Wandering around a corner of Axiomatic Set Theory

Deepest Appreciations to Dr. Kakuda

By Yasuo Kanai
Yamato University

Bibliography

① Introduction to Mathematical Logic, Elliott Mendelson, Chapman and Hall/CRC

② The Consistency of the Continuum Hypothesis. Gödel, K. ,Princeton University Press, Princeton, (1940).

③ Some strong axioms of infinity incompatible with the axiom of contructibility, Frederick Rowbottom, Annals of Mathematical Logic, Volume 3, No. 1 (1971) pp. 1-44

④ Boolean-valued Models and Independence Proofs in Set Theory, J.L.Bell , Oxford University Press (1978/1/12)

⑤ Set Theory (The Third Millennium Edition, revised and expanded), Thomas Jech, Springer Monographs in Mathematics

⑥ Precipitous ideals, T. Jech, M. Magidor, W. Mitchell and K. Prikry, Journal of Symbolic Logic 45 (1980), pp.1–8

1. Ideals on Cradinals

Definition 1. Let κ be any regular uncountable cardinal and I a subset of $P(\kappa)$.

I is said to be a ***non-trivial ideal on κ*** if it satisfies that

1) $\{\xi\} \in I$, for each $\xi < \kappa$,
2) $X, Y \in I$ implies $X \cup Y \in I$,
3) $X \in I$, $Y \in P(\kappa)$ implies $X \cap Y \in I$.

The ***dual filter*** of I is denoted by I^*.

Definition 2. Let I be a non-trivial ideal on a regular uncountable cardinal κ.

Let λ be any cardinal.

(1) I is said to be ***λ-complete***
if for any family $\{ X_\xi \in I \mid \xi < \mu \}$ of cardinality $\mu < \lambda$,
$\cup X_\xi$ is in I.

(2) I is said to be ***normal***
if for each family $\{ X_\xi \in I \mid \xi < \kappa \}$ of I,
$\nabla X_\xi = \{ \alpha < \kappa \mid \mathit{for\ some}\ \beta < \alpha, \alpha \in X_\beta \}$ is in I.

(3) Moreover, if $X \in I$ *or* $\kappa - X \in I$ for each $X \subseteq \kappa$, I is said to be a ***prime ideal*** and the dual filter said to be ***an ultrafiler on κ.***

Denition 3. Given an ultrafilter U on I and L-structures A_i, $i \in I$, the ***ultraproduct*** $\Pi_U A_i$ is the unique L-structure B such that:

(1) The universe of B is the set $B = \Pi_U A_i$.

(2) For each atomic formula $\varphi(x_1, \ldots, x_k)$ which has at most one symbol from the vocabulary L, and each $f_1, \ldots, f_k \in \Pi_{i \in I} A_i$,

$B \vDash \varphi(f_{1U}, \ldots, f_{kU})$ iff $\{i \in I \mid A_i \vDash \varphi(f_1(i), \ldots, f_k(i))\} \in U$.

The ***ultrapower*** of an L-structure A modulo U, denoted by $\mathrm{Ult}_I(A,U)$, is defined as the ultraproduct $\Pi_U A = \Pi_U A_i$ where $A_i = A$ for each $i \in I$.

Definition 4. (Solovay ⑦) Suppose that I is an ideal on κ. Then, $P(\kappa)/I$ is a Boolean algebra. If we force with $P(\kappa)/I$ (without the zero element) then we get a V-ultrafilter $G_I \subseteq P(\kappa)$.

With this ultrafilter we can take the ultraproduct $\mathrm{Ult}_{\check{\kappa}}(\check{V}, G_I)$ using functions $f \in ({}^{\kappa}V)^{\check{}}$ in $\check{V}$. This gives us a generic elementary embedding

$j : V \to \mathrm{Ult}_{\check{\kappa}}(\check{V}, G_I)$.

An ideal I is ***precipitous*** if this generic ultrapower is always well-founded.

Results

Theorem 1. (7) Let λ be any cardinal $\leq \kappa^+$ and I a κ-complete non-trivial ideal on κ.
Then the following are equivalent.

(1) I is λ-saturated.

(2) Each member of $\mathrm{Ult}_{\check{\kappa}}(\check{V}, G_I)$ can be represented by a functional of cardinality less than λ.

(3) Each ordinal less than $j(\mathrm{sat}(I)\check{\ })$ in $\mathrm{Ult}_{\check{\kappa}}(\check{V}, G_I)$ can be represented by a functional of cardinality less than λ.

In the above, a functional F is a set of functions such that the set $\{ \, dom(f) \mid f \in F \, \}$ is I-disjoint.

And, j is the canonical elementary embedding of $\check{V}$ into $\mathrm{Ult}_{\check{\kappa}}(\check{V}, G_I)$ in $V[G_I]$, and sat(I) is the least cardinal μ such that I is not μ-saturated.
Recall that $\check{x}$ is a P-name for x in the ground model for any notion of forcing P.
Since each functional of cardinality $\leq \kappa$ is equal to an ordinary function in the generic extension $V[G_I]$,
we can have that:

Corollary. (7) A κ-complete non-trivial ideal I is κ^+-saturated if and only if each ordinal in $\mathrm{Ult}_{\check{\kappa}}(\check{V}, G_I)$ can be represented by an ordinary function in V.

