Quantum integrable systems of elliptic Calogero-Moser type

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Outline

1. Overview
Generalities

- Physical perspective: Systems of Calogero-Moser type are integrable one-dimensional N-particle systems that come in various versions: classical/quantum, nonrelativistic/relativistic, with special interactions given by rational/trigonometric/hyperbolic/elliptic functions.
- Harmonic analysis perspective: The quantum systems amount to commutative algebras of operators associated with root systems, with the differential/difference operator case corresponding to Lie groups/quantum groups; their symbols Poisson commute and amount to the classical versions.
- This seminar focuses on the quantum elliptic systems associated with the root systems A_{N-1} and BC_N.

1. Overview

The nr/PDO case

- The nonrelativistic/A_{N-1} quantum Calogero-Moser (CM) Hamiltonian is given by

$$H_{\mathrm{nr}} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\partial_{x_j}^2 + \frac{g(g-\hbar)}{m}\sum_{1\le j<k\le N} V(x_j - x_k),$$

where $\hbar > 0$ (Planck's constant), $m > 0$ (particle mass), $g \in \mathbb{R}$ (coupling constant), $V(x)$ pair potential of four types:

I.	$1/x^2$	(rational)
II.	$\pi^2/\alpha^2 \sinh^2(\pi x/\alpha), \quad \alpha > 0$	(hyperbolic)
III.	$r^2/\sin^2(rx), \quad r > 0$	(trigonometric)
IV.	$\wp(x; \pi/2r, i\alpha/2), \quad r, \alpha > 0$	(elliptic)

- Associated integrable system (N commuting PDOs):

$$H_1 = -i\hbar \sum_{j=1}^{N} \partial_{x_j}, \quad H_2 = mH_{\rm nr},$$

$$H_k = \frac{(-i\hbar)^k}{k} \sum_{j=1}^{N} \partial_{x_j}^k + \text{l. o.}, \quad k = 3, \ldots, N,$$

where l.o. = lower order in partials.

- Physical picture:

$$H_{\rm nr}, \ \ P_{\rm nr} = H_1, \ \ B = -m \sum_{j=1}^{N} x_j,$$

represent the Lie algebra of the Galilei group:

$$[H_{\rm nr}, P_{\rm nr}] = 0, [H_{\rm nr}, B] = i\hbar P_{\rm nr}, [P_{\rm nr}, B] = i\hbar Nm.$$

- The 'nonrelativistic'/BC_N elliptic Hamiltonian is given by

$$H_{\rm nr} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\partial_{x_j}^2 + \frac{g(g-\hbar)}{m}\sum_{\substack{1\le j<k\le N\\ \tau=+,-}}\wp(x_j-\tau x_k)$$

$$+\sum_{j=1}^{N}\sum_{t=0}^{3}\frac{g_t(g_t-\hbar)}{2m}\wp(x_j+\omega_t).$$

- It was introduced by Inozemtsev, who showed integrability of the classical version. On the quantum level there also exist $N-1$ additional pairwise commuting PDOs (Oshima/H. Sekiguchi) of orders $4,\ldots,2N$.
- The $N=1$ Schrödinger equation amounts to the Heun equation.

1. Overview

The rel/AΔO case

- The relativistic/A_{N-1} systems yield N commuting AΔOs (analytic difference operators):

$$H_k(x) = \sum_{|I|=k} \prod_{\substack{m\in I \\ n\notin I}} f_-(x_m - x_n) \cdot \prod_{m\in I} e^{-i\hbar\beta\partial_{x_m}} \cdot \prod_{\substack{m\in I \\ n\notin I}} f_+(x_m - x_n),$$

where $k = 1, \ldots, N$, $\beta > 0$, and $f_\pm(x)^2$ given by

I. $(x \pm i\beta g)/x$,

II. $\sinh(\pi(x \pm i\beta g)/\alpha)/\sinh(\pi x/\alpha)$,

III. $\sin(r(x \pm i\beta g))/\sin(rx)$,

IV. $\sigma(x \pm i\beta g; \pi/2r, i\alpha/2)/\sigma(x; \pi/2r, i\alpha/2)$.

- Physical picture: $\beta = 1/mc$ and c = light speed;

$$H_{\rm rel} = mc^2[H_1(x) + H_1(-x)], \quad P_{\rm rel} = mc[H_1(x) - H_1(-x)],$$

and B yield the Lie algebra of the Poincaré group:

$$[H_{\rm rel}, P_{\rm rel}] = 0, [H_{\rm rel}, B] = i\hbar P_{\rm rel}, [P_{\rm rel}, B] = i\hbar c^{-2} H_{\rm rel}.$$

