the spurious regression when there is a linear time trend in the model. In this section, we consider an even more general model which contains linear and quadratic time trends:

$$y_t = \alpha + \beta x_t + \gamma_1 \times t + \gamma_2 \times t^2 + u_t. \tag{5}$$

Clearly, model (5) reduces to (2) when $\gamma_2 = 0$.

We redo the simulation study of the previous section with model (2) being replaced with (5). The resulting empirical distributions for $\hat{\beta}$ and scaled $\hat{t}_{\beta}^2$ are reported in Figures 10-11, respectively. To save space, the sample size is fixed at $n = 1000$ in these figures; we have visually confirmed that the shape of the empirical distribution is stable across different sample sizes.

Our simulation results are summarized as follows. First, the conjecture of the scaling factor of $\hat{\beta}$ and $\hat{t}_{\beta}^2$ does not change from the previous section; in other words, Table 1 entirely applies to the cases associated with model (5). An intuition of this result, again drawn from (4), is that adding the quadratic time trend to model (2) does not resolve the issue of the uncaptured stochastic trend $\sum_{\tau=1}^{t} \epsilon_{y\tau}$.

Second, the empirical distributions of $\hat{\beta}$, which is presumably of $O_p(1)$ for all cases, have slightly higher peaks and thinner tails than those associated with the linear time trend scenario. The height of the peak is roughly 1 for the linear trend scenario, while it is roughly 1.2 for the quadratic trend scenario; compare Figures 1-9 and Figure 10. This result suggests that adding the quadratic time trend has an impact on the asymptotic distribution of $\hat{\beta}$, though not on the stochastic order. In view of these findings, yet another implication could be obtained by formulating a regression model with a K-dimensional polynomial of time trends:

$$y_t = \alpha + \beta x_t + \sum_{k=1}^{K} \gamma_k \times t^k + u_t. \tag{6}$$

Given the simulation results of the linear and quadratic trend scenarios, our conjecture on (6) is that $\hat{\beta} = O_p(1)$ given a fixed value of K; as $K \to \infty$, the asymptotic distribution of $\hat{\beta}$ may become degenerate for each n. We leave it a future task to investigate these scenarios.[1]

Third, the empirical distributions of the scaled $\hat{t}_{\beta}^2$ are almost identical to those associated with the linear time trend scenario (Figure 11). This result suggests that adding the quadratic time trend does not have an impact on either the stochastic order or the asymptotic distribution of $\hat{t}_{\beta}^2$.

References

Phillips, P. C. B. (1998): "New Tools for Understanding Spurious Regressions," *Econometrica*, 66, 1299–1325.

Phillips, P. C. B., X. Wang, and Y. Zhang (2019): "HAR Testing for Spurious Regression in Trend," *Econometrics*, 7, #50.

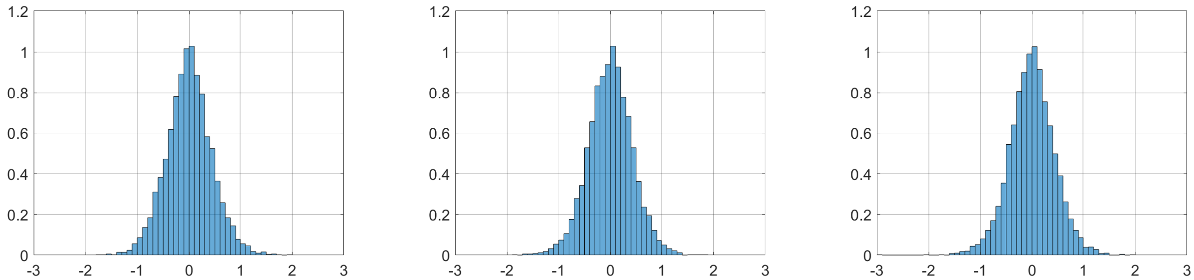
[1] Some useful results could be borrowed from Phillips (1998) and Phillips, Wang, and Zhang (2019), which analyze the consequence of regressing a random walk process onto a polynomial of time trends.

Table 1: Conjecture on correct scaling factors when time trends are included

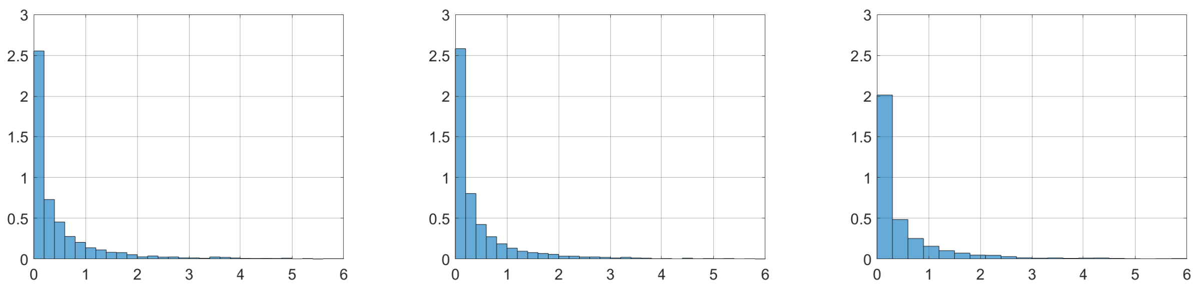
	Case X1: $d_{xn} = 0$	**Case X2**: $d_{xn} = n^{-1/2}$	**Case X3**: $d_{xn} = 1$
Case Y1: $d_{yn} = 0$	$\hat{\beta} = O_p(1)$ $\hat{t}^2_{\beta} = O_p(n)$	$\hat{\beta} = O_p(1)$ $\hat{t}^2_{\beta} = O_p(n)$	$\hat{\beta} = O_p(1)$ $\hat{t}^2_{\beta} = O_p(n^2)$
Case Y2: $d_{yn} = n^{-1/2}$	$\hat{\beta} = O_p(1)$ $\hat{t}^2_{\beta} = O_p(n)$	$\hat{\beta} = O_p(1)$ $\hat{t}^2_{\beta} = O_p(n)$	$\hat{\beta} = O_p(1)$ $\hat{t}^2_{\beta} = O_p(n^2)$
Case Y3: $d_{yn} = 1$	$\hat{\beta} = O_p(1)$ $\hat{t}^2_{\beta} = O_p(n)$	$\hat{\beta} = O_p(1)$ $\hat{t}^2_{\beta} = O_p(n)$	$\hat{\beta} = O_p(1)$ $\hat{t}^2_{\beta} = O_p(n^2)$

