Computably enumerable structures: Domain dependence

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References:

1. with Gavryushkin and Stephan in APAL (2014).
2. with Gavryushkin, Stephan, Jain in TCS (2016).
3. with Turetsky, Semukhin, Fokina in JSL (2016).
4. with Miyasnikov in Trans of the AMS (2014).

Plan:

1. Motivation.
2. Definitions and examples.
3. Reducibility $\leq_C$.
4. Case studies.

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Motivation:

Quotient sets appear all over mathematics. Here is an example:

Theorem (Homomorphism Theorem):
For any countable algebra ***A*** *there exists an onto homomorphism from the term algebra* ***F*** *onto* ***A***

$$h : \mathbf{F} \rightarrow \mathbf{A}.$$

Hence, the algebra ***A*** *is isomorphic to* ***F****/E, where*

$$E = \{ (x,y) \mid h(x) = h(y)\}.$$

So, by the last part:

1. Elements of **A** are E-equivalence classes.
2. Operations of **A** are induced by operations of **F**.

How do we view the algebra A?

View **F** as a computable algebra

$$\mathbf{F}=(\omega; f_0, f_1, \ldots, f_k).$$

Representation Theorem:

*For every countable algebra **A** there is an equivalence relation E on ω such that **A** is isomorphic to the quotient algebra*

$$\boldsymbol{F}/E= (\omega/E; f_0, f_1, \ldots, f_k).$$

So, the *domain* of **A** is ω/E, and the operations of **A** *are induced by computable operations respecting* E.

Hence, computability-theoretic complexity of **A** hides not in its Atomic diagram but rather in E (the equality relation).

E-structures

Our interest is in structures with domain ω/E.

Definition 1: *An E-structure is of the form*

$$(\omega/E; f_1, \ldots, f_k, P_1, \ldots, P_m), \text{ where}$$

- *Each f_i is induced by a computable map respecting E.*
- *Each P_j is induced by a c.e. predicate respecting E.*

A structure is c.e. if it is an E-structure for some c.e. E.
An E-structure is an E-algebra if it has no predicates.

We often assume that E is a c.e. equivalence relation.

Examples

Example 1.

Every countable algebra is an E-algebra for some E.

Example 2 (Makanin).

Let S be the semi-group generated by **a**, **b**, **c** such that

ccbb = **bbcc**, **bcccbb** = **cbbbcc**, **accbb** = **bba**,
abcccbb = **cbba**, **bbccbbbbcc** = **bbccbbbbcca**

Let E be the word problem on S; E is a c.e. relation. View {**a**, **b**, **c**}* as ω. So, the domain of S is ω/E; the concatenation respects E, and E is undecidable.

The classes $K_E(C)$

Definition 2:

Given an equivalence relation E and a class C of structures, set

$$\boldsymbol{K_E(C)}$$

be the class of all E-structures (isomorphic to a structure) from C.

Definition 3:

*If a structure **A** belongs to $\boldsymbol{K}_E(C)$ then E realises **A**. Otherwise, we say that E omits **A**.*

What do these definitions tell us?

1. Let us fix E. The set

$K_E(C)$ = {**A** | **A** is in C and isomorphic to an E-structure}

represents the algebraic content of E.

2. Let us fix a class C. The set

K_C = {E | E realises all structures from C}

represent computability-theoretic content of C.

The class $K_E(C)$

Let C be the class of all structures. Consider:

$K_E(C)$ = {**A** | **A** is in C and is isomorphic to an E structure}.

Here are types of questions one might ask:

1. Does $K_E(C)$ contain a linear order?
2. Does $K_E(C)$ contain a finitely generated algebra?
3. Are there groups, rings or Boolean algebras in $K_E(C)$?
4. Can we say anything reasonable about structures in the class $K_E(C)$? Can we describe them?

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Example 1: Implications of non-computability

Let E be non computable equivalence relation. Then the class K_E**(**C**)** excludes the following structures:

- Finitely generated structures whose all nontrivial quotients are finite (Malcev).
- All structures with finitely many congruencies only, such as fields (Ershov).
- Noetherian rings (Bour).
- Finitely presented and residually finite algebras (Malcev, McKenzie).
- Complete infinite graph (Khoussainov, Stephan).

Example 2: Implications of an algebraic assumption

Assumption: The class K_E**(**C**)** possesses an algebra **A** whose all nontrivial quotients are finite. Then:

- Either E is computable or tr **(**E**)** is hyperimmune.
- If E is not computable then
  - (1) every E-algebra is locally finite.
  - (2) every E-algebra is residually finite.
  - (3) the language of the algebra **A** must contain a function symbol of airty > 1.

Example 3: Varia

1. If E is pre-complete then E realises no linear order.
2. If E realises a finitely branching directed tree, then each equivalence class is computable.
3. If any two distinct E-equivalence class are not recursively separable and E realises a linear order L, then L must be dense.

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Reducibility $\leq_C$

Definition 4: *Let C be a class of structures.*
Let E_1 and E_2 be c.e. equivalence relations.

Say that E_1 is C-reducible to E_2, written $E_1 \leq_C E_2$, if all structures in C realised by E_1 are also realised by E_2.