＝ 集合論のはじまり ＝

カントール 1

三角級数の表す関数の一意性を証明する (1870)

不連続点からなる集合 A を含む関数の三角級数表現の一意性(1872)

α次導集合(1880)
(derived set)

実数全体の非可算性(1873)

超限順序数,基数(1895,1897)

超越数全体の非可算性

ハウスドルフ
非可算順序型
(1908)

巨大基数理論の起こり

ボレル集合の超限階層(1905)

ベール関数
(1899)

カントール　2

リーマン積分

α次導集合
(1872)

連続体は非可算
(1874)

ジョルダン測度

連続体仮説
(1878)

2次元ハルナック集合(1885)

ヒルベルト
23問題の最初
(1900)

記述集合

ルベーグ測度
(1902)

完全集合(1884)

ボレル集合
(1898)

完全集合は連続体と同じ濃度をもつ(1884)

ハウスドルフ,
アレクサンドロフ (1916)

完全集合性質

ボレル集合の集まりだけで考えると連続体仮説が成り立つ

デデキント　　カントール　3

集合による数学の記述(1871)

ルベーグ積分(1902)

デデキントの(切断による)実数の定義(1872)

ボレル集合：実数の記述部分集合(1898)

ベールの関数，カテゴリ(1899)

フレーゲの論理の形式的体系(1879)

ツェルメロの選択公理の考察(1904)

ラッセルのパラドックス(1901)

クラトフスキの順序対

ウィナーの順序対

ハウスドルフの順序対(1914)

ツェルメロの選択公理の考察(1904)

(置換公理, 分離公理)・整列可能定理

ヴィタリ(1908)

選択公理より(ルベーグ)可測でない集合を導く

ツェルメロの集合論の公理(1908)

ツェルメロ集合論

1) 外延性公理
2) 空集合の公理
3) 対集合の公理
4) 和集合の公理
5) 冪集合の公理
6) 選択公理
7) 無限公理
8) 分離公理

ハウスドルフ(1914)

1914年に『集合論基礎』(*Grundzüge der Mengenlehre*)を出版

フランケル(1921-22), スコーレム(1923) 集合論の公理

ツェルメロの集合論の公理 + 置換公理

超限順序数 → 超限帰納法 → ミリマノフの累積的階層(1917)

Bibliographies

⑦ Real-valued measurable cardinals
Axiomatic Set Theory, R.M. Solovay
(Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967), Amer. Math. Soc., Providence, R.I. (1971), pp. 397–428

⑧ **Saturated ideals in Boolean extensions,**
Yuzuru Kakuda, Nagoya Math. J.
Volume 48 (1972), 159-168.

⑨ **On a condition for Cohen extensions which preserve precipitous ideals, Yuzuru Kakuda, The Journal of Symbolic Logic, Volume 46(2)：296-300, 1981.**

⑩ **Saturation of Ideals and Pseudo-Boolean Algebras of Ideals on Sets, Yuzuru Kakuda,**
Mathematics seminar notes,
Volume 6(2), 269-321, 1978.

⑪ On splitting stationary subsets of large cardinals, Structural properties of ideals, J. E. Baumgartner, A. D. Taylor, and S. Wagon, The Journal of Symbolic Logic, Volume 42：203-214, 1977.

⑫ Saturation properties of ideals in generic extensions Ⅰ・Ⅱ, J.E.Baumgartner and A.Taylor, Trans. Amer. Math. Soc., vol.270, pp557-574 (1982).

⑬ Flipping properties: A unifying thread in the theory of large cardinals, F.G.Abramson, L.A.Harrington, E.M. Kleinberg, W.S. Zwicker, Annals of Mathematical Logic, Volume 12, pp 25-58,1977,

⑭ The evolution of large cardinal axioms in set theory, A. Kanamori and M. Magidor, in：'Higher set theory (G. Müller and D. Scott, eds)' Lecture Notes in Mathematics, vol.669, Springer-Verlag, Berlin, pp99-275,1978,

**⑮ Mathematical Logic, Joseph R. Shoenfield,
A K Peters/CRC Press**

2. Properties of Ideals

Definition 5. Let I be a non-tricial κ-complete ideal on κ. Then

1) I is said to be ***λ-saturated*** if there is no I-disjoint subfamily of $P(\kappa) - I$ (, this family is denoted by I^+) of cardinality λ, where ***I-disjoint*** means that $A \cap B$ is in I for any pair (A,B) of distinct elements of I.

2) I is said to be ***completive*** if the quotient algebra $P(\kappa)/I$ is complete.

3) I is said to be ***λ-distributive*** if the quotient algebra $P(\kappa)/I$ is λ-distributive.

Here we introduce the special definable ideals.

$\boldsymbol{BD}_\kappa = \{ X \in P(\kappa) \mid |X| < \kappa \}$

This ideal is ***the bounded ideal*** on κ.

A subset C of κ is said to be ***a closed unbouded set*** if it satisfies that for any limit ordinal $\alpha < \kappa$, $\sup(C \cap \alpha) = \alpha$ and for any $\xi < \kappa$, there is $\alpha \in C$ with $\xi < \alpha$.

$\boldsymbol{NS}_\kappa = \{ X \in P(\kappa) \mid$ *for some closed unbounded set* C, $X \cap C = \varnothing \}$

This ideal is *the non-stationary ideal* on κ.

Definition 6. κ is said to be a ***stationary cardinal*** if $\{\, \mathbf{M}(X) \mid X \in NS_{\kappa}^{+} \,\}$ generates a proper κ-complete normal filter.

In the above, **M** is an operation defined by
$\mathbf{M}(X) = \{\, \xi < \kappa \mid cf(\xi) > \omega \text{ and } X \cap \xi \in NS_{\xi}^{+} \,\}$

Definition 7. (1) An ideal J on κ is said to be an **M**-***ideal*** if $A \in J^*$ implies $\mathbf{M}(A) \in J^*$.
(2) An extension J of I is said to be ***μ-I-closed*** generated by a subset S of I^{+}
if $J = \{\, X \subseteq \kappa \mid \textit{for some } A \subseteq S,\ |A| < \mu$
$\textit{and } [X]_I \leq \bigvee_{Y \in A} [Y]_I \,\}$.