The DGP is $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is either $y_t = \alpha + \beta x_t + \gamma t + u_t$ (i.e., linear time trend) or $y_t = \alpha + \beta x_t + \gamma_1 t + \gamma_2 t + u_t$ (i.e., quadratic time trend). $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_{\beta}$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This table summarizes a simulation-based conjecture on the order of stochastic convergence or divergence of $\hat{\beta}$ and $\hat{t}^2_{\beta}$. The conjecture is the same whether the linear or quadratic time trend is included in the model.

Figure 1: Empirical distributions of $\hat{\beta}$ and $\hat{t}_{\beta}^2$ when time trend t is included (Case Y1X1)



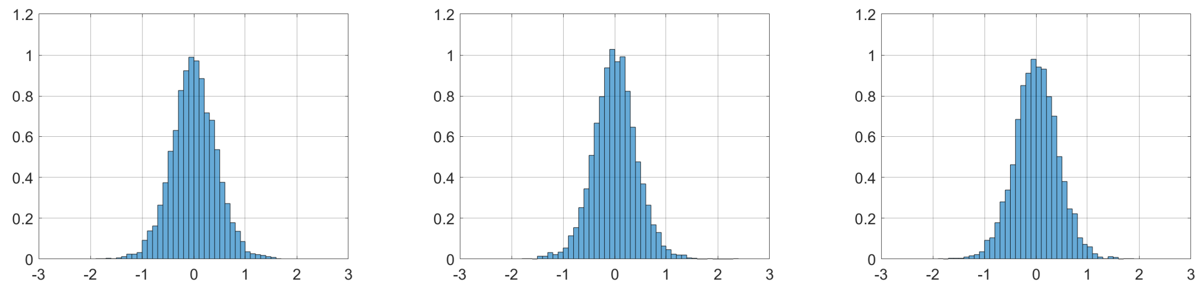
(a) $\hat{\beta}$ $(n = 100)$ (b) $\hat{\beta}$ $(n = 500)$ (c) $\hat{\beta}$ $(n = 1000)$



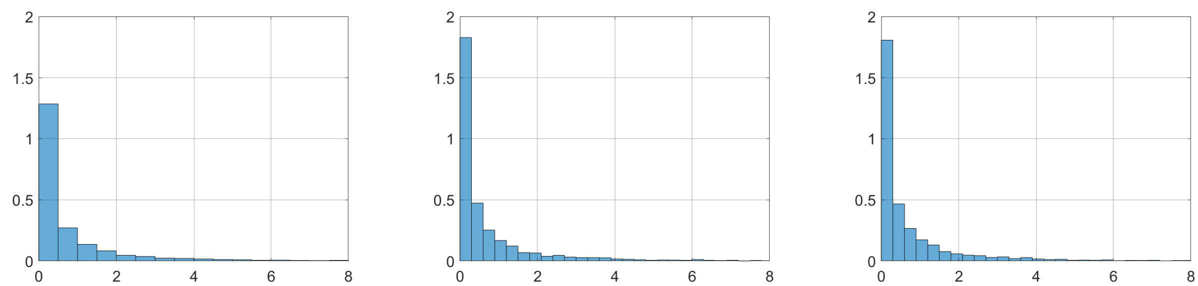
(d) $n^{-1}\hat{t}_{\beta}^2$ $(n = 100)$ (e) $n^{-1}\hat{t}_{\beta}^2$ $(n = 500)$ (f) $n^{-1}\hat{t}_{\beta}^2$ $(n = 1000)$

We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $d_{yn} = 0$, $d_{xn} = 0$, $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$, and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is $y_t = \alpha + \beta x_t + \gamma t + u_t$. $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_{\beta}$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This figure presents the empirical distributions of $\hat{\beta}$ and $n^{-1}\hat{t}_{\beta}^2$.