Say that E_1 and E_2 have the same C-degree, written $E_1 =_C E_2$, if $E_1 \leq_C E_2$ and $E_2 \leq_C E_1$.

The reducibility $\leq_C$ induces the partial order on the set of all C-degrees.

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Case Study 1: Linear orders

Let X be a co-infinte c.e. subset of ω. Consider

E(X)={(n,k) | n=k or both n, k are in X}.

Theorem 1:

- *E(X) realises a linear order L with X representing a isolated point of L iff X is recursive.*
- *E(X) realises a linear order with X being an end point iff X is semirecursive (also C. Jockusch).*
- *E(X) realises a linear order iff X is one-one reducible to the join of two c.e. semirecursive sets.*

Case Study 1: Linear Orders

Corollary: *If X is maximal, r-maximal, creative or simple but not hyper-simple then E(X) realises no linear order.*

Assume X is simple.

Theorem 2:

- *If X is not 1-to-1 redicible to a join of two semirecursive sets then E(X) realises no linear order.*
- *If X is semirecursive then E(X) realises the following linear orders:* $n+\omega$, ω^*+n, $\omega+1+\omega^*$.
- *If X is 1-to-1 reducible to a join of two semirecrsive sets then E(X) realises* $\omega+1+\omega^*$ *only.*

Beyond E(X)

Theorem 3:

For every $n>0$ theer exists a c.e. equivalence relation E that realises exactly n linearly ordered sets.

Corollary:

There exists a c.e. equivalence relation such that the only linear order realised by E is the order of rational numbers.

Case Study 2: Class Alg of algebras

Definition 5:

An algebra **A** is ***trivial*** if each operation of **A** is either a constant function or a projection.

We have the order $\leq_{\mathrm{Alg}}$ among equivalence relations.

Theorem 4:

1. *The order $\leq_{Alg}$ has a minimal element E. Moreover, E can be made computably enumerable.*
2. *The order $\leq_{Alg}$ has ω many maximal elements.*

Case Study 3: Isle graphs

Definition 6:
An ***isle*** is a countable graph that has infinitely many isolated points. If an isle has finitely many edges only then we call the isle **finitary**.

So, we can consider the partial order $\leq_{\text{Isle}}$.

Theorem 5:
The partial order $\leq_{Isle}$ has the least element. Any c.e. equivalence relation with cohesive transversal represents the minimal element.

Case Study 3: Isle graphs

Recall that $E_0 \leq_{FF} E_1$ if there exists a computable function f such that for all n,m we have

$$(n,m) \text{ is in } E_0 \text{ if and only if } (f(n), f(m)) \text{ is in } E_1 .$$

Theorem 6:

If $E_0 \leq_{FF} E_1$ then then $E_0 \leq_{Isle} E_1$. Hence, the partial order $\leq_{Isle}$ has the largest element.

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Atoms for partial order $\leq_{Isle}$

Theorem 7:
The partial order $\leq_{Isle}$ possesses a unique atom.

The proof uses the notion of e-state borrowed from the construction of maximal sets.

Case Study 4: Partition graphs

Definition 7:

A graph $G = (V, Edge)$ is a **partition graph** if there is a partition $A_0, A_1, \ldots$ Of V such that $\{x, y\} \in Edge$ iff no k exists for which $x, y \in A_k$.

We call $A_0, A_1, \ldots$ the anti-clique components of the graph. There are two trivial partition graphs:

- The complete graph.
- The graph whose all vertices are isolated.

Case Study 4: Partition graphs

Denote the class of partition graphs by *Part* .
So, we have the partial order $\leq_{Part}$.

Theorem 8:
The equivalence relation id_{ω} *is the largest element of the partial order* $\leq_{Part}$.

Theorem 9:
The pre-complete equivalence relation is the least element in the partial order $\leq_{Part}$.

Finitary partition graphs

Definition 8:

A partition graph is **finitary** if it possesses finitely many anti-clique components only.

Let *G* be a finitary partition graph. The isomorphism type of *G* is determined by:

1. The number of its infinite anti-cluqie components.
2. The number of its finite anti-clique components and their cardinalites.

Finitary partition graphs

Let ***F*** be the set of all *E* equivalence relations that realise finiatary partition graphs.

Definition 9:

An equivalence relation *E* has type (*n*, *m*) if *n* and *m* are the largest integers such that for all $1 \leq i < n$, $j < m$, *E* realises finitary partition graphs with *i* infinite components and *j* finite components.

Theorem 10:

For each n and m there exists an E of type (n, m).

Full description of F

Theorem 11:

*The partial order **F** is isomorphic to the two-dimensional grid-order*

$$(\{(n,m) \mid n,m \text{ are in } \omega\} \cup \{\omega\}; \leq),$$

where ≤ is the component-wise order on the set of pairs.

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Open Problem(s)

Select your favorite class C of structures (e.g. n-ary trees, planar graphs, groups, rings, semigroups, lattices, Boolean algebras).

- Study C-reducibility for these classes.

- Study degrees of E that realise all structures from C.

- Let E be an equivalence relations. Describe structures from class C that are realised by E.

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