Technical Appendix for
'Testing for Granger Causality with Mixed Frequency Data'

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This appendix presents technical details and an example concerning several covariance matrices.

E.1 Framework and Assumptions

The stacked HF and LF variables are

$$\boldsymbol{X}(\tau_L) = [\boldsymbol{x}_H(\tau_L,1)', \ldots, \boldsymbol{x}_H(\tau_L,m)', \boldsymbol{x}_L(\tau_L)']'. \tag{E.1.1}$$

The required assumptions follow.

Assumption 2.1. *The process $\boldsymbol{X}(\tau_L)$ is governed by a VAR(p) for some $p \geq 1$:*

$$\boldsymbol{X}(\tau_L) = \sum_{k=1}^{p} \boldsymbol{A}_k \boldsymbol{X}(\tau_L - k) + \boldsymbol{\epsilon}(\tau_L).$$

The coefficients $\boldsymbol{A}_k$ are $K \times K$ matrices for $k = 1, \ldots, p$. The $K \times 1$ error vector $\boldsymbol{\epsilon}(\tau_L) = [\epsilon_1(\tau_L), \ldots, \epsilon_K(\tau_L)]'$ is a strictly stationary martingale difference with respect to increasing $\mathcal{F}_{\tau_L} \subset \mathcal{F}_{\tau_L+1}$, where $\boldsymbol{\Omega} \equiv E[\boldsymbol{\epsilon}(\tau_L)\boldsymbol{\epsilon}(\tau_L)']$ is positive definite.

Assumption 2.2. *All roots of the polynomial $\det(\boldsymbol{I}_K - \sum_{k=1}^{p} \boldsymbol{A}_k z^k) = 0$ lie outside the unit circle.*

Assumption 2.3. *$\boldsymbol{X}(\tau_L)$ and $\boldsymbol{\epsilon}(\tau_L)$ are α-mixing: $\sum_{h=0}^{\infty} \alpha_{2^h} < \infty$.*

The (p,h)-autoregression (cfr. Dufour, Pelletier, and Renault (2006)):

$$\boldsymbol{X}(\tau_L + h) = \sum_{k=1}^{p} \boldsymbol{A}_k^{(h)} \boldsymbol{X}(\tau_L + 1 - k) + \boldsymbol{u}^{(h)}(\tau_L),$$

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where

$$\boldsymbol{A}_k^{(1)} = \boldsymbol{A}_k \quad \text{and} \quad \boldsymbol{A}_k^{(i)} = \boldsymbol{A}_{k+i-1} + \sum_{l=1}^{i-1} \boldsymbol{A}_{i-l}\boldsymbol{A}_k^{(l)} \text{ for } i \geq 2$$

$$\boldsymbol{u}^{(h)}(\tau_L) = \sum_{k=0}^{h-1} \boldsymbol{\Psi}_k \boldsymbol{\epsilon}(\tau_L - k).$$

Now stack

$$\boldsymbol{W}(\tau_L, p) = \left[\boldsymbol{X}(\tau_L)', \boldsymbol{X}(\tau_L - 1)', \ldots, \boldsymbol{X}(\tau_L - p + 1)'\right]' \in \mathbb{R}^{pK \times 1} \tag{E.1.2}$$

and define:

$$\boldsymbol{Y}(\tau_L + h, p) \equiv \text{vec}\left[\boldsymbol{W}(\tau_L, p)\boldsymbol{u}^{(h)}(\tau_L + h)'\right]. \tag{E.1.3}$$

E.2 Covariance Matrix Characterization

Recall the least squares asymptotic covariance matrix from Theorem 2.1 of the main paper:

$$\boldsymbol{\Sigma}_p(h) \equiv (\boldsymbol{I}_K \otimes \boldsymbol{\Gamma}_{p,0}^{-1})\boldsymbol{D}_p(h)(\boldsymbol{I}_K \otimes \boldsymbol{\Gamma}_{p,0}^{-1})',$$

where

$$\boldsymbol{\Gamma}_{p,0} \equiv E\left[\boldsymbol{W}(\tau_L, p)\boldsymbol{W}(\tau_L, p)'\right]$$

$$\boldsymbol{D}_{p,T_L^*}(h) \equiv \text{Var}\left[\frac{1}{\sqrt{T_L^*}} \sum_{\tau_L=0}^{T_L^*-1} \boldsymbol{Y}(\tau_L + h, p)\right]$$

$$\boldsymbol{D}_p(h) \equiv \lim_{T_L^* \to \infty} \boldsymbol{D}_{p,T_L^*}(h).$$