Figure 2: Empirical distributions of $\hat{\beta}$ and $\hat{t}_{\beta}^2$ when time trend t is included (Case Y1X2)



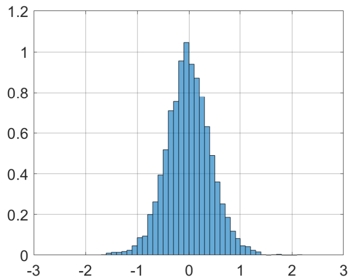
(a) $\hat{\beta}$ $(n = 100)$ (b) $\hat{\beta}$ $(n = 500)$ (c) $\hat{\beta}$ $(n = 1000)$



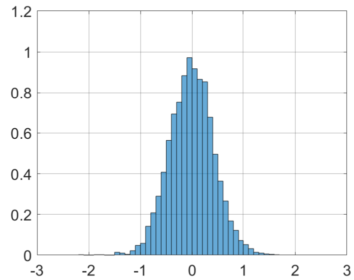
(d) $n^{-1}\hat{t}_{\beta}^2$ $(n = 100)$ (e) $n^{-1}\hat{t}_{\beta}^2$ $(n = 500)$ (f) $n^{-1}\hat{t}_{\beta}^2$ $(n = 1000)$

We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $d_{yn} = 0$, $d_{xn} = n^{-1/2}$, $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$, and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is $y_t = \alpha + \beta x_t + \gamma t + u_t$. $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_{\beta}$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This figure presents the empirical distributions of $\hat{\beta}$ and $n^{-1}\hat{t}_{\beta}^2$.

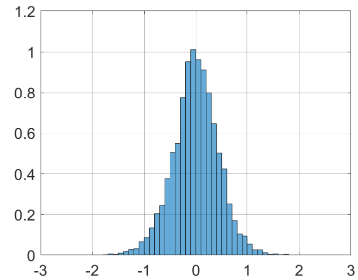
Figure 3: Empirical distributions of $\hat{\beta}$ and $\hat{t}_{\beta}^{2}$ when time trend t is included (Case Y1X3)



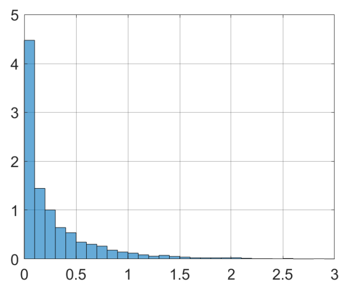
(a) $\hat{\beta}$ $(n = 100)$



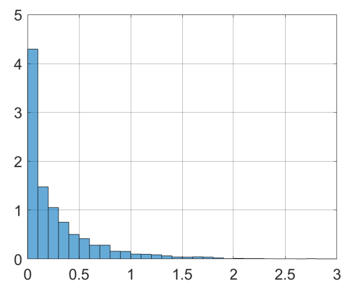
(b) $\hat{\beta}$ $(n = 500)$



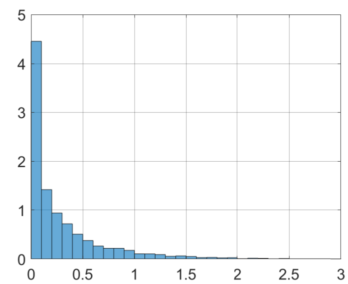
(c) $\hat{\beta}$ $(n = 1000)$



(d) $n^{-2}\hat{t}_{\beta}^{2}$ $(n = 100)$



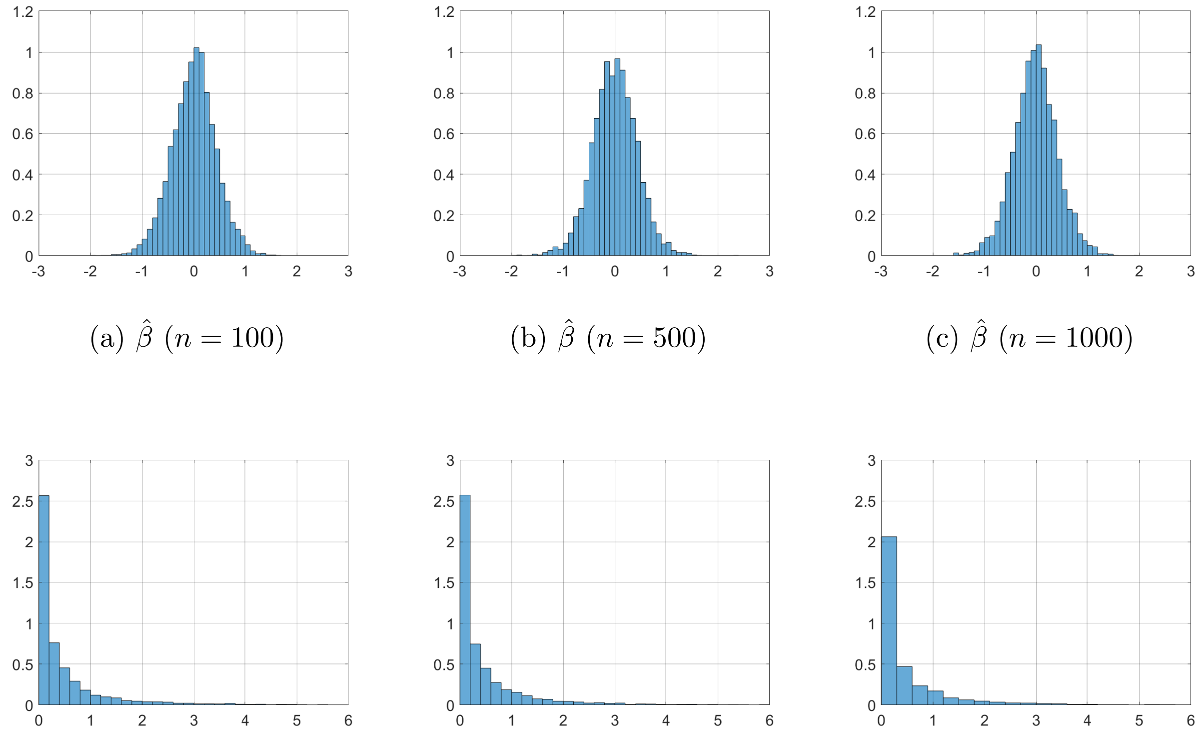
(e) $n^{-2}\hat{t}_{\beta}^{2}$ $(n = 500)$



(f) $n^{-2}\hat{t}_{\beta}^{2}$ $(n = 1000)$

We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $d_{yn} = 0$, $d_{xn} = 1$, $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$, and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is $y_t = \alpha + \beta x_t + \gamma t + u_t$. $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_{\beta}$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This figure presents the empirical distributions of $\hat{\beta}$ and $n^{-2}\hat{t}_{\beta}^{2}$.