Definition 8. We define a sequence in $\boldsymbol{NS}_\kappa$, called a ***canonical Mahlo sequence*** $\boldsymbol{< M_\alpha : \alpha < \theta(\kappa) >}$ on κ, defined by recursion on α as follows:
$M_0 = \kappa$; if $\alpha = \beta + 1$ and $\mathbf{M}(M_\beta)$ is stationary in κ, $M_\alpha = \mathbf{M}(M_\beta)$; and if α is limit, M_α is any stationary subset of κ such that $[M_\alpha]_{\boldsymbol{NS}\kappa} = \bigwedge_{\beta<\alpha}[M_\beta]_{\boldsymbol{NS}\kappa}$.
If such a set does not exist, M_α is left undefined and set $\theta(\kappa) = \alpha$.

Definition 9. A ***λ-closed Mahlo family*** is a sequence $N = < A_\alpha : \alpha \leq \delta >$ of subsets of $\boldsymbol{NS}_\kappa$ satisfying the following conditions.

(1) $A_0 = \boldsymbol{NS}_\kappa{}^*$, for all $\alpha < \delta$, $\varnothing \notin A_\alpha$, $A_\alpha \neq A_{\alpha+1}$ and $A_\alpha \subseteq A_{\alpha+1}$.

(2) For each $\alpha < \delta$, $X \in A_{\alpha+1}$ iff $X \in A_\alpha$ or for some $Y \in A_\alpha$, $\mathbf{M}(Y) - X \in \boldsymbol{NS}_\kappa$.

(3) If α is a limit ordinal less than δ, $X \in A_\alpha$ iff for some subset B of $\bigcup_{\beta<\alpha} A_\beta$ with $|B| < \lambda$ and for some $Y \in \boldsymbol{NS}_\kappa{}^+$, $Y - X \in \boldsymbol{NS}_\kappa$ and $[Y]_{NS\kappa} = \bigwedge_{Z \in B} [Z]_{NS\kappa}$.

(4) For any set $B \subseteq \bigcup_{\alpha \leq \delta} A_\alpha$ with $|B| < \lambda$, if $\bigwedge_{Z \in B} [Z]_{NS\kappa}$ exists and is equal to $[X]_{NS\kappa}$, then $X \in A_\alpha$ for some $\alpha \leq \delta$.

δ is ***the length of*** N denoted by $l(N)$, and N is simply called ***a Mahlo family*** if $\lambda = |\boldsymbol{NS}_\kappa{}^+|^+$.

Definition 10.

(1) κ is said to be ***greatly Mahlo*** if $\theta(\kappa) \geq \kappa^+$.

(2) κ is said to be ***super Mahlo*** if there is a Mahlo family.

Results

Theorem 2. (13) Let λ be any cardinal $\geq \kappa^+$. Then, there is a λ-closed Mahlo family if and only if κ bears a λ-$\boldsymbol{NS}_\kappa$-closed **M**-ideal.

Corollary. (13) (1) κ is super Mahlo if and only if κ bears a $\boldsymbol{NS}_\kappa$-closed **M**-ideal.
(2) κ is greatly Mahlo if and only if κ bears a κ^+-$\boldsymbol{NS}_\kappa$-closed, i.e. normal **M**-ideal.

Lemma 3. (13) Let J be an ideal on κ extending $\boldsymbol{NS}_\kappa$. Then we have:

(1) J is κ-complete if and only if J is κ-$\boldsymbol{NS}_\kappa$-closed.

(2) J is normal and κ-complete

if and only if J is κ^+-$\boldsymbol{NS}_\kappa$-closed

Lemma 4. (Baumgartner, Taylor and Wagon ⑪ or Kakuda ⑩) If I is an **M**-ideal on κ, then for any stationary subset A of κ, $I \neq \boldsymbol{NS}_\kappa \restriction A$.

Theorem 5. (Baumgartner, Taylor and Wagon ⑪ or Kakuda ⑩)

(1) I is κ-saturated if and only if the only non-trivial κ-I-closed ideals extending I are of the form $I{\restriction}A$ for some $A \in I^+$.

(2) Assume that I is normal. Then I is κ^+-saturated if and only if the only non-trivial κ^+-I-closed ideals extending I are of the form $I{\restriction}A$ for some $A \in I^+$.

Theorem 6. **(12)** Let $\lambda \geq \kappa^+$ be any cardinal.
I is λ-completive if and only if whenever J is a non-trivial $|D|$-I-closed extension of I generated by $D \subseteq I^+$ with $|D| < \lambda$, $J = I \restriction A$ for some $A \in I^+$.

Corollary 1. **(12)** I is completive if and only if the only non-trivial I-closed ideals extending I are of the form $I \restriction A$ for some $A \in I^+$.

Corollary 2. **(12)** If κ is a super Mahlo cardinal, then the non-stationary ideal $\boldsymbol{NS}_\kappa$ is not completive.

Assume that κ is a stationary cardinal and
H the κ-complete normal filter generated by
$\{ \mathbf{M}(X) \mid X \in \boldsymbol{NS}_{\kappa}^{+} \}$.
Let $A = \{ \alpha < \kappa \mid \alpha$ is weakly inaccessible $\}$.
Then, we have the following.

Lemma 7. **(12)** A is stationary in κ, in fact, is in H^{*}.

Theorem 8. **(12)** Every stationary cardinal is super Mahlo.

Corollary. **(12)** (1) If κ is a weakly compact cardinal, then $P(\kappa) / \boldsymbol{NS}_{\kappa}$ is not complete.

(2) If κ carries a κ-complete κ-saturated ideal, then $P(\kappa) / \boldsymbol{NS}_{\kappa}$ is not complete.

