From Noumi's representation to elliptic K-matrices

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1. M. Noumi, H. Yamada, K. Mimachi, *Finite-dimensional representations of the quantum group* $\mathrm{GL}_q(n, \mathbb{C})$ *and the zonal spherical functions on* $\mathrm{U}_q(n-1)\backslash\mathrm{U}_q(n)$, Japanese J. Math. **19** (1993), no. 1, 31–80.

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2. M. Noumi, *Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces*, Adv. Math. **123** (1996), no. 1, 16–77.

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2. M. Noumi, *Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces*, Adv. Math. **123** (1996), no. 1, 16–77.
3. M. Noumi, *Macdonald-Koornwinder polynomials and affine Hecke rings*, Surikaisekikenkyusho Kokyuroku **919** (1995), 44–55.

This talk:

highlight the role of Noumi's representation of the affine Hecke algebra in:

1. solving the system of basic hypergeometric difference equations (non-polynomial theory).

   Main references:

   1. J.V. Stokman, *The c-function expansion of a basic hypergeometric function associated to root systems*, Ann. of Math. (2) **179** (2014), no. 1, 253–299.
   2. J.V. Stokman, *Connection coefficients for basic Harish-Chandra series*, Adv. Math. **250** (2014), 351–386.

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2. integrable lattice models with boundaries.

   Main references:

   1. J.V. Stokman, B.H.M. Vlaar, *Koornwinder polynomials and the XXZ spin chain*, arXiv:1310.5545, J. Approx. Th. (to appear).
   2. J.V. Stokman, *Connection problems for quantum affine KZ equations and integrable lattice models*, arXiv:1410.4383, Comm. Math. Phys. (to appear).

Notations

Coxeter graph

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i. affine Weyl group $W = \langle s_0, \ldots, s_n \rangle$,

ii. affine Hecke algebra $H_{\underline{k}} = H_{k_0,k,k_n} = \mathbb{C}\langle T_0, \ldots, T_n \rangle$.

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ii. affine Hecke algebra $H_{\underline{k}} = H_{k_0,k,k_n} = \mathbb{C}\langle T_0, \ldots, T_n \rangle$.

Braid relations according to the Coxeter graph and quadratic relations:

$$s_j^2 = 1, \qquad (T_j - k_j)(T_j + k_j^{-1}) = 0$$

for $0 \leq j \leq n$, with $k_i := k$ if $1 \leq i < n$.

Difference-reflection operators

W-action on $\mathbb{C}^n$:

$$s_0\underline{z} := (1 - z_1, z_2, \ldots, z_n),$$
$$s_i\underline{z} := (z_1, \ldots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \ldots, z_n),$$
$$s_n\underline{z} := (z_1, \ldots, z_{n-1}, -z_n)$$

for $1 \leq i < n$. Contragredient action $(w \cdot f)(\underline{z}) := f(w^{-1}\underline{z})$ on the field $\mathcal{M}$ of meromorphic functions on $\mathbb{C}^n$.

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for $1 \leq i < n$. Contragredient action $(w \cdot f)(\underline{z}) := f(w^{-1}\underline{z})$ on the field $\mathcal{M}$ of meromorphic functions on $\mathbb{C}^n$.

Decomposition: $W = W_0 \ltimes \tau(\mathbb{Z}^n)$ with

i. $W_0 = \langle s_1, \ldots, s_n \rangle$ acting on $\mathcal{M}$ by permutations and sign changes of the variables (hyperoctahedral group),

ii. Free rank n Abelian subgroup $\tau(\mathbb{Z}^n)$ of W acting on $\mathbb{C}^n$ by

$$\tau(\lambda)\underline{z} := \underline{z} + \lambda, \qquad \lambda \in \mathbb{Z}^n.$$

Remark: $\tau(\epsilon_i) = s_{i-1} \cdots s_1 s_0 s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_i$.

Difference-reflection operators

Definition

The algebra $\mathcal{D}$ of difference-reflection operators is defined as follows:

i. $\mathcal{D} = \mathcal{M} \otimes \mathbb{C}[W]$ *as a complex vectorspace;*

ii. *For* $D = \sum_{v \in W} a_v v$, $D' = \sum_{w \in W} b_w w \in \mathcal{D}$ $(a_v, b_w \in \mathcal{M})$:

$$DD' := \sum_{u \in W} \Big(\sum_{v,w: vw=u} a_v (v \cdot b_w) \Big) u.$$

Remark: $\mathcal{D}$ canonically acts on $\mathcal{M}$ as difference-reflection operators:

$$Df := \sum_{v \in W} a_v (v \cdot f)$$

for $D = \sum_{v \in W} a_v v \in \mathcal{D}$.

