Several values in Cichoń's diagram

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We are interested in *the meager ideal* $\mathcal{M}$ and *the null ideal* $\mathcal{N}$ on $\mathbb{R}$.

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(3) and (4) by **Bartoszynski (1987)**.

Cichoń's diagram

Inequalities: **Bartoszyński, Fremlin, Miller, Rothberger, Truss**.
Completeness: **Bartoszyński, Judah, Miller, Shelah**.

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r, dashed] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u, dashed] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r, dashed] \arrow[u, dashed] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Also $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

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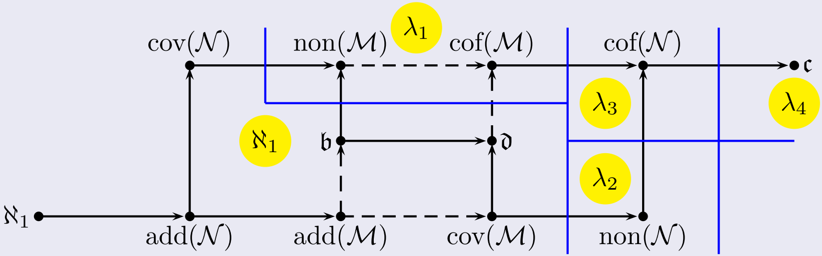
- *csi of proper forcing* only allows to assign $\aleph_1$ and $\aleph_2$.
- Many models are obtained from *FS (finite support) iterations of ccc posets*, but such an iteration forces $\mathrm{non}(\mathcal{M}) \leq \mu \leq \mathrm{cov}(\mathcal{M})$ where μ is the cofinality of the length of the iteration (when μ has uncountable cofinality).

A non FS example

Theorem (A. Fischer, Goldstern, Kellner and Shelah)

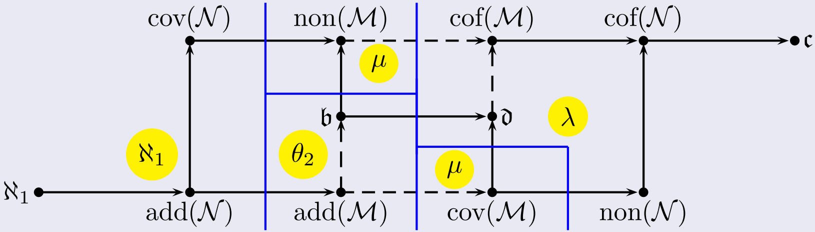
If $\aleph_1 < \lambda_1, \lambda_2 < \lambda_3 < \lambda_4$ *are pairwise distinct cardinals such that* $\lambda_i^{\aleph_0} = \lambda_i$ *for* $i = 1, 2, 3, 4$, *then it is consistent that*



Consistency examples (1)

Theorem (Brendle; Judah-Shelah's FS techniques 1990's)

If $\theta_2 \leq \mu$ are uncountable regular cardinals and $\lambda \geq \mu$ such that $\lambda^{<\theta_2} = \lambda$, then it is consistent that



Sketch

Perform a FS iteration of length $\lambda\mu$ (ordinal product) using

(i) $\mathbb{E}$ (standard ccc poset that adds and eventually diferent real in ω^ω)

$$\begin{tikzcd} V = V_0 \arrow[r, dashed] & V_{\lambda\alpha} \arrow[r, "\mathbb{E}"] & V_{\lambda\alpha+1} \end{tikzcd}$$

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$$V = V_0 \dashrightarrow V_{\lambda\alpha} \xrightarrow{\mathbb{E}} V_{\lambda\alpha+1} \xrightarrow{\mathbb{D}^{N_{\lambda\alpha+1}}} V_{\lambda\alpha+2} \xrightarrow{\mathbb{D}^{N_{\lambda\alpha+2}}} V_{\lambda\alpha+3} \dashrightarrow V_{\lambda(\alpha+1)}$$

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How to obtain $\mathfrak{b} \leq \theta_2$, $\mathrm{cov}(\mathcal{N}) = \aleph_1$, and the other equalities in $V_{\lambda\mu}$?

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- $V_{\lambda\mu} \models \mathrm{cov}(\mathcal{N}) = \aleph_1$. In V_{ω_1} it is added a family of null sets of size $\aleph_1$ that covers the reals, and this family still covers in $V_{\lambda\mu}$.

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- $V_{\lambda\mu} \models \lambda \leq \mathrm{non}(\mathcal{N})$.
- $V_{\lambda\mu} \models \mathrm{cov}(\mathcal{M}) \leq \mu \leq \mathrm{non}(\mathcal{M})$ because of the eventually different reals added by $\mathbb{E}$.