Figure 4: Empirical distributions of $\hat{\beta}$ and $\hat{t}_\beta^2$ when time trend t is included (Case Y2X1)

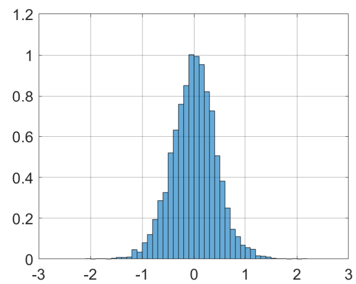


(a) $\hat{\beta}$ $(n = 100)$ (b) $\hat{\beta}$ $(n = 500)$ (c) $\hat{\beta}$ $(n = 1000)$

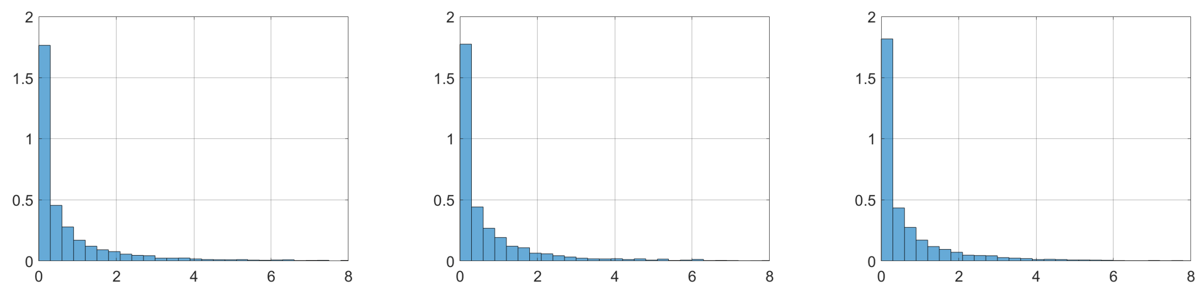
(d) $n^{-1}\hat{t}_\beta^2$ $(n = 100)$ (e) $n^{-1}\hat{t}_\beta^2$ $(n = 500)$ (f) $n^{-1}\hat{t}_\beta^2$ $(n = 1000)$

We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $d_{yn} = n^{-1/2}$, $d_{xn} = 0$, $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$, and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is $y_t = \alpha + \beta x_t + \gamma t + u_t$. $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_\beta$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This figure presents the empirical distributions of $\hat{\beta}$ and $n^{-1}\hat{t}_\beta^2$.

Figure 5: Empirical distributions of $\hat{\beta}$ and $\hat{t}_\beta^2$ when time trend t is included (Case Y2X2)



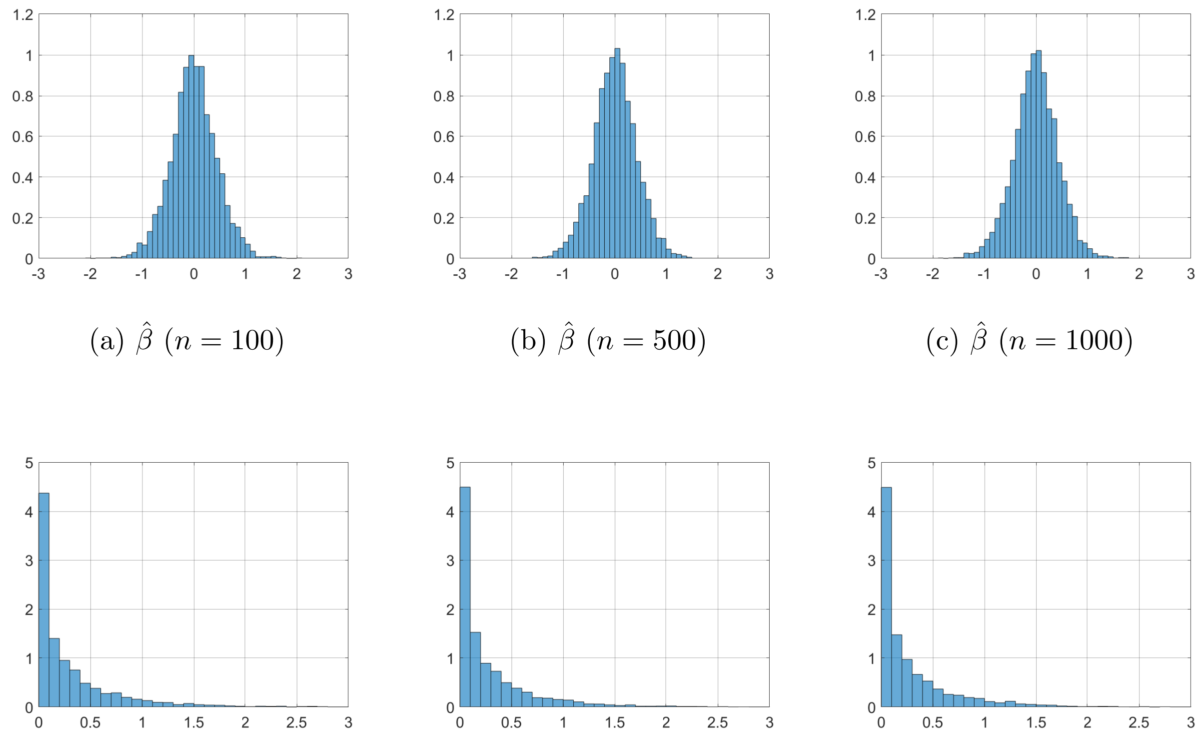
(a) $\hat{\beta}$ $(n = 100)$ (b) $\hat{\beta}$ $(n = 500)$ (c) $\hat{\beta}$ $(n = 1000)$



