Dynamics of Duality:
How Duality Emerges, Changes, and Breaks

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Outline

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Wittgenstein's Conception of Space

Wittgenstein gives a fresh look at the issue of the relationships between space and points:

> *What makes it apparent that space is not a collection of points, but the realization of a law? (Philosophical Remarks, p.216)*

Wittgenstein's intensional view on space is a compelling consequence of his persistent disagreement with the set-theoretical extensional view of mathematics:

> *Mathematics is ridden through and through with the pernicious idioms of set theory. One example of this is the way people speak of a line as composed of points. A line is a law and isn't composed of anything at all. (Philosophical Grammar, p.211)*

Duality exists b/w point-set and point-free spaces (cf. Newton vs. Leibniz).

Hilbert's Programme

Coquand et al. assert:

> *A partial realisation of Hilbert's programme has recently proved successful in commutative algebra [..] One of the key tools is Joyal's point-free version of the Zariski spectrum as a distributive lattice [..] (Spectral schemes as ringed lattices, 2009)*

In this paper they contrive a constructive version of Gro.'s schemes.

- In my view, $\mathrm{Spec} : \mathbf{Alg} \to \mathbf{Sp}$ is the introduction of ideal elements in Hilbert's sense; its adjoint functor is their elimination.

Duality has contributed to Hilbert's programme and constructivism. The point-free Tychonoff theorem is AC-free; this is classic. Yet the state-of-the-art goes far beyond it. Not just general topology.

The Duality Picture, Mathematically

The duality picture:

	Ontic	Epistemic	
Complex Geometry	*Complex Surface*	*Function Field*	Riemann
Algebraic Geometry	*Variety/Scheme*	*k-Algebra/Ring*	Hilbert-Grothendieck
Representation Th.	*Group*	*Representations*	Pontryagin-Tannaka
Topology	*Topological Space*	*Algebra of Opens*	Isbell-Papert
Convex Geometry	*Convex Space*	*Semantic Domain*	M. (2011; 2013)
Logic	*Space of Models*	*Algebra of Theories*	Stone
Computer Science	*System*	*Observable Properties*	Abramsky-Smyth
System Scicence	*Controllability*	*Observability*	Kalman
Quantum Physics	*State Space*	*Alg. of Observables*	von Neumann

There are a variety of duality theories available to unite them.
Conceptually, they are all *ontic-epistemic* dualities.

Lawvere on Categories and Philosophy

In Lawvere's terms, duality arises between *the formal* and *the conceptual*, which, he believed, is relevant to Hegelian dialectics.

> *[A]dvances forged by category theorists will be of value to dialectical philosophy, lending precise form [..] to ancient philosophical distinctions such as [..] objective vs. subjective, being vs. becoming, space vs. quantity [..] (Lawvere 1992)*

Categorical philosophy, if not dialectics, is gradually growing in both continental and analytic traditions (e.g., A. Rodin and J. Ladyman).

- I argued for "categorical logical positivism" and "pluralistic unified science" as its goal in my recent *Synthese* paper.

The Duality Picture, Philosophically

A philosophical perspective on the ontic-epistemic duality, or put another way, the realism-antirealism duality (cf. Dummett):

	Ontic	***Epistemic***	
Descartes	*Matter*	*Mind*	Cartesian Dualism
Kant	*Thing-in-itself*	*Appearance*	Idealism
Cassirer	*Substance*	*Function*	Logical Idealism
Heidegger	*Essence*	*Existence*	Analysis of *Dasein*
Whitehead	*Reality*	*Process*	Holism/Organicism
Wittgenstein	*World*	*Language*	Logical Philosophy
Searle	*Intentionality*	*Simulatability*	Philosophy of Mind
Dummett	*Truth*	*Verification*	Theory of Meaning

The ontic and the epistemic could be united, at least in the case of Dummett's dualism b/w truth and verification conditional semantics.