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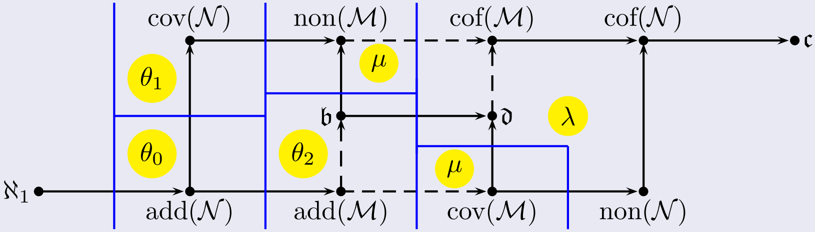
- $V_{\lambda\mu} \models \lambda \leq \mathrm{non}(\mathcal{N})$.
- $V_{\lambda\mu} \models \mathrm{cov}(\mathcal{M}) \leq \mu \leq \mathrm{non}(\mathcal{M})$ because of the eventually different reals added by $\mathbb{E}$.
- $V_{\lambda\mu} \models \mathrm{non}(\mathcal{M}) \leq \mu \leq \mathrm{cov}(\mathcal{M})$ because of the Cohen reals added at limit stages.

Consistency examples (2)

Theorem (From Brendle; Judah-Shelah's FS techniques 1990's)

If $\theta_0 \leq \theta_1 \leq \theta_2$ are uncountable regular cardinals and $\lambda^{<\theta_2} = \lambda$, then it is consistent that



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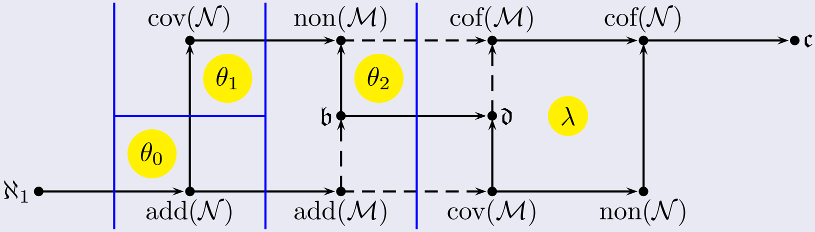
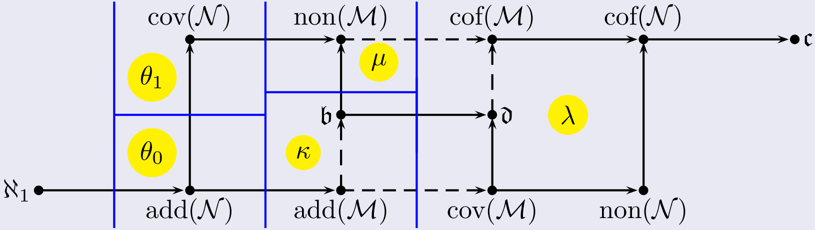


Consistency examples (4)

Theorem (Goldstern - M. - Shelah 2016)

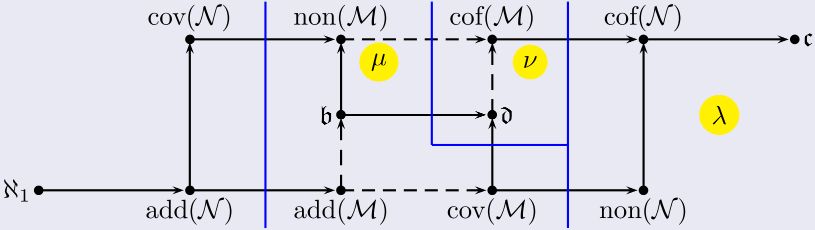
Let $\theta_0 \leq \theta_1 \leq \kappa = \kappa^{\aleph_0} \leq \mu = \mu^{\aleph_0}$ *be uncountable regular cardinals,* $\mu < \lambda = \lambda^{<\mu} \leq 2^\kappa$. *Then, there is a ccc poset forcing*



Consistency examples (5)

Theorem (M. 2013)

Let $\mu \leq \nu$ *be uncountable regular cardinals,* $\nu \leq \lambda$ *such that* $\lambda^{\aleph_0} = \lambda$. *Then, there is a ccc poset forcing*



Consistency examples (6)

Theorem (Fischer - Friedman - M. - Montoya)

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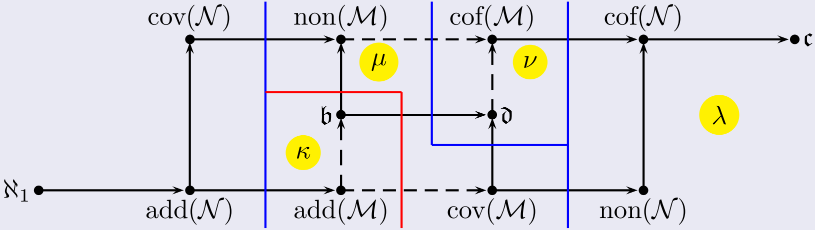

Coherent FS iterations

Definition

Let M be a transitive model of ZFC^*. $\mathbb{P} \in M$ and $\mathbb{Q}$ posets. We say that $\mathbb{P}$ *is a complete suborder of* $\mathbb{Q}$ *with respect to* M, denoted by $\mathbb{P} \lessdot_M \mathbb{Q}$, if $\mathbb{P} \subseteq \mathbb{Q}$ and any maximal antichain of $\mathbb{P}$ in M is also a maximal antichain of $\mathbb{Q}$.