Noumi's representation

Fixed pair $\underline{u} = (u_0, u_n)$ of nonzero complex numbers and $0 < q < 1$.

Notation:

$$c_0(\underline{z}) := k_0^{-1} \frac{(1 - q^{\frac{1}{2}} k_0 u_0 q^{-z_1})(1 + q^{\frac{1}{2}} k_0 u_0^{-1} q^{-z_1})}{(1 - q^{1-2z_1})},$$

$$c_i(\underline{z}) := k^{-1} \frac{(1 - k^2 q^{z_i - z_{i+1}})}{(1 - q^{z_i - z_{i+1}})}, \qquad 1 \leq i < n,$$

$$c_n(\underline{z}) := k_n^{-1} \frac{(1 - k_n u_n q^{z_n})(1 + k_n u_n^{-1} q^{z_n})}{(1 - q^{2z_n})}$$

Theorem (Noumi)

There exists a unique monomorphism $\iota_{\underline{k}}^{\underline{u},q} : H_{\underline{k}} \hookrightarrow \mathcal{D}$ *such that*

$$\iota_{\underline{k}}^{\underline{u},q}(T_j) = k_j + c_j(s_j - 1), \qquad 0 \leq j \leq n.$$

Bernstein-Zelevinsky-Lusztig

Structure of the affine Hecke algebra $H_{\underline{k}}$:

i. The Hecke algebraic versions

$$Y_i := T_{i-1}^{-1} \cdots T_1^{-1} T_0 T_1 \cdots T_{n-1} T_n T_{n-1} \cdots T_i$$

of $\tau(\epsilon_i) \in W$ pairwise commute in $H_{\underline{k}}$ $(1 \leq i \leq n)$;

ii. The multiplication map is a linear isomorphism

$$H_{\underline{k};0} \otimes \mathcal{A}_Y \xrightarrow{\sim} H_{\underline{k}}$$

where $H_{\underline{k};0} = \mathbb{C}\langle T_1, \ldots, T_n \rangle$ and $\mathcal{A}_Y = \mathbb{C}\langle Y_1^{\pm 1}, \ldots, Y_n^{\pm 1} \rangle$.

iii. $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{W_0} \simeq Z(H_{\underline{k}})$ by $p \mapsto p(Y_1, \ldots, Y_n)$.

The commuting difference operators

Notations:

i. DO $:= \mathcal{M}\#\tau(\mathbb{Z}^n) \subset \mathcal{D}$ subalgebra of difference operators.

ii. Restriction map $\mathrm{Res}: \mathcal{D} \to \mathrm{DO}$:

$$\mathrm{Res}\Big(\sum_{u\in W_0,\lambda\in\mathbb{Z}^n} a_{\lambda,u}\tau(\lambda)u\Big) := \sum_{\lambda\in\mathbb{Z}^n}\Big(\sum_{u\in W_0} a_{\lambda,u}\Big)\tau(\lambda).$$

Theorem (Noumi)

The W_0-equivariant difference operators

$$D_p := \mathrm{Res}\Big(\iota_{\underline{k}^{-1}}^{\underline{u}^{-1},q}(p(Y_1,\ldots,Y_n))\Big) \in \mathrm{DO}^{W_0} \qquad (p \in \mathbb{C}[t_1^{\pm 1},\ldots,t_n^{\pm 1}]^{W_0})$$

pairwise commute.

Remark: The Koornwinder second-order difference operator and the Van Diejen higher order difference operators are of the form D_p for suitable p.

The basic hypergeometric system of difference equations

For $\underline{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ write $q^{\underline{\xi}} := (q^{\xi_1}, \ldots, q^{\xi_n})$.

Definition

The basic hypergeometric system of difference equations with spectral parameter $q^{\underline{\xi}}$ is the system of difference equations

$$D_p f = p(q^{\underline{\xi}}) f \qquad \forall\, p \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{W_0}$$

for an unknown meromorphic function $f \in \mathcal{M}$. The set of solutions is denoted by $\mathcal{S}(q^{\underline{\xi}})$.