Stone-type Dualities

The algebra of propositions is dual to the space of models:

- Classical logic: 0-dim. compact T_2 spaces.
  - Props. are closed opens, for which LEM holds.
- Intuitionistic logic: non-T_2 sp. Compact sober sp. such that its compact opens form a basis, and the interiors of their boolean combinations are compact (M.-Sato; cf. Lurie's HTT).
  - Props. are compact opens. The topo. meaning of LEM is 0-dim.
- Modal logic: Vietoris coalgebra over space.
  - Modal operators amount to Vietoris hyperspaces.
  - Abramsky-Kupke-Kurz-Venema duality; see my JPAA paper.

The existence of unit ensures duals spaces are compact; otherwise they are locally compact. The same holds for Gelfand duality as well.

Stone-type Dualities (cont'd)

- First-order logic. Two approaches: topological groupoids (spaces of models with automorphisms) and indexed/fibrational Stone spaces (duals of Lawvere hyperdoctrines).
  - Higher-order logic. Indexed Stone sp. still work (duals of triposes).
- Infinitary logic. Not even locally compact spaces. Adjunctions.
  - May not be enough models/points to separate non-equiv. props.
  - No need for AC due to infinitary operations, i.e., no need to reduce infinitaries on the topological side into finitaries on the algebraic.
- Many-valued logics. It depends. Rational polyhedra for Ł. Mostly subsumed under the framework of dualities induced by Ω (or Chu sp.), which may be multiple truth values (we shall get back to this).

Outline

1. Introduction
   - Wittgenstein's conception of space
   - Hilbert's programme via duality
   - A bird's-eye view of logical dualities
2. Duality in Logic and Algebraic Geometry
   - Completeness as duality
   - Nullstellensatz as duality
   - Completeness $\simeq$ Nullstellensatz
3. Answering the Three Questions
   - Duality theories via CT and UA
   - Chu duality theory
   - When duality breaks

Completeness as Duality

Logical Completeness:

- $Form \circ Mod(T) = T$.
  - $Form(\mathcal{M}) :=$ the formulae valid in any $M \in \mathcal{M}$.
  - $Form \circ Mod(T)$ = the formulae valid in any model of T.
- Nullstellensatz: $I \circ V(J) = \sqrt{J}$.
  - V gives zeros; I gives polynomials vanishing on varieties.
  - No $\sqrt{\ }$ in logic? If $\varphi \wedge \varphi$ is refutable, so is φ (contraction).
- It tells us order-theoretic duality b/w theories and models.
  - $T \subset T'$ iff $Mod(T) \supset Mod(T')$. The same holds for *Form*.
  - *Form* and *Mod* give a pair of (dually) adjoint functors.
- Adding non-incl. arrows and equipping $Mod(T)$ with a topology, it becomes Stone duality.

Nullstellensatz as Duality

Hilbert Nullstellensatz:

- $I \circ V(J) = \sqrt{J}$ for an ideal $J \subset k[x_1, ..., x_n]$ with k an ACF.
  - $\sqrt{J} := \{p \mid \exists n \in \mathbb{N}\ p^n \in J\}$. J is a radical ideal iff $\sqrt{J} = J$.
- It tells us order-theoretic duality b/w the radical ideals of $k[x_1, ..., x_n]$ and the affine varieties over k.
  - By adding more arrows, it becomes cat. equiv. b/w finitely generated reduced k-algebras and affine varieties over k.
- This extends to the scheme-theoretical duality.

Theories $=$ Ideals. Models $=$ Zeros. This is known since Joyal.

- My claim: Completeness $=$ Nullstellensatz;
  Stone duality $=$ Hilbert duality, both up to $\mathbb{GF}(p^n)$.

Dualities in Logic and Algebraic Geometry

Completeness: $T =$ the formulas valid on $Mod(T)$.
Nullstellensatz: $J =$ the polynomials vanishing on $V(J)$
where J is a radical ideal (this always holds in a Bool. ring).