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If $N \supseteq M$ is a transitive model of ZFC^* and $\mathbb{Q} \in N$, $\mathbb{P} \lessdot_M \mathbb{Q}$ implies that, if G is $\mathbb{Q}$-generic over N then $G \cap \mathbb{P}$ is $\mathbb{P}$-generic over M and $M[G \cap \mathbb{P}] \subseteq N[G]$.

$$N \xrightarrow[\mathbb{Q}]{} N[G]$$

$$M \xrightarrow[\mathbb{P}]{} M[G \cap \mathbb{P}]$$

Coherent FS iterations

Lemma

In the context of the previous definition, assume that $\dot{\mathbb{P}}' \in M$ is a $\mathbb{P}$-name and $\dot{\mathbb{Q}}' \in N$ is a $\mathbb{Q}$-name, both of posets. If $\mathbb{P} \lessdot_M \mathbb{Q}$ and $\mathbb{Q}$ forces (over N) that $\dot{\mathbb{P}}' \lessdot_{M^{\mathbb{P}}} \dot{\mathbb{Q}}'$, then $\mathbb{P} * \dot{\mathbb{P}}' \lessdot_M \mathbb{Q} * \dot{\mathbb{Q}}'$.

$$\begin{tikzcd} N \arrow[r, "\mathbb{Q}"'] & N^{\mathbb{Q}} \arrow[r, "\mathbb{Q}'"'] & N^{\mathbb{Q}*\dot{\mathbb{Q}}'} \\ M \arrow[r, "\mathbb{P}"'] & M^{\mathbb{P}} \arrow[r, "\mathbb{P}'"'] & M^{\mathbb{P}*\dot{\mathbb{P}}'} \end{tikzcd}$$

Coherent FS iterations

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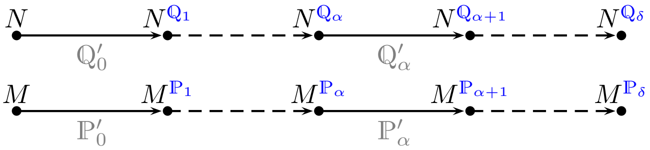
Lemma (Brendle-Fischer 2011)

Let $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}'_\alpha : \alpha < \delta \rangle$ and $\mathbb{Q}_\delta = \langle \mathbb{Q}_\alpha, \dot{\mathbb{Q}}'_\alpha : \alpha < \delta \rangle$ be FS iterations in M and N, respectively. If $\mathbb{P}_\alpha \lessdot_M \mathbb{Q}_\alpha$ and $\Vdash_{\mathbb{Q}_\alpha, N} \dot{\mathbb{P}}'_\alpha \lessdot_{M^{\mathbb{P}_\alpha}} \dot{\mathbb{Q}}'_\alpha$ for all $\alpha < \delta$, then $\mathbb{P}_\delta \lessdot_M \mathbb{Q}_\delta$



Preservation of unbounded reals

Let $M \subseteq N$ be transitive models of ZFC^*, $c \in \omega^\omega \cap N$ *unbounded over* M (that is, no member of $\omega^\omega \cap M$ dominates c) and a coherent pair of FS iterations as below.

$$\begin{tikzcd}
c \in N \arrow[r, "\mathbb{Q}'_0"'] & N^{\mathbb{Q}_1} \arrow[r, dashed] & N^{\mathbb{Q}_\alpha} \arrow[r, "\mathbb{Q}'_\alpha"'] & N^{\mathbb{Q}_{\alpha+1}} \arrow[r, dashed] & N^{\mathbb{Q}_\delta} \\
M \arrow[r, "\mathbb{P}'_0"'] & M^{\mathbb{P}_1} \arrow[r, dashed] & M^{\mathbb{P}_\alpha} \arrow[r, "\mathbb{P}'_\alpha"'] & M^{\mathbb{P}_{\alpha+1}} \arrow[r, dashed] & M^{\mathbb{P}_\delta}
\end{tikzcd}$$

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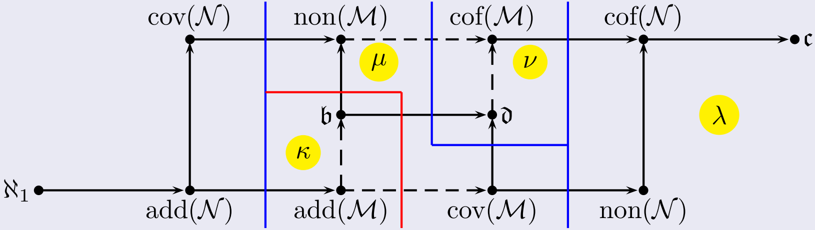
M \arrow[r, "\mathbb{P}'_0"'] & M^{\mathbb{P}_1} \arrow[r, dashed] & M^{\mathbb{P}_\alpha} \arrow[r, "\mathbb{P}'_\alpha"'] & M^{\mathbb{P}_{\alpha+1}} \arrow[r, dashed] & M^{\mathbb{P}_\delta}
\end{tikzcd}$$

We are interested in preserving c unbounded, i.e., to obtain c unbounded over $M^{\mathbb{P}_\delta}$. The relevant theory is known from **Blass-Shelah 1984; Brendle-Fischer 2011; M. 2013**.