Remarks:

i. $\mathcal{S}(q^{\underline{\xi}}) \subset \mathcal{M}$ is W_0-invariant, and a vector subspace over the field $F := \mathcal{M}^{\tau(\mathbb{Z}^n)}$ of translation invariant meromorphic functions.

ii. For appropriate discrete values of $\underline{\xi}$ (indexed by partitions of length $\leq n$), the basic hypergeometric system of difference equations has a W_0-invariant Laurent polynomial solution in the q^{z_i}: **Koornwinder polynomial**.

Solving the spectral problem

Fixed generic parameters $\underline{k}, \underline{u}$ and generic spectral parameters $\underline{\xi}$.

Basic Harish-Chandra series (in joint works with Letzter, van Meer): $\Phi_{\underline{\xi}} \in \mathcal{S}(q^{\underline{\xi}})$, characterized by the requirement that $\Phi_{\underline{\xi}}(\underline{z})$ tends to an appropriate plane wave function $W_{\underline{\xi}}(\underline{z})$ when $\Re(\underline{z}) \to \infty$ (where $\Re(\underline{z}) \to \infty$ means $\Re(z_i - z_{i+1}), \Re(z_n) \to \infty$):

Theorem

$$\mathcal{S}(q^{\underline{\xi}}) = \bigoplus_{w \in W_0} F \Phi_{w\underline{\xi}}.$$

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Basic hypergeometric function: q-analogue $\phi_{\underline{\xi}} \in \mathcal{S}(q^{\underline{\xi}})^{W_0}$ of the Heckman-Opdam hypergeometric function, defined as an explicit series in Koornwinder polynomials.

c-**function expansion**: explicit expression for $c_{\underline{\xi}} \in F$ as product of theta functions such that

$$\phi_{\underline{\xi}} = \sum_{w \in W_0} c_{w\underline{\xi}} \Phi_{w\underline{\xi}}.$$

Baxterization of affine Hecke algebra modules

Theorem

Let $\pi : H_{\underline{k}} \to \mathrm{End}(V)$ be a representation of $H_{\underline{k}}$. The affine Weyl group W acts on the space $\mathcal{M} \otimes V$ of V-valued meromorphic functions on V by

$$\big(\nabla(s_j)f\big)(\underline{z}) := C_j(\underline{z})f(s_j\underline{z}), \qquad C_j(\underline{z}) := \frac{\pi(T_j) + c_j(\underline{z}) - k_j}{c_j(\underline{z})}.$$

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Proof (sketch): Write $H = \iota_{\underline{k}}^{u,q}(H_{\underline{k}}) \subset \mathcal{D}$ and view V as H-module. We have:

1. $s_j = c_j^{-1}(\iota_{\underline{k}}^{u,q}(T_j) + c_j - k_j)$ in $\mathcal{D}$,
2. $\mathcal{D} \simeq \mathcal{M} \otimes H$ as vector spaces by the multiplication map.

The $\mathcal{D}$-action on

$$\mathrm{Ind}_H^{\mathcal{D}}(V) = \mathcal{D} \otimes_H V \simeq \mathcal{M} \otimes V$$

gives the desired W-action.

The boundary quantum KZ equations

Definition (Cherednik)

Let $\pi : H_{\underline{k}} \rightarrow \mathrm{End}(V)$ *be a representation. The boundary quantum Knizhnik-Zamolodchikov (bqKZ) equations are the equations*

$$\nabla(\tau(\lambda))f = f \qquad \forall\, \lambda \in \mathbb{Z}^n$$

for an unknown meromorphic V*-valued function* $f \in \mathcal{M} \otimes V$*. We write* Sol_V *for the space* $\big(\mathcal{M} \otimes V\big)^{\nabla(\tau(\mathbb{Z}^n))}$ *of solutions of the bqKZ equations.*

Remark:

i. BqKZ equations form a compatible system of difference equations:

$$\big(\nabla(\tau(\lambda))f\big)(\underline{z}) = C_{\tau(\lambda)}(\underline{z})f(\underline{z} - \lambda), \qquad \lambda \in \mathbb{Z}^n$$

for suitable $C_{\tau(\lambda)}(\underline{z}) \in \mathrm{End}(V)$ (called transport operators).

ii. Sol_V is $\nabla(W_0)$-invariant, and a F-vector subspace of $\mathcal{M} \otimes V$.