Theorem 9. (13) Assume that κ is a strongly compact cardinal, I is a non-trivial normal κ-complete ideal on κ and B is an I-regular complete Boolean algebra. Then if I is completive, it is B-valid that for some $A \subseteq \hat{\kappa}$, $\mathbf{J} \restriction A$ is completive.

Corollary 1. (13) Let M be a transitive model of ZFC and in M, let κ be a strongly compact cardinal and λ a regular uncountable cardinal less than κ. Then there exists a generic extension $M[G]$ in which
$\kappa = \lambda^+$ and κ carries a non-trivial κ-complete ideal I which is completive but not κ^+-saturated.

Corollary 2. (13))(2000) If *ZFC* + ``there is a strongly compact cardinal" is consistent, so is *ZFC* + ``there is a regular uncountable cardinal κ which bears a non-trivial
κ-complete ideal I such that the quotient algebra $P(\kappa)/I$ is complete but not κ^+-saturated.

It should be noticed that
if κ carries a non-trivial κ-complete ideal I which is completive but not κ^+-saturated, then $\kappa^+ < 2^\kappa$.

Theorem 10. (Kanamori and Shelah(1995))
If *ZFC* + ``there is a Woodin cardinal'' is consistent,
then so is *ZFC* + ``there is a completive ideal I on $\aleph_1$, $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_3$ (hence I is not $\aleph_2$-saturated)''.

Theorem 11. (Gitik and Shelah(1997)) For any regular cardinal $\kappa \geq \aleph_2$, NS_κ is not κ^+-saturated.

Problems

Is it true that $\boldsymbol{NS}_\lambda$ is not completive for any regular cardinal $\lambda \geq \aleph_2$?

＝ 測度問題 ＝

ルベーグ(1905)

ボレル集合はベール関数による開区間の逆像

ボレル集合(1898)

ルベーグ積分

ルベーグ可測

ルジン，ススリンの解析集合(1917)

ルベーグの測度問題(1904)

ジョルダン‘測度’(1902)

区間 $I = [0,1]$ のすべての部分集合上で定義された(負の値をとらない)測度 m で次の条件を満たす ものが存在するか？

1) A と B が(平行移動で)合同ならば $m(A) = m(B)$

2) $m(X) = 1$

3) $m(\cup_{n=1}^{\infty} S_n) = \Sigma_{n=1}^{\infty} m(S_n)$

ただし，S_n は互いに共通部分をもたない。(完全加法性)

ジョルダンの面積 = ジョルダン‘測度’ J は 次の2つを満たす。

(1) $J(A) \geq 0$, $J(\varnothing) = 0$

(2) $A \cap B = \varnothing$ ならば $J(A \cup B) = J(A) + J(B)$（有限加法性）

この2つが，「面積とは何か？」の答えである。（ルベーグ「積分・長さおよび面積」）

ハウスドルフ 測度の大域的問題(1914)

ハウスドルフ 測度の大域的問題(1914)

n次元ユークリッド空間の各有界集合 E に負でない実数 $m(E)$ を対応させる，次の条件を満たす関数 m は存在するか？

1) $m(I) = 1$　ただし，I は単位立法体

2) $E_1 \cap E_2 = \varnothing$ ならば $m(E_1 \cup E_2) = m(E_1) + m(E_2)$

3) E_1 と E_2 が合同ならば　$m(E_1) = m(E_2)$

↓

バナッハの n = 1,2 に対する肯定的解決(1923)

↓

バナッハ，タルスキーの定理(1924)

選択公理を仮定して，$n \geq 3$ では上記ハウスドルフの問題を否定的に解決

=バナッハ-タルスキーの逆理

〈ルベーグ可測〉

ボレル集合(1898)

↓

ルジン，ススリンの解析集合(1917)

↓

ルジン，シェルピンスキーの射影集合(1925)

バナッハ，クラトフスキーの定理(1929)

連続体仮説を仮定すると，区間 $I = [0,1]$ のすべての部分集合上で定義された完全加法的測度 m で
1) 1点の測度は 0 である
2) $m(I) = 1$ を満たすものは存在しない。

→ 連続体仮説への疑念

連続体仮説を仮定すると，数直線上で定義された非可測関数で，高々可算集合を除いて連続となるものが存在する。

ウラムの定理(1930)

集合 E の濃度が $\aleph_1, \aleph_2, \aleph_3, \cdots, \aleph_n, \cdots, \aleph_\omega$ のいずれであっても，E のすべての部分集合上で定義された完全加法的測度 m で
1) 1点の測度は 0 である
2) $m(I) = 1$ を満たすものは存在しない。

→ 測度問題の巨大基数の必要性

＝ イデアルとは ＝

ボレルのσ-イデアル
(強ルベーグ測度零
イデアル) (1919)

デデキントのイデアル論(1871)

ウラムのω上の
超フィルター(1929)

ブール環における
イデアル(1929)

ウラムのフィルター,
超フィルター(1930)

ストーン(1936)
ブール環におけるイデアル分析

ブルバキの一員
であるカルタンのイデアル (1936)

タルスキーのイデアル(1940?)

3. Distributive Ideals on Boolean Algebras

Definition 11. Let B be any Boolean algebra and I a subset of B.

I is said to be an ***ideal on B*** if it satisfies that

1) $\mathbf{0} \in I$,

2) a, $b \in I$ implies $a \vee b \in I$,

3) $a \in I$, $b \in B$ implies $a \wedge b \in I$,

where 0 is the least element, and $\vee$ (join) and $\wedge$ (meet) are the Boolean operations.