- The correspondence looks clear, and yet there are some complications involved.
- Models are in Ω^κ and varieties are in k^n.
- $\mathbb{F}_2$ in logic and $\bar{\mathbb{F}}_2$ in the corresponding geometry.

Completeness from Nullstellensatz

Assume $v(\varphi(x_1, ..., x_n)) = 0$ for any $\{0, 1\}$-valuation v.

- We can naturally consider $\varphi \in \mathbb{F}_2[x_1, ..., x_n]$ by rewriting φ using 0,1, XOR (addition) and AND (multiplication) only.

Then, $\varphi \in I\,(\mathbb{F}_2^n) = I \circ V(J)$ where $J := \langle x_1^2 - x_1, ..., x_n^2 - x_n \rangle$.

- Nullstellensatz over $\bar{\mathbb{F}}_2$ tells us: $I \circ V(J) = \sqrt{J}$.

Hence, $\varphi = 0$ in $\mathbb{F}_2[x_1, ..., x_n]/\sqrt{J}$. This implies:

- $\neg\varphi$ is provable in any standard calculus for CL.
    - $\mathbb{F}_2[x_1, ..., x_n]/\sqrt{J}$ can be seen as a calculus and be shown to be equivalent w.r.t. provability to LK, NK, etc.

Completeness thus follows from Nullstellensatz over $\bar{\mathbb{F}}_2$.

Nullstellensatz from Completeness

Conversely, Nullstellensatz follows from Completeness.

- For example, consider two infinitary geometries over $\mathbb{F}_2$.
  - One is induced by infinite coordinates $(x_1, x_2, ..., x_n, ...)$.
  - The other is induced by infinitary multiplication.
- A Nullstellensatz-type thm. in the former follows from the (strong) completeness of classical propositional logic.
- A Nullstellensatz-type thm. in the latter follows from the completeness of infinitary logic (w.r.t. infinitary calculus).
  - We assume J in each Nullstellensatz contains $x_i^2 - x_i$.
  - Nullstellensatz can fail in some ∞-dim. geometries.

The link b/w completeness and Nullstellensatz extends to $\mathbb{GF}(p^n)$.

Further Interactions b/w Logic and Alg. Geom.

This allows us to go back and fourth b/w Logic and Alg. Geom. E.g.,

- A variety is irreducible iff its coord. ring is an integral dom. For a theory T, $Mod(T)$ is irreducible iff T is complete.
- A variety has a unique, minimal, irreducible decomposition. If T contains finitely many atomic propositions, $Mod(T)$ has a unique, minimal, irreducible decomposition.
- $X \subset Fml$ is satisfiable iff so is any finite subset of it. $X \subset \mathbb{F}_2[x_1, ..., x_n, ...]$ has a common zero iff so is any finite subset.
  - This is not trivial, since the ring is not Noetherian. Logic would contribute to ∞-dimensional alg. geometry.

Stone duality for $\mathbb{GF}(p^n)$-val. logic is Hilbert duality for geometry over $\mathbb{GF}(p^n)$.

Outline

Duality Induced by Janusian Objects

Lawvere says:

> *A potential duality arises when a single object lives in two different categories.*

Let Ω be in **Alg** and **Sp**, general categories of algebras and of spaces.

- If the harmony condition holds (in my formulation), there is a dual adjunction b/w **Alg** and **Sp**, given by $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega)$ and $\mathrm{Hom}_{\mathbf{D}}(\text{-}, \Omega)$.
  - Harmony basically means algebraic operations are continuous.
- This includes Gelfand duality ($\Omega = \mathbb{C}$) and Pontryagin duality ($\Omega = \{z \in \mathbb{C} \ ; \ |z| = 1$) as well. Ω may be multiple truth values.

This allowed me to solve an open problem by B. Jacobs on duality for algebras of the distribution monad (*Categorical Duality Theory*, 2013).

Natural Duality Theory

Term functions mean functions definable by given basic operations.