(d) $n^{-1}\hat{t}_\beta^2$ $(n = 100)$ (e) $n^{-1}\hat{t}_\beta^2$ $(n = 500)$ (f) $n^{-1}\hat{t}_\beta^2$ $(n = 1000)$

We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $d_{yn} = n^{-1/2}$, $d_{xn} = n^{-1/2}$, $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$, and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is $y_t = \alpha + \beta x_t + \gamma t + u_t$. $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_\beta$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This figure presents the empirical distributions of $\hat{\beta}$ and $n^{-1}\hat{t}_\beta^2$.

Figure 6: Empirical distributions of $\hat{\beta}$ and $\hat{t}_{\beta}^2$ when time trend t is included (Case Y2X3)

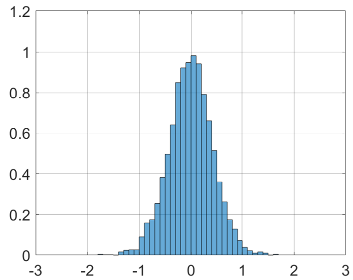


(a) $\hat{\beta}$ $(n = 100)$ (b) $\hat{\beta}$ $(n = 500)$ (c) $\hat{\beta}$ $(n = 1000)$

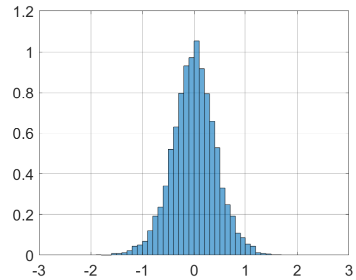
(d) $n^{-2}\hat{t}_{\beta}^2$ $(n = 100)$ (e) $n^{-2}\hat{t}_{\beta}^2$ $(n = 500)$ (f) $n^{-2}\hat{t}_{\beta}^2$ $(n = 1000)$

We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $d_{yn} = n^{-1/2}$, $d_{xn} = 1$, $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$, and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is $y_t = \alpha + \beta x_t + \gamma t + u_t$. $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_{\beta}$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This figure presents the empirical distributions of $\hat{\beta}$ and $n^{-2}\hat{t}_{\beta}^2$.

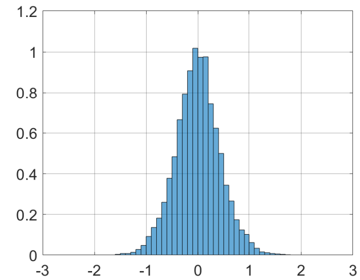
Figure 7: Empirical distributions of $\hat{\beta}$ and $\hat{t}_{\beta}^{2}$ when time trend t is included (Case Y3X1)



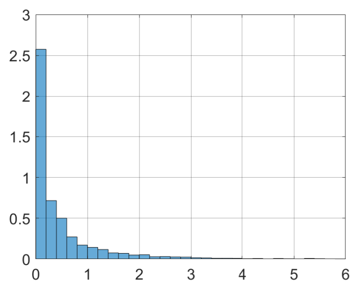
(a) $\hat{\beta}$ $(n = 100)$



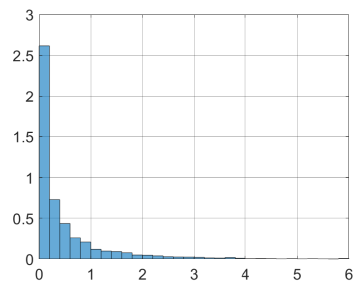
(b) $\hat{\beta}$ $(n = 500)$



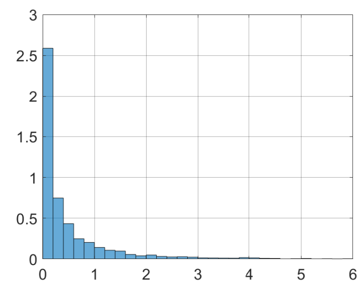
(c) $\hat{\beta}$ $(n = 1000)$



(d) $n^{-1}\hat{t}_{\beta}^{2}$ $(n = 100)$



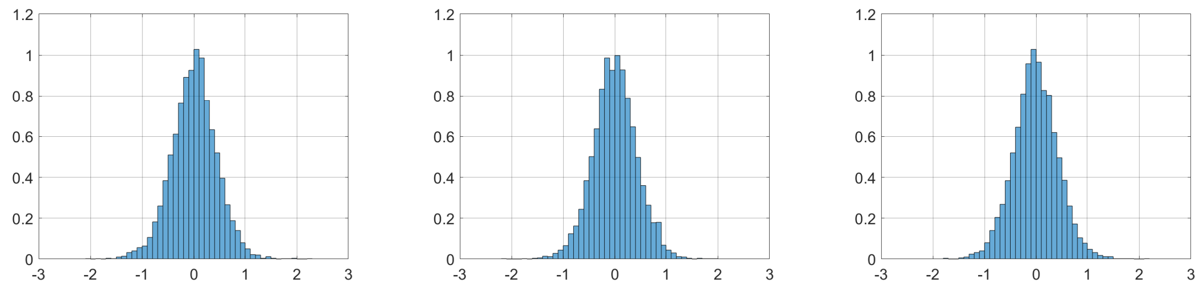
(e) $n^{-1}\hat{t}_{\beta}^{2}$ $(n = 500)$