Definition 12. (Smith & Tarski (1956))
A Boolean algebra A is ***(α,β)-distributive*** if the following is satisfied:
Given any double sequence $a \in \mathrm{A}^{\alpha\times\beta}$
such that all the sums $\Sigma_{\eta<\beta}\, a_{\xi,\eta}$ for $\xi < \alpha$,
their product $\Pi_{\xi<\alpha}\Sigma_{\eta<\beta}\, a_{\xi,\eta}$,
and all the products $\Pi_{\xi<\alpha}\, a_{\xi,f(\xi)}$ for $f \in \beta^{\alpha}$ exist,
then the sum $\Sigma_{f\in\beta^{\alpha}}\Pi_{\xi<\alpha}\, a_{\xi,f(\xi)}$ also exists,
and we have $\Pi_{\xi<\alpha}\Sigma_{\eta<\beta}\, a_{\xi,\eta} = \Sigma_{f\in\beta^{\alpha}}\Pi_{\xi<\alpha}\, a_{\xi,f(\xi)}$.

Generalized Distributivity

Definition 13. Let B be any Boolean algebra and f any function into $P(B)$.

B is $\langle\lambda, f\rangle$***-distributive*** if B satisfies that

for all b in B, if for each a in dom(F) $0 < b \leq \bigvee f(a)$, then there is v in Πf such that for any t in $[\mathrm{dom}(f)]^{<\lambda}$ $(b \wedge \bigwedge_{a\in t} v(a) > \mathbf{0}$)).

Quote from my doctoral Dissertation

When we construct and develop a powerful set theory based on Zermelo-Fraenkel set theory, it happens quite often to find out one condition, say $h(\alpha)$, from each set of conditions, say A_α, whose disjunction is consistent (i.e., $\vee_{\boldsymbol{\alpha<\kappa}} \boldsymbol{A_\alpha} = \mathbf{1}$ in Boolean terms) and arrange them into one consistent condition (i.e., $\wedge_{\boldsymbol{\alpha<\kappa}} \boldsymbol{h(\alpha)} > \mathbf{0}$ in Boolean terms).

Results

Lemma 12. (Pierce) Let f be any function and let I be a λ-complete ideal in a μ-complete f-distributive Boolean algebra B, where λ and μ are cardinals such that $|\Sigma f| < \lambda$ and $|\Pi f| < \mu$. Then the following are equivalent.

(1) I is f-distributive.

(2) I is $|\Pi f|^{+}$-complete.

(3) $|\Pi f| < \lambda$ holds.

Corollary. Let f be any function on a cardinal η and let I be a λ-complete ideal in a μ-complete $\langle \nu, f \rangle$-distributive Boolean algebra B, where λ , μ and ν are cardinals such that $\nu < \eta$, $|\sum f \upharpoonright X| < \lambda$ and) $|\prod f \upharpoonright X| < \mu$ for all X in $P_{<\nu}(\eta)$. Then if I is $\langle \nu, f \rangle$-distributive, $\sup._{X \in P_{<\nu}(\eta)} |\prod f \upharpoonright X|^{+} < \lambda$ holds.

Theorem 13. (14) The following are equivalent in *ZF* set theory.

(1) The κ-Axiom of Choice.

(2) Every power set algebra is $\langle 2, \kappa \rangle$-distributive.

Theorem 14. (14) The following are equivalent in *ZF* set theory.

(1) The Principle of Dependent Choice.

(2) Every Boolean algebra is $\langle \omega,\omega \rangle$-distributive.

Theorem 15. (11) Let κ be any cardinal and let B be a κ-complete Boolean algebra of cardinality λ. Then the following are equivalent.

(1) There exists a κ-complete prime ideal in B.

(2) There exists a $\langle \kappa, C_{\lambda,2} \rangle$-distributive ideal in B.

In the above, $C_{\lambda,2}$ indicates the function on λ whose range is the singleton $\{2\}$.

Corollary. (11) (F.G. Abramson, L.A. Harrington, E.M. Kleinberg and W.S. Zwicker, C.A. DiPrisco and W.S. Zwicker ⑬) Let κ be any regular uncountable cardinal. Then we have:
(1) κ is weakly compact if and only if BDκ is $\langle \kappa, C_{\kappa,2} \rangle$-distributive.
(2) κ is measurable if and only if BDκ is $\langle \kappa, C_{2^\kappa,2} \rangle$-distributive.
(3) κ is strongly compact if and only if for each regular λ ≥ κ, BDλ is $\langle \kappa, C_{2^\lambda,2} \rangle$-distributive.

Theorem 16. (11) The following are equivalent.

(1) Whenever σ is a function on S satisfying the conditions (*) and $< t_a : a \in S >$ is a sequence with $t_a \subseteq \sigma(a)$ for each $a \in S$, there exists a set t such that for any $a \in S$ there is a $b \in S$ with $a < b$ and $t \cap \sigma(a) = t_b \cap \sigma(a)$.

(2) There exists a fine κ-complete $\langle \kappa, f \rangle$- distributive ideal on S for any $f : S \to \kappa$.

Theorem 17. **(12)** Let σ be any function of S into $P(T)$ such that for a, b in S, $\mu_a = |P(\sigma(a))| < \kappa$ and if $a <_S b$ then $\sigma(a) \subseteq \sigma(b)$. Assume that I is a $<_S$-fine κ-complete $<_S$-normal $\langle 3, f \rangle$-distributive ideal on S, where f is the function on $H = (\{0\} \times S) \cup (\{1\} \times T)$ defined by $f(0,a) = P(\sigma(a))$ and $f(1,t) = T$.

Moreover, we assume that

$R = \{ a \in S \mid cf_{<T}(\sigma(a)) > \aleph_0 \}$ has positive I-measure,

$\{ a \in S \mid t \in \sigma(a) \}$ has I-measure one for each $t \in T$

and if g is a function on $A \in I^+$ with $g(a) \in \sigma(a)$ then there exists a subset B of A of positive I-measure such that $g \restriction B$ is constant.