E.2.1 Derivation of $D_p(h)$

Under Assumption 2.1 $\boldsymbol{\epsilon}(\tau_L)$ is a stationary mds with respect to $\mathcal{F}_{\tau_L}$, where

$$E\left[\boldsymbol{\epsilon}(\tau_L)\boldsymbol{\epsilon}(\tau_L)'\right] \equiv \boldsymbol{\Omega} \text{ is positive definite.}$$

Trivially, therefore, $\boldsymbol{\epsilon}(\tau_L)$ has a continuous, bounded and positive spectral density. Hence by stationarity Assumption 2.2, $\boldsymbol{X}(\tau_L)$ has a continuous, bounded and everywhere positive spectral density. Therefore $\{\boldsymbol{Y}(\tau_L + h, p)\}_{\tau_L}$ is a zero mean L_2-bounded stationary process with continuous, everywhere positive spectrum, and therefore auto-covariances

$$\boldsymbol{\Delta}_{p,s}(h) \equiv E\left[\boldsymbol{Y}(\tau_L + h + s, p)\boldsymbol{Y}(\tau_L + h, p)'\right]$$

that satisfy

$$\boldsymbol{\Delta}_{p,0}(h) \text{ is positive definite } \forall h \geq 0, \text{ and } \boldsymbol{\Delta}_{p,s}(h) = \boldsymbol{0}_{pK^2 \times pK^2} \quad \forall s \geq h.$$

Analytical characterizations of $\boldsymbol{\Gamma}_{p,0}$ and $\boldsymbol{\Delta}_{p,s}(h)$, and a proof that $\boldsymbol{\Delta}_{p,s}(h) = \mathbf{0}_{pK^2 \times pK^2}$ $\forall s \geq h$ are presented in Section E.2.2. The partial sum variance of $\boldsymbol{Y}(\tau_L + h, p)$ is therefore:

$$\begin{aligned}
\boldsymbol{D}_{p,T_L^*}(h) &\equiv \text{Var}\left[\frac{1}{\sqrt{T_L^*}}\sum_{\tau_L=0}^{T_L^*-1}\boldsymbol{Y}(\tau_L + h, p)\right] \\
&= \boldsymbol{\Delta}_{p,0}(h) + \sum_{s=1}^{h-1}\left[1 - \frac{s}{T_L^*}\right] \times \left[\boldsymbol{\Delta}_{p,s}(h) + \boldsymbol{\Delta}_{p,s}(h)'\right] \\
&= \boldsymbol{\Delta}_{p,0}(1) \text{ if } h = 1.
\end{aligned}$$

We define $\boldsymbol{D}_p(h)$ as the long-run variance of $\boldsymbol{Y}(\tau_L + h, p)$:

$$\begin{aligned}
\boldsymbol{D}_p(h) &\equiv \lim_{T_L^* \to \infty} \boldsymbol{D}_{p,T_L^*}(h) = \boldsymbol{\Delta}_{p,0}(h) + \sum_{s=1}^{h-1}\left[\boldsymbol{\Delta}_{p,s}(h) + \boldsymbol{\Delta}_{p,s}(h)'\right] \\
&= \boldsymbol{\Delta}_{p,0}(1) \text{ if } h = 1. \qquad \text{(E.2.1)}
\end{aligned}$$

Observe that $\boldsymbol{D}_{p,T_L^*}(h)$ for T_L^* sufficiently large is positive definite, hence $\boldsymbol{D}_p(h)$ is positive definite. Simply note that by stationarity and spectral density positiveness for $\boldsymbol{X}(\tau_L)$, it follows $\boldsymbol{a}'\boldsymbol{Y}(\tau_L + h, p)\boldsymbol{a}$ is for any conformable $\boldsymbol{a} \neq \mathbf{0}$, $\boldsymbol{a}'\boldsymbol{a} = 1$, stationary and has a continuous, bounded everywhere positive spectral density $f_a(\lambda)$. Therefore $\boldsymbol{a}'\boldsymbol{D}_{p,T_L^*}(h)\boldsymbol{a} = 2\pi f_a(0) + o(1) > 0$ for T_L^* sufficiently large (see eq. (1.7) in Ibragimov (1962)).