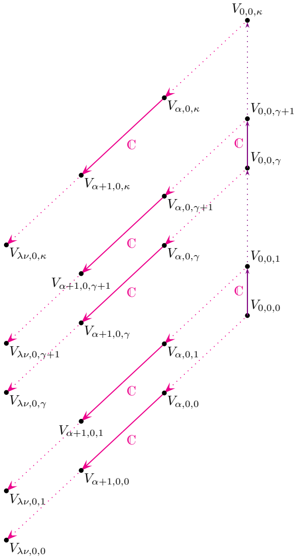
Consistency examples (6)

Theorem (Fischer - Friedman - M. - Montoya)

Let $\kappa \leq \mu \leq \nu$ be uncountable regular cardinals, $\nu \leq \lambda$ such that $\lambda^{\aleph_0} = \lambda$. Then, there is a ccc poset forcing



$$
\begin{tikzcd}
V_{0,0,\kappa} \\
V_{0,0,\gamma+1} \arrow[u, dotted] \\
V_{0,0,\gamma} \arrow[u, "\mathbb{C}"] \\
V_{0,0,1} \arrow[u, dotted] \\
V_{0,0,0} \arrow[u, "\mathbb{C}"]
\end{tikzcd}
$$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$V_{0,0,\gamma+1}$
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$\mathbb{C}$
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$V_{0,0,1}$
$V_{\alpha+1,0,\gamma+1}$
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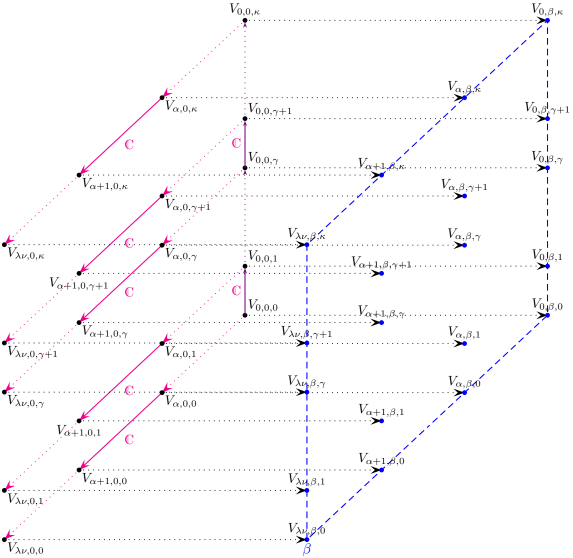
$V_{\alpha+1,0,\gamma}$
$V_{\lambda\nu,0,\gamma+1}$
$V_{\alpha,0,1}$
$\mathbb{C}$
$V_{\lambda\nu,0,\gamma}$
$V_{\alpha,0,0}$
$V_{\alpha+1,0,1}$
$\mathbb{C}$
$V_{\alpha+1,0,0}$
$V_{\lambda\nu,0,1}$
$V_{\lambda\nu,0,0}$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$\mathbb{C}$
$\mathbb{C}$
$V_{0,0,\gamma}$
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$V_{\alpha+1,0,\kappa}$
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$V_{\alpha+1,\beta,\gamma+1}$
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$\mathbb{C}$
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$V_{\alpha+1,\beta,0}$
$V_{\alpha+1,0,0}$
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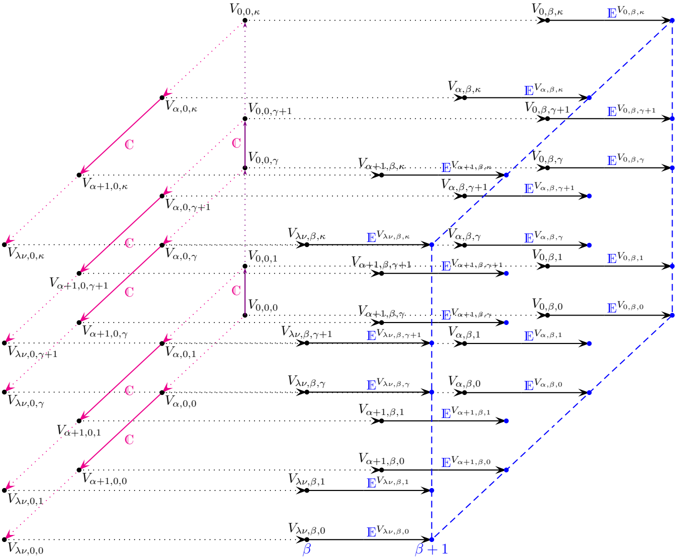
$V_{\lambda\nu,\beta,0}$
$V_{\lambda\nu,0,0}$
β