- The nonrelativistic limit $c \to \infty$ gives

$$P_{\mathrm{rel}} \to P_{\mathrm{nr}}, \quad H_{\mathrm{rel}} - Nmc^2 \to H_{\mathrm{nr}}.$$

- The hyperbolic and elliptic regimes have two length scales, namely

$$a_+ \equiv \alpha, \quad (\text{imaginary period/interaction length}),$$

and

$$a_- \equiv \hbar/mc, \quad (\text{shift step size/Compton wave length}).$$

- The above family of AΔOs H_k with a_+ and a_- interchanged yields a second family commuting with the first one. Hence, eigenfunctions of one family that are symmetric under interchange of a_+ and a_- (modular-invariant) are joint eigenfunctions of both families. (In the hyperbolic case this can be tied in with the modular quantum groups introduced by Faddeev.)

- To bring out modular symmetry and another $\mathbb{Z}_2$ symmetry, it is crucial to reparametrize the commuting AΔOs $H_1, \ldots, H_N$. To this end (and also for later purposes) we need the elliptic gamma function $G(z)$ and allied functions. We have

$$G(z) := \prod_{m,n=0}^{\infty} \frac{1 - q_+^{2m+1} q_-^{2n+1} e^{-2irz}}{1 - q_+^{2m+1} q_-^{2n+1} e^{2irz}},$$

where $q_\pm := \exp(-ra_\pm)$. It corresponds to two elliptic curves with real period π/r and imaginary periods ia_+, ia_-.
- We also need the RHS functions in the AΔEs to which G is the minimal solution:

$$\frac{G(z + ia_\delta/2)}{G(z - ia_\delta/2)} = R_{-\delta}(z), \quad \delta = +, -,$$

$$R_\delta(z) = \prod_{l=0}^{\infty}(1 - q_\delta^{2l+1} e^{2irz})(z \to -z).$$

(Thus R_δ is even and π/r-periodic.)

- Next, we need a Harish-Chandra function

$$c(z) := G(z + ia - ib)/G(z + ia), \quad a := (a_+ + a_-)/2,$$

weight function $w(z) := 1/c(z)c(-z)$ and scattering function

$$u(z) := c(z)/c(-z).$$

Their multi-variate versions are

$$F(x) := \prod_{1\leq j<k\leq N} f(x_j - x_k), \quad f = c, w, u.$$

- Setting

$$\rho_{\delta,\pm}(z) := R_\delta(z \pm (ia_\delta/2 - ib))/R_\delta(z \pm ia_\delta/2),$$

we introduce $2N$ commuting Hamiltonians

$$H_{k,\delta}(x) := \sum_{|I|=k} \prod_{\substack{m\in I \\ n\notin I}} \Big(\rho_{\delta,+}(x_m - x_n)\rho_{\delta,-}(x_m - x_n - ia_{-\delta})\Big)^{1/2}$$

$$\times \prod_{m\in I} e^{-ia_{-\delta}\partial_{x_m}}, \quad k = 1, \ldots, N, \quad \delta = +, -.$$

- Now $H_{k,+}$ amounts to the previous H_k up to a multiplicative constant. The present normalization entails invariance under $b \mapsto 2a - b$.
- We also need $2N$ AΔOs

$$A_{k,\delta}(x) := W(x)^{-1/2} H_{k,\delta}(x) W(x)^{1/2}.$$

Using the G-AΔEs they can be written as

$$A_{k,\delta}(x) = \sum_{|I|=k} \prod_{\substack{m \in I \\ n \notin I}} \rho_{\delta,+}(x_m - x_n) \cdot \prod_{m \in I} e^{-ia_{-\delta}\partial_{x_m}}.$$

They are not invariant under $b \mapsto 2a - b$, since $W(x)$ is not. But since $U(x)$ is invariant, the AΔOs

$$\mathcal{A}_{k,\delta} := U(x)^{-1/2} H_{k,\delta} U(x)^{1/2} = C(x)^{-1} A_{k,\delta} C(x),$$

are invariant. Each of these three AΔO-families is crucial for further developments.

