Higher stationarity *P.D.Welch, Kyoto, 29.ix.2016*

Lecture dedicated to the memory of Professor Yuzuru Kakuda

• This talk aims to extend work of Bagaria-Magidor-Sakai[1] who defined, and initiated an investigation into notions of *higher forms of stationarity*.
• The definitions and results here, when not otherwise stated, are due to H. Brickhill and will be part of her thesis.
• The motivations came originally from J. Bagaria, and were related to finding topological models for certain kinds of modal logic with multiple modal operators. He found that certain spaces amongst the topologies on ordinals with sufficient reflection properties were what he needed.

[1] [BaMS] J. Bagaria and M. Magidor and H. Sakai, *"Reflection and Indescribability in the Constructible Universe"*, Israel J. of Maths., 2015

Stationary closure

Definition

Let $C \subseteq \eta \in On$. C is said to be *closed* if

$$\text{For } \alpha < \eta : \quad C \cap \alpha \text{ unbounded in } \alpha \rightarrow \alpha \in C.$$

(ii) $S \subseteq \eta$ is said to be *stationary* if

$$\text{For } \forall C \subseteq \eta : \quad C \text{ closed and unbounded (club) in } \eta \rightarrow S \cap C \neq \emptyset.$$

As a basis for an inductive definition:

- We are going to call "unboundedness" in η, "being 0-stationary in η"
- Thus C is closed and unbounded in η iff it is closed and 0-stationary in η.
- We are going to call "club in η", "being 0-club in η"
- Then for, *e.g.* , η a cardinal of uncountable cofinality, the club subsets of η form a filter F^0_η - the *club filter* on η.

Definition

Let $S \subseteq \eta \in On$. S is said to be *stationary closed* if

$$\text{For } \alpha < \eta: \quad S \cap \alpha \text{ stationary in } \alpha \to \alpha \in S.$$

Base Case again in this terminology:

Definition

$S \subseteq \eta$ is (1-)*stationary*, if it meets every club (= 0-club) in η; that is

$$\forall C \subseteq \eta, C \text{ 0-club in } \eta \to S \cap C \neq \emptyset.$$

Definition

$C \subseteq \eta$ is 1-*club in* η if C is stationary in η and is stationary closed.

Definition

$S \subseteq \eta$ is 2-*stationary*, if it meets every 1-club in η; that is

$$\forall C \subseteq \eta, C \text{ 1-club in } \eta \to S \cap C \neq \emptyset.$$

Definition (Brickhill)

Let n be an ordinal and $S, C \subseteq \eta$ sets of ordinals.

1. S is 0-*stationary in* η if S is unbounded in η.
2. C is *n-stationary-closed* if $\forall \alpha(C \cap \alpha$ is n-stationary $\rightarrow \alpha \in C)$.
3. C is *n-club* if C is n-stationary-closed below η and n-stationary in η.
4. η is *n-reflecting* if η is n-stationary in η, and,

$$\forall A, B \subseteq \eta(A, B\, n\text{-stationary } \rightarrow \exists \lambda < \eta(A \cap \lambda\, B \cap \lambda\, , n\text{-stationary.}))$$

5. S is *n-stationary in* η if:

$$\forall n' < n\ : (\eta \text{ is } n'\text{-reflecting} \wedge \forall C \subseteq \eta(C\, n'\text{-club} \rightarrow S \cap C \neq \text{ffl})).$$

Definition (n'th derivative)

$d_n(A) =_{df} \{\alpha \mid A \cap \alpha \text{ is } n\text{-stationary}\}.$

Definition (Brickhill)

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1. S is 0-*stationary in* η if S is unbounded in η.
2. C is n*-stationary-closed* if $(d_n(C) \subseteq C)$.
3. C is n*-club* if $(d_n(C) \subseteq C)$ and is n-stationary in η.
4. η is n*-reflecting* if η is n-stationary in η, and,

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Definition (Brickhill)