Relation to spectral problem

Definition (Minimal principal series)

Let $\underline{\xi} \in \mathbb{C}^n$. *The minimal principal series with central character* $W_0 q^{\underline{\xi}}$ *is*

$$V(q^{\underline{\xi}}) := \mathrm{Ind}_{\mathcal{A}_Y}^{H_{\underline{k}}}\big(Y_i \mapsto q^{\xi_i}\big).$$

Notation. Standard basis $\{v_w(q^{\underline{\xi}})\}_{w \in W_0}$ of $V(q^{\underline{\xi}})$:

$$v_w(q^{\underline{\xi}}) := (T_{i_1} \cdots T_{i_r}) \otimes_{\mathcal{A}_Y} 1$$

with $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ a reduced expression.

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Theorem (Difference Cherednik-Matsuo correspondence)

For generic parameters $\underline{k}, \underline{u}$ *and generic central character* $\underline{\xi}$*, the linear map* $\chi : \mathcal{M} \otimes V(q^{\underline{\xi}}) \to \mathcal{M}$*, given by* $\sum_{w\in W_0} \psi_w \otimes v_w(q^{\underline{\xi}}) \mapsto \sum_{w\in W_0} k_w \psi_w$ *with* $k_w = k_{i_1}k_{i_2}\cdots k_{i_r}$*, restricts to a* F*-linear* W_0*-equivariant isomorphism*

$$\chi : \mathrm{Sol}_{V(q^{\underline{\xi}})} \xrightarrow{\sim} \mathcal{S}(q^{\underline{\xi}}).$$

Spin representation

Definition

There exists a unique representation $\pi_\alpha : H_{\underline{k}} \to \mathrm{End}\big((\mathbb{C}^2)^{\otimes n}\big)$ *satisfying*

$$\pi_\alpha(T_i) = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & k - k^{-1} & 0 \\ 0 & 0 & 0 & k \end{pmatrix}_{i,i+1},$$

$$\pi_\alpha(T_0) = \begin{pmatrix} k_0 - k_0^{-1} & 1 \\ 1 & 0 \end{pmatrix}_1, \qquad \pi_\alpha(T_n) = \begin{pmatrix} 0 & q^{-\alpha} \\ q^\alpha & k_n - k_n^{-1} \end{pmatrix}_n$$

for $1 \leq i < n$.

Relation to spin chains with boundaries

Baxterization of the spin representation takes on the following form: $f \in \mathcal{M} \otimes (\mathbb{C}^2)^{\otimes n}$,

$$\big(\nabla(s_0)f\big)(\underline{z}) = K^l\big(\frac{1}{2} - z_1\big)_1 f(s_0\underline{z}),$$
$$\big(\nabla(s_i)f\big)(\underline{z}) = P_{i,i+1}R(z_i - z_{i+1})_{i,i+1}f(s_i\underline{z}),$$
$$\big(\nabla(s_n)f\big)(\underline{z}) = K^r(z_n)_n f(s_n\underline{z})$$

with

i. $P \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ the permutation operator,

ii. $R(x) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ an explicit solution of the quantum Yang-Baxter equation,

iii. $K^l(x), K^r(x) \in \mathrm{End}(\mathbb{C}^2)$ an explicit solution of the corresponding left and right reflection equations.

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- **iii.** $K^l(x), K^r(x) \in \mathrm{End}(\mathbb{C}^2)$ an explicit solution of the corresponding left and right reflection equations.

Remark: such triples (K^l, R, K^r) define an integrable one-dimensional quantum spin chain with boundaries on both ends (Sklyanin). In the present case: Heisenberg XXZ spin-$\frac{1}{2}$ chain with boundaries.

Solutions of bqKZ equations

BqKZ equations

$$C_{\tau(\lambda)}(\underline{z})f(\underline{z}-\lambda)=f(\underline{z}) \qquad \forall\,\lambda\in\mathbb{Z}^n$$

associated to the spin representation $\left(\pi_\alpha,\left(\mathbb{C}^2\right)^{\otimes n}\right)$.

Asymptotic version:

1. Asymptotic transport operators $C^\infty_{\tau(\lambda)}\in\mathrm{End}\left(\left(\mathbb{C}^2\right)^{\otimes n}\right)$:

$$C^\infty_{\tau(\lambda)}:=\lim_{\underline{z}\to\infty} C_{\tau(\lambda)}(\underline{z}),$$

   with $\underline{z}\to\infty$ meaning $\Re(z_i-z_{i+1}),\Re(z_n)\to\infty$ as before.