E.2.2 Derivation of $\Gamma_{p,0}$ and $\Delta_{p,s}(h)$

We now explicitly characterize the covariance matrices $\boldsymbol{\Gamma}_{p,0} \equiv E[\boldsymbol{W}(\tau_L, p)\boldsymbol{W}(\tau_L, p)']$ and $\boldsymbol{\Delta}_{p,s}(h) \equiv E[\boldsymbol{Y}(\tau_L + h + s, p)\boldsymbol{Y}(\tau_L + h, p)']$. Recall that under Assumption 2.2 a unique stationary and ergodic solution to (E.1.1) exists:

$$\boldsymbol{X}(\tau_L) = \sum_{k=0}^{\infty} \boldsymbol{\Psi}_k \boldsymbol{\epsilon}(\tau_L - k),$$

where $\boldsymbol{\Psi}_k$ satisfies $\boldsymbol{\Psi}_0 = \boldsymbol{I}_K$, $\boldsymbol{\Psi}_k = \sum_{s=1}^{p} \boldsymbol{A}_s \boldsymbol{\Psi}_{k-s}$ for $k \geq 1$ and $\boldsymbol{\Psi}_k = \mathbf{0}_{K \times K}$ for $k < 0$, and $|\boldsymbol{\Psi}_k| = O(\rho^k)$ for some $\rho \in (0,1)$. Denote the auto-covariances of $\boldsymbol{X}(\tau_L)$ as

$$\boldsymbol{\Upsilon}_s = [\upsilon_{ij,s}]_{i,j=1}^{K} \equiv E\left[\boldsymbol{X}(\tau_L + s)\boldsymbol{X}(\tau_L)'\right] = \begin{cases} \sum_{k=0}^{\infty} \boldsymbol{\Psi}_{s+k}\boldsymbol{\Omega}\boldsymbol{\Psi}_k' & \text{if } s \geq 0 \\ \boldsymbol{\Upsilon}_{-s}' & \text{if } s < 0. \end{cases} \qquad \text{(E.2.2)}$$

In view of $|\boldsymbol{\Psi}_k| = O(\rho^h)$ for $\rho \in (0,1)$, and $||\boldsymbol{H}(\tau_L)||_{2+\delta} \in (0,\infty)$, it follows $||\boldsymbol{\Omega}|| < \infty$ and therefore $\sum_{s=-\infty}^{\infty} |\upsilon_{ij,s}| < \infty$ for any i,j. The process $\{\boldsymbol{W}(\tau_L, p)\}_{\tau_L}$ defined by (E.1.2) therefore has auto-covariances

$$\boldsymbol{\Gamma}_{p,s} \equiv E\left[\boldsymbol{W}(\tau_L+s,p)\boldsymbol{W}(\tau_L,p)'\right] = \begin{bmatrix} \boldsymbol{\Upsilon}_s & \boldsymbol{\Upsilon}_{s+1} & \cdots & \boldsymbol{\Upsilon}_{s+p-1} \\ \boldsymbol{\Upsilon}_{s-1} & \boldsymbol{\Upsilon}_s & \cdots & \boldsymbol{\Upsilon}_{s+p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Upsilon}_{s-p+1} & \boldsymbol{\Upsilon}_{s-p+2} & \cdots & \boldsymbol{\Upsilon}_s \end{bmatrix}. \tag{E.2.3}$$

Further, $\boldsymbol{u}^{(h)}(\tau_L)$ has auto-covariances

$$\boldsymbol{Q}_s(h) \equiv E\left[\boldsymbol{u}^{(h)}(\tau_L+s)\boldsymbol{u}^{(h)}(\tau_L)'\right] = \begin{cases} \sum_{k=0}^{h-s-1} \boldsymbol{\Psi}_{s+k}\boldsymbol{\Omega}\boldsymbol{\Psi}_k' & \text{if } 0 \leq s < h \\ \boldsymbol{Q}_{-s}(h)' & \text{if } -h < s < 0 \\ \boldsymbol{0}_{K\times K} & \text{if } |s| \geq h. \end{cases} \tag{E.2.4}$$

Using (E.2.4) and $\boldsymbol{Y}(\tau_L+h,p) \equiv (\boldsymbol{I}_K \otimes \boldsymbol{W}(\tau_L,p))\boldsymbol{u}^{(h)}(\tau_L+h)$, the auto-covariances $\boldsymbol{\Delta}_{p,s}(h)$ of $\boldsymbol{Y}(\tau_L+h,p)$ can now be fully characterized:

$$\boldsymbol{\Delta}_{p,s}(h) \equiv E[\boldsymbol{Y}(\tau_L+h+s,p)\boldsymbol{Y}(\tau_L+h,p)'] = \begin{cases} \boldsymbol{Q}_0(h) \otimes \boldsymbol{\Gamma}_{p,0} & \text{if } s = 0 \\ \boldsymbol{\Delta}_{p,-s}(h)' & \text{if } -h < s < 0 \\ \boldsymbol{0}_{pK^2\times pK^2} & \text{if } |s| \geq h. \end{cases} \tag{E.2.5}$$

Note that $\boldsymbol{Y}(\tau_L+h,p)$ is serially uncorrelated at lag $|s| \geq h$, although in general we cannot say $\boldsymbol{Y}(\tau_L+h,p)$ is $h-1$ dependent. Evidently a convenient expression for $\boldsymbol{\Delta}_{p,s}(h)$ does not exist when $s \in \{1,\ldots,h-1\}$.