Then if X is a $<_T$-stationary subset of T,

$$R - M_\sigma(X) \text{ has } I\text{-measure zero.}$$

In the above, $M_\sigma(X)$ is defined by

$$M_\sigma(X) = \{\, a \in S \mid cf_{<T}(\sigma(a)) > \aleph_0 \text{ and } X \cap \sigma(a) \text{ is } <_T\text{-stationary in } \sigma(a) \,\}$$

In Theorem 17, if we put $T = P_{<\mu}(\lambda)$ and $\sigma(a) = P_{<\mu}(a)$, we get the next theorem.

Theorem 18. **(12)** Let $S = P_{<\eta}(\lambda)$ and $T = P_{<\mu}(\lambda)$, where $\aleph_0 < \mu < \eta \leq \kappa \leq \lambda$ and $2^{(\nu^{<\mu})} < \kappa$ for any $\nu < \eta$. Assume that there exists a $\prec$-fine κ-complete $\prec$-normal $\langle \aleph_1, C_{S,\tau} \rangle$-distributive ideal I on S, where $\tau = max.\{ \lambda^{<\mu}, 2^{(\eta^{<\mu})} \}$.

Then, if X is a $\prec_T$-stationary subset of T,

$$M_{\sigma^1}(X) = \{ a \in S \mid cf_{\prec T}(P_{<\mu}(a)) > \aleph_0 \text{ and } X \cap P_{<\mu}(a) \text{ is } \prec_T\text{-stationary in } P_{<\mu}(a) \}$$

has I-measure one.

Theorem 19. (Feng and Magidor) Assume that κ is λ-supercompact with $\lambda \geq \kappa$ regular.
Then for every stationary $S \subseteq P_{<\omega^1}(\lambda)$ and for every tight and unbounded $A \subseteq P_{<\kappa}(\lambda)$, there is an $X \in A$ such that $S \cap P_{<\omega^1}(X)$ is stationary in $P_{<\omega^1}(X)$.

Problems

How strong is the condition that there is a κ-complete non-trivial $\langle \kappa, C_{2^\kappa,\eta} \rangle$-distributive ideal on κ with $\kappa \leq \eta$?

Bibliography

⑯ Formal Logic: or, The Calculus of Inference, Necessary and Probable, De Morgan, A., Taylor and Walton, London, 1847.

⑰ Mathematical Analysis of Logic, Boole, G., MacMillan, Barclay & MacMillan, Cambridge, 1847. Reprint Open Court, La Salle, 1952.

⑱ An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities, Boole, G., Walton and Maberly, London, 1854.

⑲ Ueber die von drei Moduln erzeugte Dualgruppe, Dedekind, Mathematische Annalen, vol. 53 (1900), pp. 371-403?

⑳ The theory of representations for Boolean algebras, M.H.Stone,Transactions of the American Mathematical Society, vol. 40 (1936), pp. 37-111

㉑ The theory of representations for Boolean algebras, M.H.Stone,Transactions of the American Mathematical Society, vol. 40 (1936), pp. 37-111

㉒ Distributive postulates for systems like Boolean algebras, George D. Birkhoff and Garrett Birkhoff, Transactions of the American Mathematical Society, Vol. 60, No. 3 (1956), pp. 3-11

㉓ A Distributivity Condition for Boolean Algebras,Edgar C. Smith, Jr., Annals of Mathematics, Second Series, Vol. 64, No. 3 (1956), pp. 551-561

㉔ Distributivity in Boolean algebras, R. S. Pierce, Pacific Journal of Mathematics, Vol. 7, No. 1 (1957), pp. 983-992

㉕ Higher Degrees of Distributivity and Completeness in Boolean Algebras, E. C. Smith, Jr. and Alfred Tarski, Transactions of the American Mathematical Society, Vol. 84, No. 1 (1957), pp. 230-257

㉖ The independence of certain distributive laws in Boolean algebras,
D. Scott, Trans. Amer. Math. Soc. vol. 84 (1957),pp.258-261

㉗ Distributivity and representability, R. Sikorski, Fund. Math., Vol. 48 (1957), pp. 105-117

4. Cardinal Arithmetic

Definition 14. (1983?) Let κ be a measurable cardinal. If F_1 and F_2 are non-trivial κ-complete normal ultrafilters on κ and define a relation $<$ by :

$F_1 < F_2$ if and only if $F_1 \in \mathrm{Ult}_\kappa(V, F_2)$.

This relation well-founded, and can give the rank of a non-trivial κ-complete normal ultrafilter U on κ in $<$. This rank is called the order of U, and the hight of $<$ is called the order of κ, denoted by $o(\kappa)$.

Results (1)

Theorem 20.(Cantor (1891))
For every set X, $|X| < |P(X)|$

Theorem 21.(Cantor, Bernstein(1))
If $|X| \leq |Y|$ and $|X| \geq |Y|$, then $|X| = |Y|$.

Theorem 22.(Bernstein(1901))
For every ordinal α and μ,

$$\aleph_\mu^{\aleph_\alpha} = 2^{\aleph_\alpha} \cdot \aleph_\mu.$$

But this is incorrect when $\alpha = 0$ and $\mu = \omega$.

Theorem 23. (Hausdorff (1904))

For any ordinals α and β, ${\aleph_{\alpha+1}}^{\aleph_\beta} = {\aleph_\alpha}^{\aleph_\beta} \cdot \aleph_{\alpha+1}$.

Theorem 24.(Konig(1905))

$2^{\aleph_0}$ cannot equal $\aleph_{\alpha+\omega}$.

Theorem 25. (Konig $\langle\gamma < \omega\rangle$(1905), Jourdain$\langle\gamma \geq \omega\rangle$(1908), Zermelo $\langle\gamma$ any set $\rangle$(1908))

For any $\alpha < \gamma$ $m_\alpha < n_\alpha$, $\sum_{\alpha<\gamma} m_\alpha < \prod_{\alpha<\gamma} m_\alpha$.