(f) $n^{-1}\hat{t}_{\beta}^{2}$ $(n = 1000)$

We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $d_{yn} = 1$, $d_{xn} = 0$, $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$, and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is $y_t = \alpha + \beta x_t + \gamma t + u_t$. $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_{\beta}$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This figure presents the empirical distributions of $\hat{\beta}$ and $n^{-1}\hat{t}_{\beta}^{2}$.

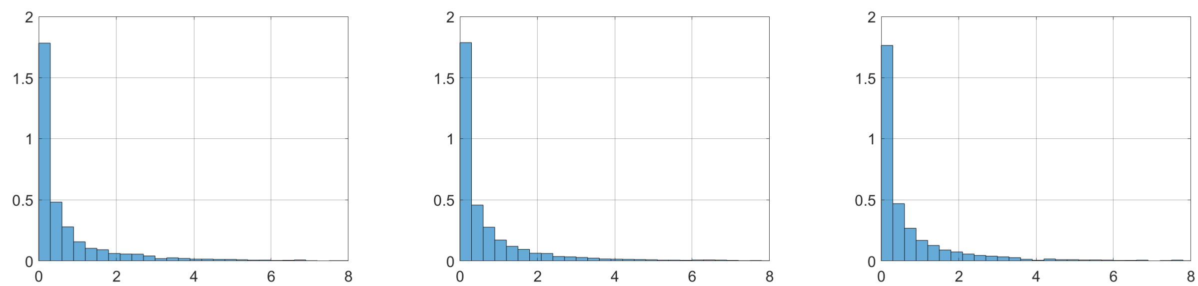
Figure 8: Empirical distributions of $\hat{\beta}$ and $\hat{t}_\beta^2$ when time trend t is included (Case Y3X2)



(a) $\hat{\beta}$ $(n = 100)$

(b) $\hat{\beta}$ $(n = 500)$

(c) $\hat{\beta}$ $(n = 1000)$



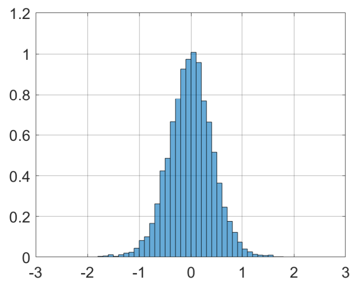
(d) $n^{-1}\hat{t}_\beta^2$ $(n = 100)$

(e) $n^{-1}\hat{t}_\beta^2$ $(n = 500)$

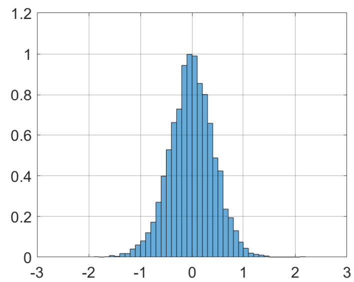
(f) $n^{-1}\hat{t}_\beta^2$ $(n = 1000)$

We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $d_{yn} = 1$, $d_{xn} = n^{-1/2}$, $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$, and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is $y_t = \alpha + \beta x_t + \gamma t + u_t$. $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_\beta$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This figure presents the empirical distributions of $\hat{\beta}$ and $n^{-1}\hat{t}_\beta^2$.

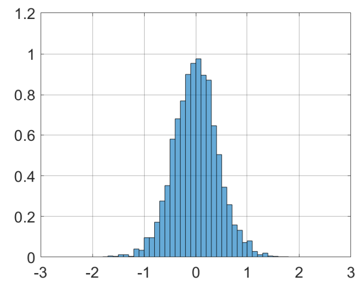
Figure 9: Empirical distributions of $\hat{\beta}$ and $\hat{t}_{\beta}^{2}$ when time trend t is included (Case Y3X3)



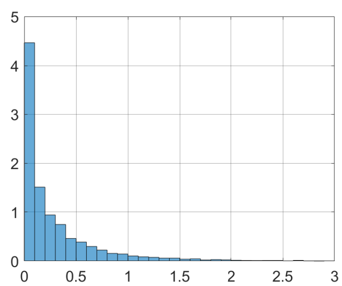
(a) $\hat{\beta}$ ($n = 100$)



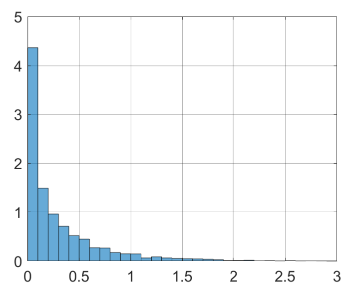
(b) $\hat{\beta}$ ($n = 500$)



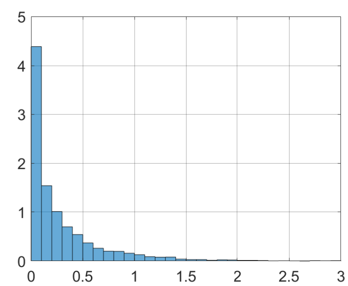
(c) $\hat{\beta}$ ($n = 1000$)