2. Asymptotic bqKZ equations:

$$C^\infty_{\tau(\lambda)}f(\underline{z}-\lambda)=f(\underline{z}) \qquad \forall\,\lambda\in\mathbb{Z}^n.$$

Solutions of bqKZ equations

1. There exists a basis $\{b_{\underline{\epsilon}}\}_{\underline{\epsilon}\in\{\pm\}^{\times n}}$ of $\left(\mathbb{C}^2\right)^{\otimes n}$ consisting of common eigenvectors of the asymptotic transport operators $C^{\infty}_{\tau(\lambda)}$ $(\lambda \in \mathbb{Z}^n)$.
2. F-basis $\{\mathcal{W}_{\underline{\epsilon}}(\underline{z})b_{\underline{\epsilon}}\}_{\underline{\epsilon}}$ of asymptotic bqKZ equations for suitable scalar plane wave functions $\mathcal{W}_{\underline{\epsilon}}$ (compensating for the eigenvalues of $b_{\underline{\epsilon}}$).
3. "Asymptotically free" basis of solutions of bqKZ equations:

$$\mathrm{Sol}_{(\mathbb{C}^2)^{\otimes n}} = \bigoplus_{\underline{\epsilon}} F\Psi_{\underline{\epsilon}}$$

with $\Psi_{\underline{\epsilon}}(\underline{z}) \sim \mathcal{W}_{\underline{\epsilon}}(\underline{z})b_{\underline{\epsilon}}$ if $\underline{z} \to \infty$.

Remark: In special cases: construction of solutions of bqKZ equations as quantum correlation functions of semi-infinite Heisenberg XXZ spin-$\frac{1}{2}$ chain (Jimbo, Kedem, Konno, Miwa, Weston).

Connection problem

For $w \in W_0$,

$$\big(\nabla(w)\Psi_{\underline{\epsilon}'}\big)(\underline{z}) = \sum_{\underline{\epsilon}} M^{w}_{\underline{\epsilon},\underline{\epsilon}'}(\underline{z};\alpha)\Psi_{\underline{\epsilon}}(\underline{z})$$

for unique $M^{w}_{\underline{\epsilon},\underline{\epsilon}'}(\,.\,;\alpha) \in F$.

Definition

Fix $\{v_+, v_-\}$ basis of $\mathbb{C}^2$ and write $v_{\underline{\epsilon}} := v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n}$. The connection matrix $M^w(\,\cdot\,;\alpha)$ is the F-linear operator on $F \otimes \big(\mathbb{C}^2\big)^{\otimes n}$ defined by

$$M^w(\underline{z};\alpha)v_{\underline{\epsilon}'} := \sum_{\underline{\epsilon}} M^{w}_{\underline{\epsilon},\underline{\epsilon}'}(\underline{z};\alpha)v_{\underline{\epsilon}}.$$

Connection problem: compute the matrix coefficients of $M^w(\underline{z};\alpha)$ explicitly in terms of theta functions (by *cocycle property* $M^{vw}(\underline{z};\alpha) = M^v(\underline{z};\alpha)M^w(v^{-1}\underline{z};\alpha)$ it suffices to compute $M^{s_j}(\underline{z};\alpha)$ $(0 \leq j \leq n)$).

The bulk connection matrices $M^{s_i}(x;\alpha)$

Notations:

1. $h:\mathbb{C}^2\rightarrow\mathbb{C}^2$ linear: $hv_\epsilon=\epsilon v_\epsilon$.
2. $\theta(x_1,\ldots,x_r):=\prod_{i=1}^r\theta(x_i)$ with $\theta(x)=\big(x,q/x;q\big)_\infty$.

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1. $h : \mathbb{C}^2 \to \mathbb{C}^2$ linear: $hv_\epsilon = \epsilon v_\epsilon$.
2. $\theta(x_1, \ldots, x_r) := \prod_{i=1}^r \theta(x_i)$ with $\theta(x) = \big(x, q/x; q\big)_\infty$.