$V_{0,0,\kappa}$
$V_{0,\beta,\kappa}$
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$V_{\alpha,0,\kappa}$
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$\mathbb{E}^{V_{\alpha,\beta,\kappa}}$
$V_{0,0,\gamma+1}$
$V_{0,\beta,\gamma+1}$
$\mathbb{E}^{V_{0,\beta,\gamma+1}}$
$\mathbb{C}$
$V_{0,0,\gamma}$
$V_{\alpha+1,\beta,\kappa}$
$\mathbb{E}^{V_{\alpha+1,\beta,\kappa}}$
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$\mathbb{E}^{V_{\lambda\nu,\beta,\kappa}}$
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$\mathbb{E}^{V_{\alpha,\beta,\gamma}}$
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$V_{\alpha+1,0,1}$
$V_{\alpha+1,\beta,0}$
$\mathbb{E}^{V_{\alpha+1,\beta,0}}$
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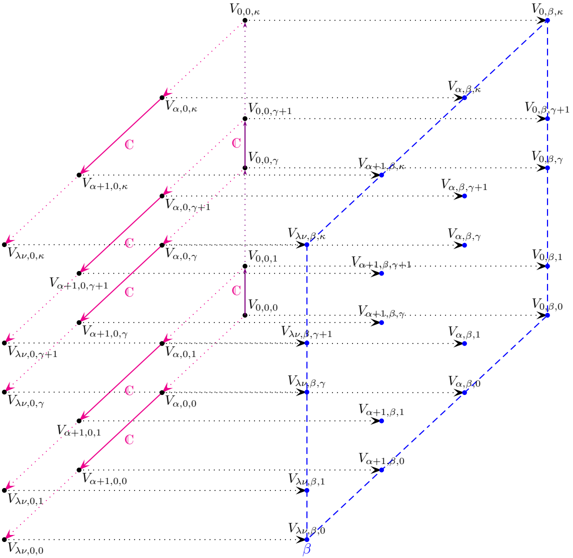
$\mathbb{E}^{V_{\lambda\nu,\beta,0}}$
$V_{\lambda\nu,0,0}$
β
$\beta+1$



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$\mathbb{C}$
$V_{0,0,\gamma}$
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β
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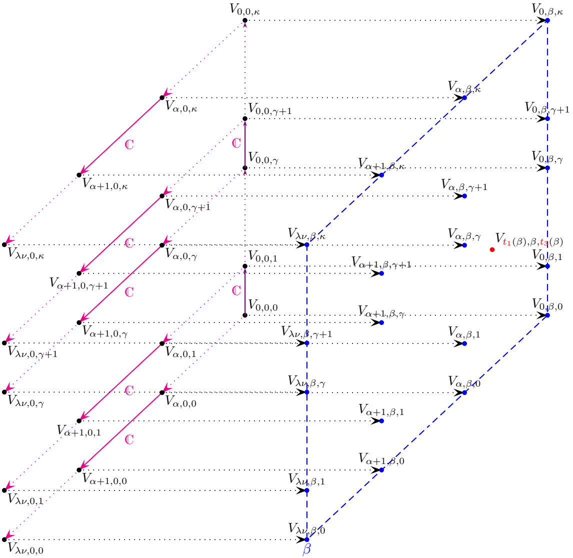
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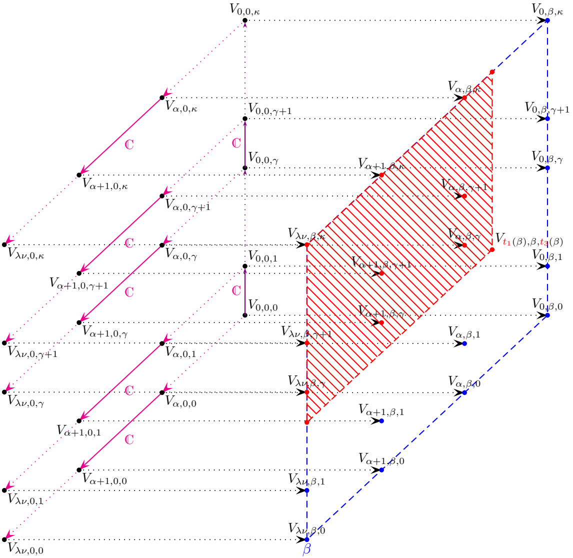
β



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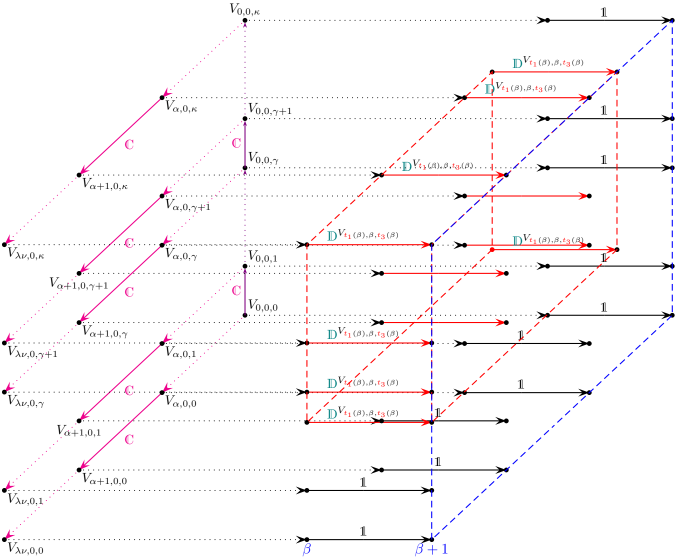
$V_{\lambda\nu,0,0}$