(d) $n^{-2}\hat{t}_{\beta}^{2}$ ($n = 100$)



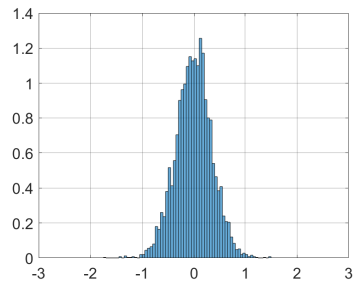
(e) $n^{-2}\hat{t}_{\beta}^{2}$ ($n = 500$)



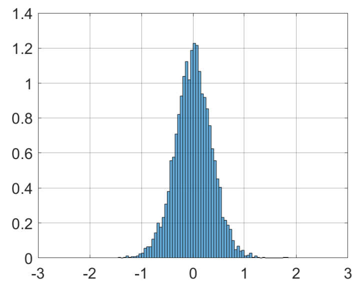
(f) $n^{-2}\hat{t}_{\beta}^{2}$ ($n = 1000$)

We generate $J = 5000$ Monte Carlo samples of size $n \in \{100, 500, 1000\}$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $d_{yn} = 1$, $d_{xn} = 1$, $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$, and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. The regression model is $y_t = \alpha + \beta x_t + \gamma t + u_t$. $\hat{\beta}$ is the least squares estimator for β. $\hat{t}_{\beta}$ is the conventional t-statistic with respect to $H_0 : \beta = 0$. This figure presents the empirical distributions of $\hat{\beta}$ and $n^{-2}\hat{t}_{\beta}^{2}$.

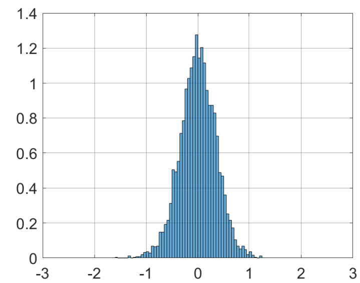
Figure 10: Empirical distributions of $\hat{\beta}$ when time trends (t, t^2) are included