Theorem 26. (Gödel (1938))
If ***ZF*** is consistent, so is ***ZFC* + *GCH***.

Theorem 27. (P. Cohen (1963))
If ***ZF*** is consistent, so are ***ZF* + ¬*AC*** and ***ZFC* + ¬ *CH***.

Theorem 28. (W. Easton (1964))
Assume ***GCH*** and F is a class function from the class of regular cardinals to cardinals such that for regular crdinals κ and λ with $\kappa \leq \lambda$, $F(\kappa) \leq F(\lambda)$ and $\kappa < cf(F(\kappa))$. Then there is a forcing extension preserving cofinalities in which $2^\kappa = F(\kappa)$ for every regular cardinal κ.

The simplest possibility is when $2^{cf(\kappa)} < \kappa$ implies $\kappa^{cf(\kappa)} = \kappa^+$. This is known as the **Singular Cardinal Hypothesis (*SCH*)**.

Theorem 29. (J.Silver (1975)) If κ is a singular cardinal of uncountable cofinality, and if $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, $2^\kappa = \kappa^+$.

Theorem 30. (Galvin and Hajnal (1975))
If $\aleph_\lambda$ is a strong limit cardinal of uncountable cofinality then $2^{\aleph_\lambda} < \aleph_{(2^\lambda)^+}$

Theorem 31. (Jensen (1974)) If $0^\#$ does not exist then every uncountable set of ordinals can be covered by a constructible set of the same cardinality

Theorem 32. (T. Jech and K. Prikry (1976)) Let κ be a regular uncountable cardinal which bears a κ-complete non-trivial κ^+-saturated ideal. If $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$.

Theorem 33. **T**he Covering Theorem shows that unless $0^\#$ exists, $2^{cf\,\kappa} < \kappa$ implies $\kappa^{cf\,\kappa} = \kappa^+$, i.e. ***SCH*** holds.

Thus in order to violate ***SCH*** we need large cardinals.

Theorem 34. (Solovay (1974)) If κ is a strongly compact cardinal and $\lambda > \kappa$ is singular then $\lambda^{cf\,\lambda} = \lambda^+$. This means that the ***SCH*** holds above the least strongly compact cardinal.

Theorem 35. (J.Silver) If there is a supercompact cardinal, there is a transitive model ***ZFC*** in which κ is a strong limit cardinal, $cf\,\kappa = \omega$, and $2^\kappa > \kappa^+$.

Theorem 36. (Magidor)
If there is a supercompact cardinal, there is a transitive model ***ZFC*** in which $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} > \aleph_{\omega+1}$.

Theorem 37. (Magidor)
If there is a *2*-huge cardinal, there is a transitive model ***ZFC*** in which ***GCH*** holds below $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.

Theorem 38. (Magidor(1977),Shelah(1983))
Assume that there exists a supercompact cardinal.
(1) There is a generic extension in which ***GCH*** holds below $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+\alpha+1}$, where α is any countable ordinal.
(2) There is a generic extension in which $\aleph_{\omega 1}$ is strong limit and $2^{\aleph_{\omega_1}} = \aleph_{\omega 1+\alpha+1}$, where α is any ordinal $< \omega_2$.

Theorem 39. (Woodin,Gitik(1989))
If there is a measurable cardinal κ of Mitchell order κ^{++}, then there exists a generic extension in which ***GCH*** holds below $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.

Theorem 40. (S. Shelah(1987))
MM*(Marutin's Maximum)** implies ***RP.

Theorem 41. (S. Shelah(1989))
1. If $\aleph_\omega$ is strong limit,
   then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.
2. For any limit ordinal ξ, $\aleph^{|\xi|}_{\xi} < \aleph_{(2^{|\xi|})^+}$.
3. If δ is limit and $\delta = \alpha + \beta$, $\beta \neq 0$,
   then $\aleph^{cf(\delta)}_{\delta} < \aleph_{\alpha+(|\beta|^{cf(\beta)})^+}$.

Theorem 42. (S. Shelah(2008))
RP implies that $\lambda^{\aleph_0} = \lambda$, for any regular cardinal $\lambda \geq \aleph_2$.

＝巨大基数 ＝

カントールの
超限順序数

ハウスドルフの
基数計算

ハウスドルフの弱到達
不能基数(1906)

マロー 基数(1911)巨
大基数の公理の初め

ツェルメロの累積階
層の集合論モデル

カントールの 連続体仮説，記述
集合 (perfect set，derived set)

ジョルダンの一般連続体仮
説の定式化

ルジン，シェルピンスキ
ーの連続体仮説の研究

ハウスドルフの特異基数(1907)

到達不能基数(概念：シェルピンス
キー，タルスキー)，（言葉：クラト
ウスキー)

ウラムの結果

連続体仮説を仮定せずとも,
$\aleph_1, \aleph_2, \cdots \aleph_\omega$ 上には一般化された
バナッハ問題の測度は存在しない

カントールの超限順序数

ウラム

ω上の超フィルター (1929)

可測基数

可測基数は到達不可能基数
（ウラム：1929）

到達不可能基数はウラムの意味で可測か？

可測基数の存在性は？

実数値可測基数(real-valued mesurable cardinal) はどの程度大きいか？

タルスキー (1943)

強コンパクト基数 ⇒ 可測基数
⇒ 弱コンパクト基数の証明(現代的用語)

↓

強コンパクト基数，弱コンパクト基数の定義(1962)

Results

Theorem 43. Let f and g be any functions on a non empty set S so that $|\Sigma f| \geq \aleph_0$, $F(x) \neq \emptyset$ and $|g(x)| \geq 2$ for all $x \in S$.
Assume that for each $x \in S$, there exists $y \in S$ such that $|f(x)| < |g(y)|^{|S|}$ holds.