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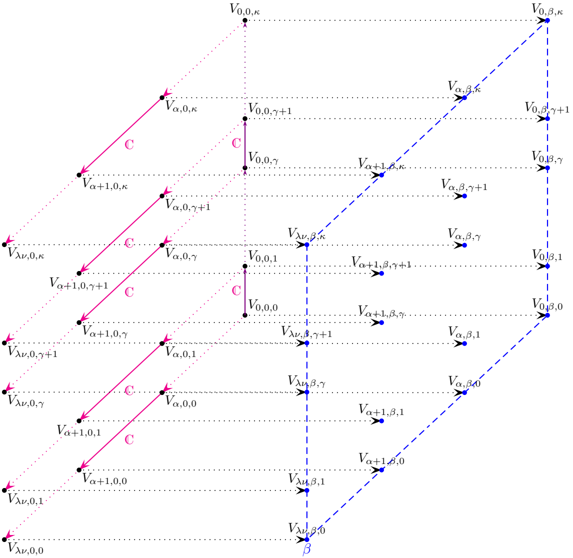
$\mathbb{D}^{V_{t_1(\beta),\beta,t_3(\beta)}}$
$\mathbb{1}$
β
$\beta+1$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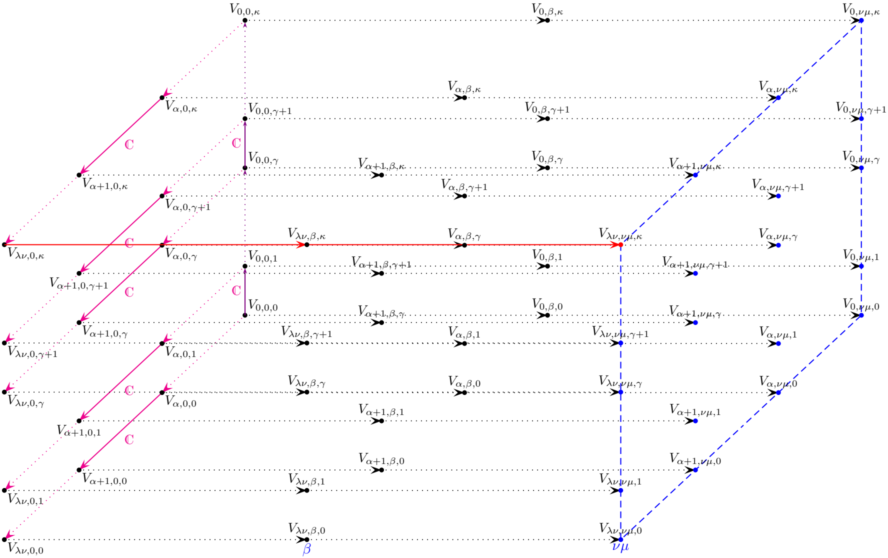
β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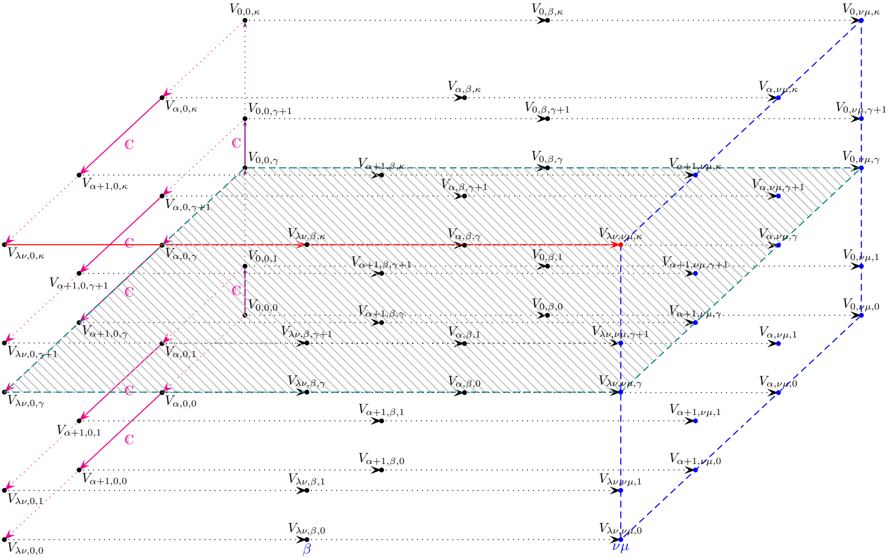
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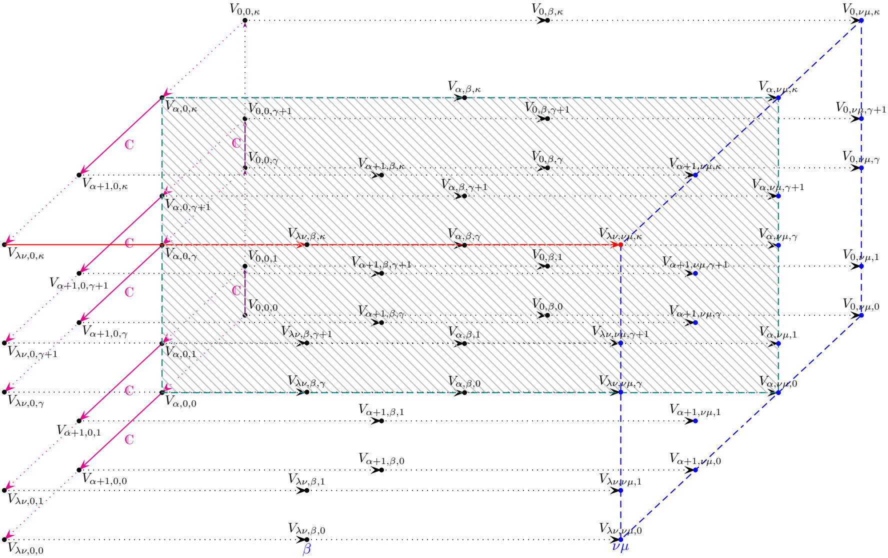
β
$\nu\mu$