- In classical logic, $\mathrm{Term}(\mathbf{2}^n, \mathbf{2}) = \mathrm{Func}(\mathbf{2}^n, \mathbf{2})$, i.e., all functions are definable by the Boolean operations.
  - This is exactly the functional completeness of classical logic.
- Primal duality thm: if all functions are term functions of L, the cat. of L-algebras is dually equivalent to the cat. of Stone spaces.
  - $\mathbb{GF}(p^n)$ is primal, and $\mathbb{GF}(p^n)$-algebras are dual to Stone spaces. This is Hilbert duality for geometry over Galois fields.
- In intuitionistic logic, $\mathrm{Term}(\mathbf{2}^n, \mathbf{2}) = \mathrm{Cont}(\mathbf{2}^n, \mathbf{2})$ where $\mathbf{2}$ is the Sierpinski sp. I have shown if all continuous functions are term functions of L, L-algebras are dually equivalent to Heyting spaces.
  - I have also shown the modal or coalg. version of primal duality.

Lambek reduced Gelfand duality to an infinitary primal duality theorem.

Chu Space Theory

A Chu space (over **Set** or SMCC) is a triple $(X, Y, e : X \times Y \to \Omega)$.

- Logic: (Mod, Fml, e). $e(M, \varphi) = 1$ iff $M \models \varphi$. May be multi-val'd.
    - Induces Stone-type dualities via $\mathrm{Hom}(\text{-}, \Omega)$.
    - Can do the same with algebras and their prime/maximal *Spec*.
- Physics: $(State, Proj, e)$. $e(\varphi, P) = \langle \varphi | P | \varphi \rangle$ for normalised φ.
    - Allows to characterise symmetries (cf. Wigner thm).
    - Induces an operationalist form of state-observable duality.
- Chu duality theory allows us to unify logical and physical dualities.

Ref: M., From Operational Chu Duality to Coalgebraic Quantum Symmetry (2013), which builds upon Abramsky's "Big Toy Models" and Pratt's "Stone Gamut".

Chu Duality Theory

- There are a broad class of T_1-type dualities, a general theory of which is given in my Chu duality paper above.
  - Varieties over an ACF are not sober, but T_1.
  - This motivates us to develop T_1-type dualities.
  - Maximal Spec plays a crucial rôle, and accordingly morphisms must change so as to preserve maximal filters/ideals.
- The most basic one is the duality between T_1 spaces (with cont. maps) and coatomistic frames (with maximal homomorphisms).
  - This is, to be surprise, rarely mentioned in the literature, most accounts of locale theory dealing with the sober-type duality alone.
- The duality of operational QM is of T_1-type as well.

When Duality Breaks

Duality-breaking is caused by either an excess of the ontic (space) or an excess of the epistemic (algebra).

- Non-commutative geometry concerns an excess of the epistemic.
  - There are too many algebras, compared to topo. spaces, which are all commutative, and cannot capture the non-commutative realm.
  - There is an impossibility thm. by Heunen and von den Berg.

My remedy: given a non-commutative alg. or substructural logic (FL), take its commutative/structural part ($\mathrm{FL}_{ewc} = IL$), dualise it as a space, and equip it with a structure sheaf. Scheme-theoretical duality theory.

Concluding Remarks

How does duality emerge, change, and break after all?

- Emergence: when one object lives in two cats., *harmoniously*.
  - In my dual adjunctions b/w algebras and spaces, the harmony condition basically means the algebraic operations are continuous.
- Mutation: the more operations, the less structures on duals.
  - If you have all (resp. cont.) functions as your term operations, duals are the simplest (resp. Heyting). Functional completeness.
  - I also explained what we have to change in order to adapt the base Stone duality to an extended situation (intuitionistic, modal, etc.).
- Breaking: caused by either an excess of the epistemic or an excess of the ontic (e.g., non-commutativity).

This ends my talk. Thank you for your patience.