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3. C is γ-*club* if $(d_\gamma(C) \subseteq C)$ and is γ-stationary in η.
4. η is γ-*reflecting* if η is γ-stationary in η, and,
$$\forall A, B \subseteq \eta(A, B\ \gamma\text{-stationary} \rightarrow d_\gamma(A) \cap d_\gamma(B) \neq \emptyset.$$
5. S is γ-*stationary in* η if:
$$\forall \gamma' < \gamma : (\eta \text{ is } \gamma'\text{-reflecting} \wedge \forall C \subseteq \eta(C\ \gamma'\text{-club} \rightarrow S \cap C \neq \emptyset)).$$

Definition (γ'th derivative)

$d_\gamma(A) =_{df} \{\alpha \mid A \cap \alpha \text{ is } \gamma\text{-stationary}\}.$

We then have if η is n-reflecting:

$$
\begin{array}{ll}
D \subseteq \eta \ is: & \text{0-club (= club)} \rightarrow \text{1-club} \rightarrow \text{2-club} \rightarrow \cdots n-\text{1-club} \rightarrow n\text{-club} \\
& \hfill \downarrow \\
\text{unbounded} & \leftarrow \text{1-stat (=stat.)} \leftarrow \text{2-stat} \leftarrow \cdots \leftarrow \text{n-stat} \leftarrow n+\text{1-stat}
\end{array}
$$

Relations to [BaMS]

- [BaMS] introduced the notion of (finite) n-stationarity, and n-reflection, and the above definitions are inspired by that. Brickhill introduced n-club, and these slightly different definitions.

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- The definitions are most easily evaluated in a regular universe such as L (and seem to be rather intractable outside of it). In L they are equivalent to [BaMS].

Indescribability

Definition

Let Φ be $\forall X_1 \exists X_2 \cdots \mathsf{Q} X_n \varphi(\vec{x}, \vec{X}, R)$ be a Π^1_n formula in the language of set theory with additional predicates $\vec{X}$ and R.

A cardinal κ is Π^1_n-*indescribable* if for any such Π^1_n Φ, and $R \subseteq V_\kappa$:
$(V_\kappa, \in, R) \models \Phi \longrightarrow \exists \zeta < \kappa\colon (V_\zeta, \in, R \cap V_\zeta) \models \Phi(R \cap V_\zeta)$.

Classically:

Theorem (Jensen, $V = L$)

If κ is inaccessible, then

$$\kappa \text{ is } \Pi^1_1\text{-indescribable} \quad \longleftrightarrow \quad \kappa \text{ reflects stationary sets.}$$

This is essentially the $n = 1$ case.

[BaMS] continued

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Theorem (BaMS, $V = L$)

If $1 < n < \omega$ and κ is inaccessible, then

$$\kappa \text{ is } \Pi^1_n\text{-indescribable} \quad \longleftrightarrow \quad \kappa \text{ reflects } n\text{-stationary sets.}$$

The n-club filter, splitting n-stationary sets &c.

- We can now generalise the notion of *n-club-filter* for finite n.

Definition (Brickhill - n-club filter)

Let $\mathcal{F}^n_\kappa =_{df} \{X \subseteq \kappa \mid X \supseteq Y, Y \text{ an } n\text{-club set}\}$

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Lemma (Br)

If κ is Π^1_n-indescribable, then $\mathcal{F}^n_\kappa$ is κ-complete.

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Lemma (Br)

If κ is Π^1_n-indescribable, then $\mathcal{F}^n_\kappa$ is κ-complete.

Theorem (Br. - Solovay splitting)

$\mathcal{F}^n_\kappa$ is κ-complete
$\Rightarrow$ *Every $n+1$-stat. $S = \bigcup_{\alpha<\kappa} S_\alpha$ with S_α $n+1$-stat. & disjoint.*

Generalising Kunen

- Kunen showed that by a forcing over L that Jensen's characterisation of weak compactness in L could fail outside of L:

Theorem (Kunen)

Assume (κ is weakly compact)^L. Then there is $\mathcal{P} \in L$ with
$L \models$"$(\kappa$ *reflects stationary sets, but is not weakly compact*$)^{V^{\mathcal{P}}}$"

This lifts to:

Theorem (Brickhill-Magidor)