Frenkel, Reshetikhin: For $1 \leq i < n$ the connection matrix $M^{s_i}(\underline{z};\alpha)$ is essentially Baxter's dynamical elliptic R-matrix for the 8-vertex face model acting on the i^{th} and $(i+1)^{st}$ tensor leg:

$$M^{s_i}(\underline{z};\alpha) = P_{i,i+1} R(z_i - z_{i+1}; 2\alpha - 2\kappa(h_1 + \cdots + h_{i-1}))_{i,i+1}$$

with $\kappa = -\log_q(k)$ and

$$R(x;\alpha) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A(x;\alpha) & B(x;\alpha) & 0 \\ 0 & B(x;-\alpha) & A(x;-\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$A(x;\alpha) := \frac{\theta(q^{2\kappa-\alpha}, q^{-x})}{\theta(q^{2\kappa-x}, q^{-\alpha})} q^{2\kappa(x-\alpha)}, \quad B(x;\alpha) := \frac{\theta(q^{2\kappa}, q^{-x-\alpha})}{\theta(q^{-\alpha}, q^{2\kappa-x})} q^{(2\kappa+\alpha)x}.$$

Boundary connection matrix $M^{s_n}(x, \alpha)$

Askey-Wilson parameters

$$\{a, b, c, d\} := \{k_n^{-1}u_n^{-1}, -k_n^{-1}u_n, q^{\frac{1}{2}}k_0^{-1}u_0^{-1}, -q^{\frac{1}{2}}k_0^{-1}u_0\}$$

and dual Askey-Wilson parameters

$$\{\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}\} := \{k_n^{-1}k_0^{-1}, -k_n^{-1}k_0, q^{\frac{1}{2}}u_n^{-1}u_0^{-1}, -q^{\frac{1}{2}}u_n^{-1}u_0\}.$$

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Notations:

$$\mathcal{C}(x;\alpha) := \frac{\theta\big(\widetilde{a}q^{\alpha}, \widetilde{b}q^{\alpha}, \widetilde{c}q^{\alpha}, dq^{\alpha-x}/\widetilde{a}\big)}{\theta\big(q^{2\alpha}, dq^{-x}\big)} q^{-(\log_q(a)-x)(\log_q(\widetilde{a})-\alpha)}$$

and

$$\widetilde{\mathcal{C}}(x;\alpha) := \frac{\theta\big(aq^{\alpha}, bq^{\alpha}, cq^{\alpha}, \widetilde{d}q^{\alpha-x}/a\big)}{\theta\big(q^{2\alpha}, \widetilde{d}q^{-x}\big)} q^{-(\log_q(\widetilde{a})-z)(\log_q(a)-\alpha)}.$$

Boundary connection matrix

Theorem

$$M^{s_n}(\underline{z};\alpha) = K(z_n;\alpha - \kappa(h_1 + h_2 + \cdots + h_{n-1}))_n$$

with

$$K(x;\alpha) := \begin{pmatrix} A_b(x;\alpha) & B_b(x;\alpha) \\ B_b(x;-\alpha) & A_b(x;-\alpha) \end{pmatrix},$$

$$A_b(x;\alpha) := \frac{\mathcal{C}(x;\alpha) - \widetilde{\mathcal{C}}(\alpha;x)}{\widetilde{\mathcal{C}}(\alpha;-x)}, \qquad B_b(x;\alpha) := \frac{\mathcal{C}(x;\alpha)}{\widetilde{\mathcal{C}}(-\alpha;-x)}.$$

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Corollary

$K(x;\alpha)$ is a 4-parameter family of solutions of the dynamical reflection equation

$$R_{21}(z_1-z_2;2\alpha)K_1(z_1;\alpha-\kappa h_2)R_{12}(z_1+z_2;2\alpha)K_2(z_2;\alpha-\kappa h_1) = \\ = K_2(z_2;\alpha-\kappa h_1)R_{21}(z_1+z_2;2\alpha)K_1(z_1;\alpha-\kappa h_2)R_{12}(z_1-z_2;2\alpha).$$

Remark:

1. Solutions of the dynamical reflection equation have been computed by direct means by many people: Inami, Konno, de Vega, Gonzalez-Ruiz, Hou, Shi, Fan, Zhang, Behrend, Pearce, Komori, Hikami, Delius, MacKay,....
2. Upshot present approach: representation theoretic interpretation of the dynamical parameter and of the 4 degrees of freedom for the solutions of the dynamical reflection equation:
   i. two boundary parameters k_0, k_n of the affine Hecke algebra;
   ii. two boundary parameters u_0, u_n arising from Noumi's representation;
   iii. dynamical parameter α arising as the representation parameter of the spin representation.