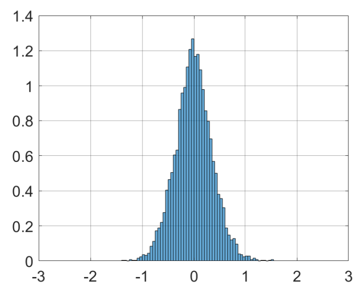
(a) Case Y1X1



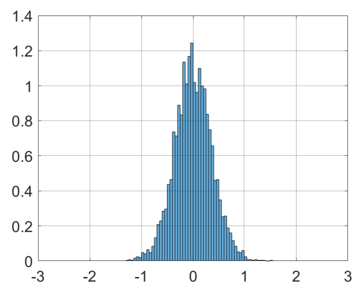
(b) Case Y1X2



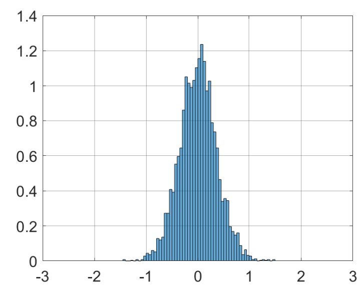
(c) Case Y1X3



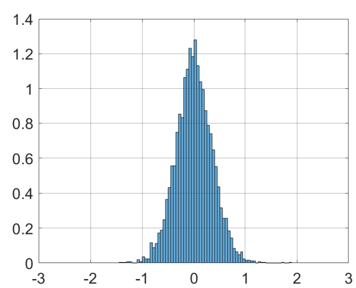
(d) Case Y2X1



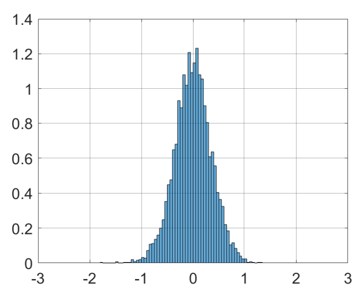
(e) Case Y2X2



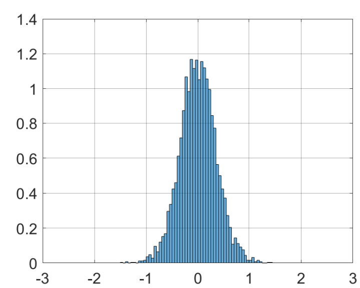
(f) Case Y2X3



(g) Case Y3X1



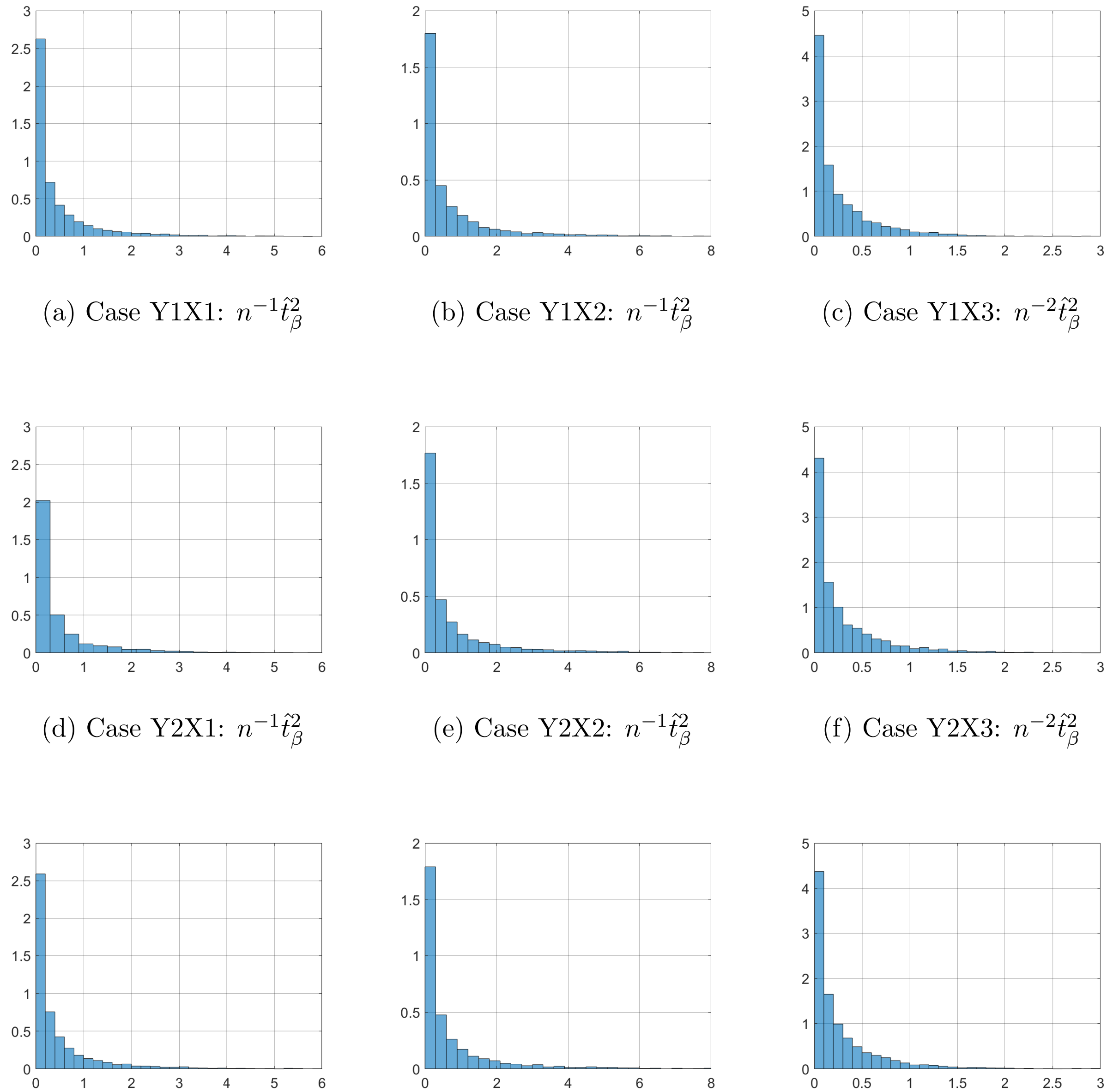
(h) Case Y3X2



(i) Case Y3X3

We generate $J = 5000$ Monte Carlo samples of size $n = 1000$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$ and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. Y1: $d_{yn} = 0$. Y2: $d_{yn} = n^{-1/2}$. Y3: $d_{yn} = 1$. X1: $d_{xn} = 0$. X2: $d_{xn} = n^{-1/2}$. X3: $d_{xn} = 1$. The regression model is $y_t = \alpha + \beta x_t + \gamma_1 t + \gamma_2 t^2 + u_t$. This figure presents the empirical distributions of $\hat{\beta}$, the least squares estimator for β.

Figure 11: Empirical distributions of scaled $\hat{t}_\beta^2$ when time trends (t, t^2) are included



(a) Case Y1X1: $n^{-1}\hat{t}_\beta^2$

(b) Case Y1X2: $n^{-1}\hat{t}_\beta^2$

(c) Case Y1X3: $n^{-2}\hat{t}_\beta^2$

(d) Case Y2X1: $n^{-1}\hat{t}_\beta^2$

(e) Case Y2X2: $n^{-1}\hat{t}_\beta^2$

(f) Case Y2X3: $n^{-2}\hat{t}_\beta^2$

(g) Case Y3X1: $n^{-1}\hat{t}_\beta^2$

(h) Case Y3X2: $n^{-1}\hat{t}_\beta^2$

(i) Case Y3X3: $n^{-2}\hat{t}_\beta^2$

We generate $J = 5000$ Monte Carlo samples of size $n = 1000$ from the DGP $y_t = d_{yn} + y_{t-1} + \epsilon_{yt}$ and $x_t = d_{xn} + x_{t-1} + \epsilon_{xt}$, where $\epsilon_{yt} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$ and $\epsilon_{xt} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. $\{\epsilon_{yt}\}$ and $\{\epsilon_{xt}\}$ are mutually independent. Y1: $d_{yn} = 0$. Y2: $d_{yn} = n^{-1/2}$. Y3: $d_{yn} = 1$. X1: $d_{xn} = 0$. X2: $d_{xn} = n^{-1/2}$. X3: $d_{xn} = 1$. The regression model is $y_t = \alpha + \beta x_t + \gamma_1 t + \gamma_2 t^2 + u_t$. This figure presents the empirical distributions of scaled $\hat{t}_\beta^2$, the squared t-statistic associated with β. The scaling factor is n for Cases X1-X2 and n^2 for Case X3.