Assume (κ is Π^1_n-indescribable)^L. Then there is $\mathcal{P} \in L$ with
$L \models$"$(\kappa$ *reflects n-stationary sets, but is not weakly compact*$)^{V^{\mathcal{P}}}$"

Ineffabilities and $\diamondsuit$'s

- There are liftings too of various definitions and results concerning ineffabilities along the lines of: *"Any $f : [\kappa]^2 \to \kappa$ has an n-stationary homogeneous set."*

- Require of a $\diamondsuit$ that it predict on an n-stationary set, not just stationary. Call this $\diamondsuit^n_\kappa$. Sample:

Theorem (Br, $V = L$)

Assume κ is Π^1_n-indescribable. Then $\diamondsuit^n_\kappa$ holds.

Jensen revisited

We saw:

Theorem (Jensen, $V = L$)

If κ is inaccessible, then

$$\kappa \quad \Pi^1_1\text{-indescribable} \quad \longleftrightarrow \quad \kappa \text{ reflects stationary sets.}$$

Actually something more is possible:

Jensen revisited

Definition (Jensen)

A $\square$ *(square) sequence to* κ is a $\langle C_\alpha \mid \alpha < \kappa, Lim(\alpha)\rangle$ so that:
(i) $C_\alpha \subseteq \alpha$ is club;
(ii) $\beta \in C^*_\alpha \to C_\beta = C_\alpha \cap \beta$.

Theorem (Jensen, Beller-Litman, $V = L$)

Let κ be inaccessible, but not Π^1_1-indescribable. Let $A \subseteq \kappa$ be stationary. Then there exists a stationary $E \subseteq A$, and a $\square$ sequence to κ *with $\beta \in C^*_\alpha \to \beta \notin E$.*

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• This is a strong characterisation of non-weak compactness, and gives the non-trivial ($\longleftarrow$) in the previous Jensen theorem.

• Beller-Litman used the theory of *Silver Machines* in their proof.

• Given the results of [BaMS] one can ask whether there is some higher level analogue of Jensen-Beller-Litman for Π^1_n-indescribability - always assuming $V = L$.

Generalising Jensen

From the last slide:

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Definition (Brickhill; $\square^\gamma$-sequences to κ)

A $\square^\gamma$*-sequence to* κ is a $\langle C_\alpha \mid \alpha \in d_\gamma(\kappa)\rangle$ so that for all α:
(i) $C_\alpha \subseteq \alpha$ is γ-club in α;
(ii) $\beta \in d_\gamma(C_\alpha) \longrightarrow C_\beta = C_\alpha \cap \beta$.

- A $\square^0$-sequence to κ is a Jensen $\square$-sequence to κ.

(NB $\beta \in C^*_\alpha \leftrightarrow \beta \in d_0(C_\alpha)$.)

Theorem (Jensen, Beller-Litman, $V = L$)

Let κ be inaccessible, but not Π^1_1-indescribable. Let $A \subseteq \kappa$ be stationary. Then there exists a stationary $E \subseteq A$, and a $\square$ sequence to κ *with* $\beta \in C^*_\alpha \to \beta \notin E$.

Theorem (Jensen, Beller-Litman, $V = L$)

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Parallel to this we have:

Theorem (Br; $V = L$)

If κ is Π^1_n-indescribable but not Π^1_{n+1}-indescribable. Let $A \subseteq \kappa$ be $n+1$-stationary. Then there exists an $n+1$-stationary $E \subseteq A$, and a $\square^n$ sequence to κ with $\beta \in d_n(C_\alpha) \to \beta \notin E$.

Transfinite Cases

- We need to define indescribability at transfinite levels, and also handle limits.

Definition ($\lambda = \omega$)

A $\square^{<\omega}$*-sequence to* κ is a $\langle (n_\alpha, C_\alpha) \mid \alpha \in d_\gamma(\kappa) \rangle$ so that for all α:
(i) $n_\alpha < \omega$, and $C_\alpha \subseteq \alpha$ is an n_α-club in α;
(ii) $\beta \in d_{n_\alpha}(C_\alpha) \longrightarrow n_\beta = n_\alpha \wedge C_\beta = C_\alpha \cap \beta$.