We now prove $\boldsymbol{\Delta}_{p,s}(h) = \boldsymbol{0}_{pK^2\times pK^2}$ for $|s| \geq h$. Assume without loss of generality that $s \geq h$. Equation (E.1.3) and the definition of $\boldsymbol{\Delta}_{p,s}(h)$ imply that

$$\boldsymbol{\Delta}_{p,s}(h) = E\left[\left(\boldsymbol{I}_K \otimes \boldsymbol{W}(\tau_L+s,p)\right)\boldsymbol{u}^{(h)}(\tau_L+s+h)\boldsymbol{u}^{(h)}(\tau_L+h)'\left(\boldsymbol{I}_K \otimes \boldsymbol{W}(\tau_L,p)'\right)\right]. \tag{E.2.6}$$

Let $I(\tau_L+s) = \sigma\{\boldsymbol{\epsilon}(\tau)|\tau \leq \tau_L+s\}$. Note that $\boldsymbol{W}(\tau_L,p)$, $\boldsymbol{W}(\tau_L+s,p)$, and $\boldsymbol{u}^{(h)}(\tau_L+h)$ are all known at period τ_L+s, while $\boldsymbol{u}^{(h)}(\tau_L+s+h)$ depends only on $\{\boldsymbol{\epsilon}(\tau_L+s+1),\ldots,\boldsymbol{\epsilon}(\tau_L+s+h)\}$ and therefore $E[\boldsymbol{u}^{(h)}(\tau_L+s+h)|I(\tau_L+s)] = \sum_{k=0}^{h-1}\boldsymbol{\Psi}_k E[\boldsymbol{\epsilon}(\tau_L+s+h-k)|I(\tau_L+s)] = \boldsymbol{0}$ by the mds Assumption 2.1. We can thus get the desired result by applying the law of iterated expectations to (E.2.6). Similarly, $\boldsymbol{\Delta}_{p,0}(h) = \boldsymbol{Q}_0(h) \otimes \boldsymbol{\Gamma}_{p,0}$ can be shown by applying the law of iterated expectations given $I(\tau_L)$ to (E.2.6).

E.2.3 Example: $h = 1$

It is useful to derive the least squares asymptotic variance $\boldsymbol{\Sigma}_p = (\boldsymbol{I}_K \otimes \boldsymbol{\Gamma}_{p,0}^{-1})\boldsymbol{D}_p(h)(\boldsymbol{I}_K \otimes \boldsymbol{\Gamma}_{p,0}^{-1})'$ for the case $h=1$. Use (E.2.2) and (E.2.3) to deduce $\boldsymbol{\Gamma}_{p,0} = \boldsymbol{\Upsilon}_0 = \sum_{k=0}^{\infty}\boldsymbol{\Psi}_k\boldsymbol{\Omega}\boldsymbol{\Psi}_k'$. Next, use (E.2.1) and (E.2.5) to deduce $\boldsymbol{D}_p(1) = \boldsymbol{\Delta}_{p,0}(1) = \boldsymbol{Q}_0(1) \otimes \boldsymbol{\Gamma}_{p,0}$, hence by (E.2.3) and (E.2.4) it follows $\boldsymbol{D}_p(1) = \boldsymbol{\Omega} \otimes \boldsymbol{\Gamma}_{p,0} = \boldsymbol{\Omega} \otimes \sum_{k=0}^{\infty} \boldsymbol{\Psi}_k \boldsymbol{\Omega} \boldsymbol{\Psi}'_k$. Kronecker product properties therefore imply $\boldsymbol{\Sigma}_p$ is identically $\boldsymbol{\Omega} \otimes \boldsymbol{\Gamma}_{p,0}^{-1} = \boldsymbol{\Omega} \otimes \boldsymbol{\Upsilon}_0^{-1} = \boldsymbol{\Omega} \otimes (\sum_{k=0}^{\infty} \boldsymbol{\Psi}_k \boldsymbol{\Omega} \boldsymbol{\Psi}'_k)^{-1}$.

References

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