Then we have that: $|\Sigma f| < |\Pi g|$.

(In Kőnig's Lemma, the assumption that $|f(x)| < |g(y)|^{|S|}$ is replaced by, simply, $|f(x)| < |g(x)|$ for $x \in S$.)

Theorem 44. (10) If there is a sequence $< f_\xi : \xi \leq \omega_2$
$>$
of functions of ω_1 into itself such that
$f_\xi <_{BD\omega_1} f_\zeta <_{NS\,\omega_1} f_{\omega_2}$ for any ξ and ζ in ω_2
with $\xi < \zeta$, then there is a sequence
$< h_\xi : \xi \leq \omega_2 >$ of functions of ω_1 into itself such that
$h_\xi <_{BD\omega_1} h_\zeta$ for any ξ and ζ with $\xi < \zeta \leq \omega_2$.

Theorem 45. (10) Assume that $\aleph_{\omega_1}$ is strong limit.
Then if there is no sequence $< f_\xi : \xi \leq \omega_2 >$ of
functions of ω_1 into itself such that $f_\xi <_{BD\,\omega_1} f_\zeta$
for $\xi < \zeta \leq \omega_2$ then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$ holds.

Theorem 46. **(12)** Assume that κ is inaccessible and there exists a $<$-fine κ-complete $<$-normal $\langle \aleph_1, C_{S,\lambda} \rangle$-distributive ideal on $S = P_{<\kappa}(\lambda)$.
Then it holds that $\lambda^{<\kappa} = \lambda$.

Corollary.(Solovay) If κ is a supercompact cardinal (strongly compact cardinal),
then for every regular cardinal $\lambda > \kappa$, $\lambda^{<\kappa} = \lambda$.

Problems

(H.Woodin) If κ is a strongly compact cardinal and $2^{\alpha} = \alpha^{+}$ for every cardinal $\alpha < \kappa$, then must be GCH hold ?

Theorem 47. (A.W. Apter) Let $V \models$"$ZFC + \kappa$ *is supercompact*". There is then a partial ordering $P \in V$ and a symmetric inner model N, $V \subseteq N \subseteq V^{P}$, so that $N \models$"$ZF + \forall\delta < \kappa\ DC_{\delta} + \kappa$ *is a strong limit cardinal* $+ \forall\delta < \kappa\ (2^{\delta} = \delta^{+}) + \kappa$ *is supercompact* + *there is a sequence* $< A_{\alpha} : \alpha < \kappa^{++} >$ *of distinct subsets of* κ".

5. References

(1) Joan Bagaria, ‘Natural Axioms of Set Theory and the Continuum Problem’ draft?, (2013)

(2) Janet Heine Barnett, ‘Origins of Boolean Algebra in the Logic of Classes:George Boole, John Venn and C. S. Peirce’ draft?, (2013)

(3) Kurt Gödel, ‘What is Cantor’s Continuum Problem?`, American Mathematical Monthly , USA, vol.54 (1947), pp515–525.

(4) 飯田隆 編, ‘リーディングス 数学の哲学 ゲーデル以降’ 勁草書房 , (1995)

(5) 角田　譲, ‘最近の集合論’ , 科学基礎論研究,vol.17, (1984) pp21-30

(6) Y. Kanai, ‘Separative Ideals and Precipitous Ideals’, *Masters Thesis*, Kobe University (1981)

(7) Y. Kanai, ‘About κ^+-saturated ideals’, Mathematics Seminar Notes, Kobe University, vol.9(1), (1981) pp65-74

(8) Y. Kanai, ‘On a result of S.Shelah’ (japanese), 京都大学数理解析研究所 講究録 441, (1981) pp. 27-42

(9) Y. Kanai, ‘ On quotient algebras in generic extensions’, Commentarii Mathematici Universitatis Sancti Pauli, vol.33(1), (1984) pp71-77

(10) Y. Kanai, ‘ On a variant of weak Chang’s Conjecture’, Zeitschrift math.Logik und Grundlagen d. Math.vol.37, (1991)

(11) On a generalization of distributivity
Kanai,Yasuo, Journal of Symbolic Logic , 59(3) 1055-1067 1994

(12) Distributivity and Stationary Reflections
Kanai,Yasuo, Proceedings of American Mathematical Society , 127(10) 3073-3080 1999

(13) Y. Kanai, ‘On Completeness of the Quotient Algebras $P(\kappa)/I$’, Archive for Mathematical Logic , vol.39(2), (2000) pp75-87

(14) On the Deductive Strength of Distributivity Axioms for Boolean Algebras in Set Theory, Kanai,Yasuo, Mathematical Logic Quarterly, 48(3) 413-426 2002

(15) A. Kanamori, ‘The Mathematical Development of Set Theory from Cantor to Cohen’, The Bulletin of Symbolic Logic, vol.2 (1996), pp1–71.

(16) A. Kanamori, ‘Introduction’, in *Handbook of Set Theory Vol. 1*, (2010), pp1–92.

(17) 功力金二郎, 村田 全 訳･解説, ‘現代数学の系譜8 G.CANTOR著 カントル 超限集合論’, 共立出版株式会社, (1979).

(18) G.H. Moore, ‘Early History of Generalized Continuum Hypothesis’, The Bulletin of Symbolic Logic, vol.17 (2011), pp489–532.

(19) 田中一之 編, ‘ゲーデルと20世紀の論理学’ 東京大学出版会 , (2007)

(20) 吉田耕作, 松原　稔 訳・解説, ‘現代数学の系譜３ H.LEBESGU著 ルベーグ 積分・長さおよび面積’, 共立出版株式会社, (1969).

ご清聴

ありがとうございました！