Theorem (Br; $V = L$)

If κ is Π^1_n-indescribable for all $n < \omega$ but not Π^1_ω-indescribable. Let $A \subseteq \kappa$ be ω-stationary. Then there exists $E \subseteq A$, E ω-stationary, and a $\square^{<\omega}$ sequence to κ *with $\beta \in d_{n_\beta}(C_\alpha) \to \beta \notin E$.*
Hence κ is not ω-reflecting.

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Hence κ is not ω-reflecting.

- But we did not yet define Π^1_γ-indescribability.

Definition (W)

Let κ be an ordinal and let $a_0, \ldots, a_{k-1} \subseteq \kappa$ and $\Phi(v_0, \ldots, v_{k+2})$ be a Σ_0-formula (without parameters).
Let $G_\kappa(\Phi, \alpha, a_0, \ldots, a_{k-1})$ be the game with two players, (Π) and (Σ), and finitely many moves s.t., setting $\alpha_{-1} = \alpha$, in the n'th round:
(Σ) first chooses some $\alpha_n < \alpha_{n-1}$ and $A_n \subseteq \kappa$; then
(Π) chooses some $D_n \subseteq \kappa$ s.t. $\Phi(\zeta, \overline{A}_n, \overline{D}_n, a_0, \ldots, a_{k-1})$ where
$\overline{D}_n = \langle D_k : k \leqslant n \rangle$ and $\overline{A}_n = \langle A_k : k \leqslant n \rangle$
The first player to be unable to move loses.

Definition (W)

(Π^1_β) Let $\alpha > 0$. Say that a property $Q(\zeta, a_0, \ldots, a_{k-1})$ of ζ, where ζ ranges over ordinals and $a_0, \ldots, a_{k-1}$ range over subsets of ζ is

- $\Pi^1_{2\cdot\alpha}$, if there is Φ s.t. for all ζ and $a_0, \ldots, a_{k-1} \subseteq \zeta$, $Q(\zeta, a_0, \ldots, a_{k-1})$ holds iff (Π) wins $G_\zeta(\Phi, \alpha, a_0, \ldots, a_{k-1})$;
- $\Pi^1_{2\cdot\alpha+1}$, if there is $\Phi(v_0, \ldots, v_{k+3})$ s.t. for all ζ and $a_0, \ldots, a_{k-1} \subseteq \zeta$, $Q(\zeta, a_0, \ldots, a_{k-1})$ holds iff for all $X \subseteq \zeta$, (Σ) wins $G_\zeta(\Phi, \alpha, a_0, \ldots, a_{k-1}, X)$.

Definition (W)

$X \subseteq \kappa$ is $\Pi^1_{2\cdot\alpha}$-*indescribable* $(0 < \alpha < \kappa)$ if for every Σ_0 formula Φ, and $a_0, \ldots, a_{k-1} \subseteq \kappa$, if (Π) wins $G_\kappa(\Phi, \alpha, a_0, \ldots, a_{k-1})$ then (Π) wins $G_\zeta(\Phi, \alpha, a_0 \cap \zeta, \ldots, a_{k-1} \cap \zeta)$ for some $\zeta \in X$.
Similarly for $\Sigma^1_{2\cdot\alpha}$, $\Pi^1_{2\cdot\alpha+1}$, and $\Sigma^1_{2\cdot\alpha+1}$.

The following generalises Levy for finite n:

Theorem (W)

If κ is Π^1_α indescribable, the $\mathcal{F}_\kappa^{\Pi^1_\alpha}$ is normal and κ complete.

Theorem (Br; $V = L$)

Suppose κ is Π^1_δ-indescribable for all $\delta < \gamma$ but not Π^1_γ-indescribable. Let $A \subseteq \kappa$ be γ-stationary. Then there exists $E \subseteq A$, E γ-stationary, and,

$(\gamma = \delta + 1)$ *a $\square^\delta$ sequence to κ,*

$(Lim(\gamma))$ *a $\square^{<\gamma}$ sequence to κ*

which avoids E. Hence κ is not γ-reflecting.