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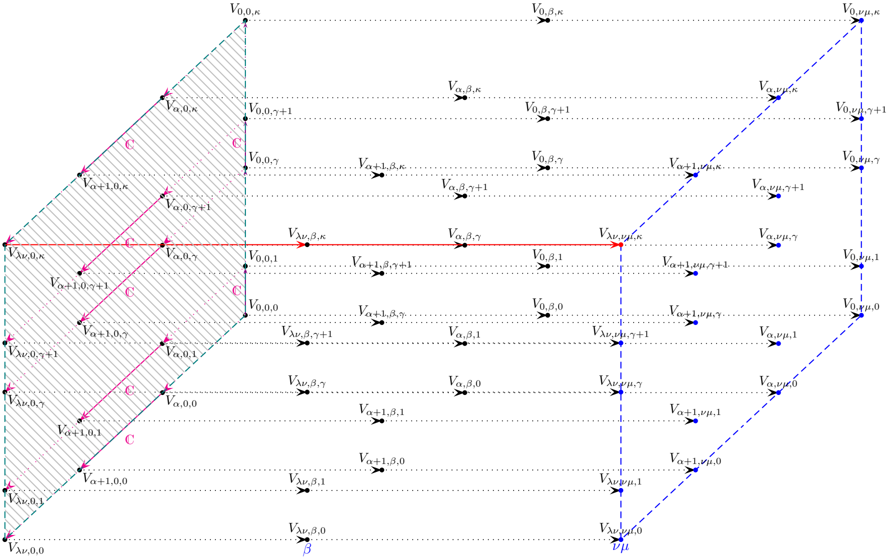
β
$\nu\mu$



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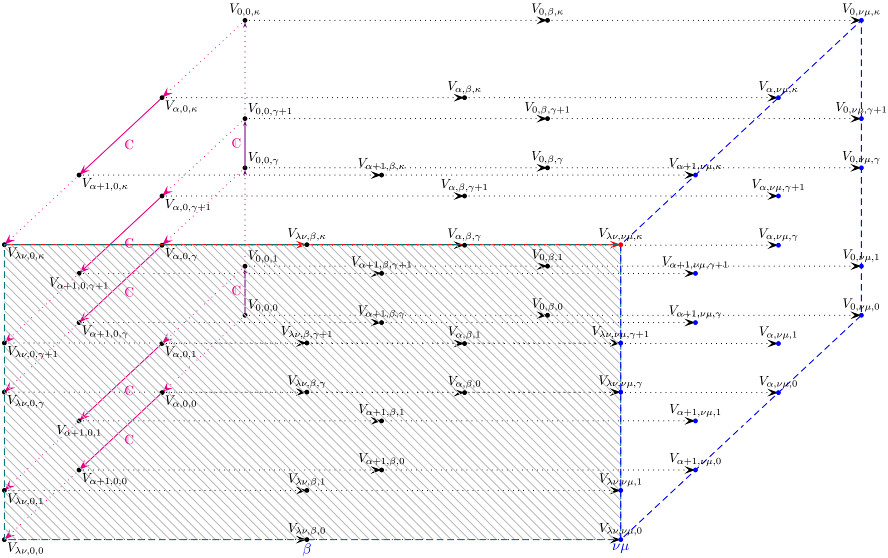
β
$\nu\mu$



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β
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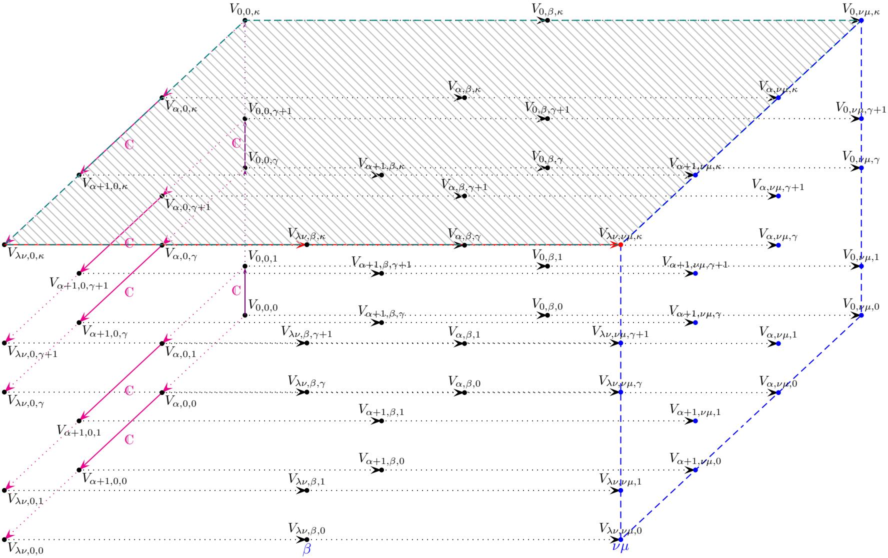
$\mathbb{C}$



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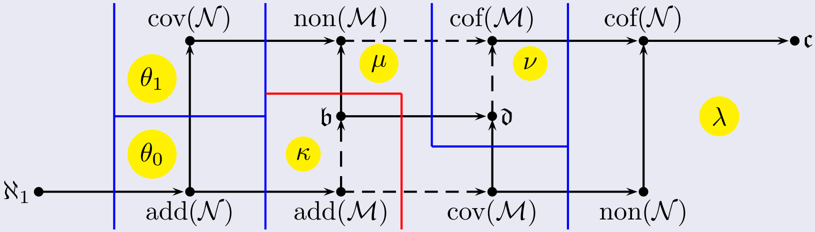
$V_{\lambda\nu,\beta,0}$
$V_{\lambda\nu,\nu\mu,0}$
$\mathbb{C}$
β
$\nu\mu$

Consistency examples (7)

Theorem (Fischer - Friedman - M. - Montoya)

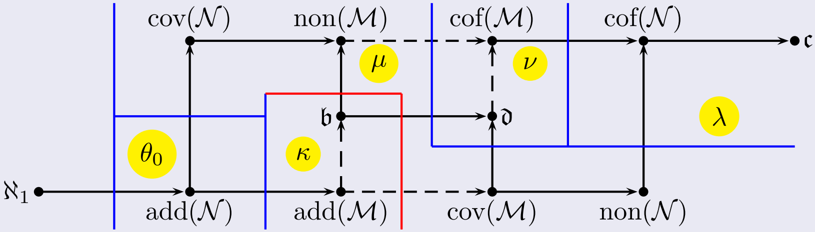
Let $\theta_0 \leq \theta_1 \leq \kappa \leq \mu \leq \nu$ be uncountable regular cardinals, $\nu \leq \lambda$ such that $\lambda^{<\theta_1} = \lambda$. Then, there is a ccc poset forcing



Consistency examples (8)

Theorem (Fischer - Friedman - M. - Montoya)

Let $\theta_0 \leq \kappa \leq \mu \leq \nu$ be uncountable regular cardinals, $\nu \leq \lambda$ such that $\lambda^{<\theta_0} = \lambda$. Then, there is a ccc poset forcing



The almost disjointness number

Theorem (Fischer - Friedman - M. - Montoya)

By slightly modifying the construction of the previous examples (except Goldstern - M. - Shelah), it can be forced, additionally, $\mathfrak{b} = \mathfrak{a}$.

The almost disjointness number

Theorem (Fischer - Friedman - M. - Montoya)

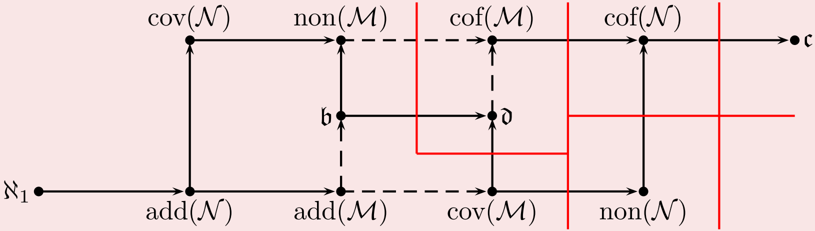
By slightly modifying the construction of the previous examples (except Goldstern - M. - Shelah), it can be forced, additionally, $\mathfrak{b} = \mathfrak{a}$.

Based in the theory of **Brendle-Fischer (2011)** to preserve mad families in matrix iterations.

Question (1)

Question

Is it consistent with ZFC that $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N})$?



Question (2)

Question

Is it consistent with ZFC that $\mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{c}$?