- A 'relativistic' Hamiltonian H_{vD} for the BC_N case is due to van Diejen; the associated $N-1$ commuting Hamiltonians were shown to exist by Hikami/Komori, and will not be considered here. As in the A_{N-1} case, we need AΔOs $H_{\pm}$, $A_{\pm}$ and $\mathcal{A}_{\pm}$, with H_+ of the form

$$H_+ = C_1 H_{vD} + C_2, \quad C_1, C_2 \in \mathbb{C}^*.$$

As before, these choices reveal non-manifest symmetries.

- In order to detail the $N=1$ AΔOs, we again need a Harish-Chandra function

$$c_e(z) := \frac{1}{G(2z+ia)} \prod_{\mu=0}^{7} G(z - i\gamma_\mu), \quad \gamma_0, \ldots, \gamma_7 \in \mathbb{C},$$

weight function $w_e(z) := 1/c_e(z)c_e(-z)$ and scattering function $u_e(z) := c_e(z)/c_e(-z)$.

- Once again, we have the relations

$$A_\delta(z) = w_e(z)^{-1/2} H_\delta(z) w_e(z)^{1/2},$$

$$\mathcal{A}_\delta(z) = u_e(z)^{-1/2} H_\delta(z) u_e(z)^{1/2} = c_e(z)^{-1} A_\delta(z) c_e(z).$$

Here, A_δ is of the form

$$A_\delta = V_\delta(z) \exp(-ia_{-\delta}\partial_z) + (z \to -z) + V_{b,\delta}(z),$$

with

$$V_\delta(z) := c_e(z)/c_e(z - ia_{-\delta}).$$

- Letting

$$V_{a,\delta}(z) := V_\delta(-z) V_\delta(z + ia_{-\delta}),$$

it follows that we have

$$H_\delta = V_{a,\delta}(z)^{1/2} \exp(ia_{-\delta}\partial_z) + (z \to -z) + V_{b,\delta}(z),$$

$$\mathcal{A}_\delta = \exp(-ia_{-\delta}\partial_z) + V_{a,\delta}(z) \exp(ia_{-\delta}\partial_z) + V_{b,\delta}(z).$$

- Using the G-AΔEs, the functions $V_\delta(z)$ and $V_{a,\delta}(z)$ can be expressed solely in terms of $R_\delta(z)$. In particular,

$$V_{a,\delta}(z) = D_\delta(z)^{-1} \prod_{\mu=0}^{7} \prod_{\tau=+,-} R_\delta(z + \tau i\gamma_\mu + ia_{-\delta}/2),$$

  with the denominator $D_\delta(z)$ a product of γ-independent R_δ-functions. As a result, $V_{a,\delta}(z)$ is elliptic in z and has B_8-symmetry in γ. (I. e., invariance under S_8 and sign flips.)
- The additive potential $V_{b,\delta}(z)$ is also elliptic and can be characterized in terms of its residues at 4 simple poles in a period cell. It admits an explicit formula from which D_8-symmetry in γ can be read off. (I. e., S_8 and even sign flips.)
- As a consequence, the AΔOs $H_\pm$ and $\mathcal{A}_\pm$ are D_8-invariant. (But $w_e(z)$ is not, so $A_\pm$ are not.)

- The generators S_0, S_1, S_2, S_3 of the Sklyanin algebra have representations (labeled by $\nu \in \mathbb{C}^*$) as AΔOs acting on even meromorphic functions. In these representations the quadratic part of the algebra is 9-dimensional. It can be viewed as the linear combinations of the van Diejen AΔOs $A_+(z)$ (with $\sum_\mu \gamma_\mu$ fixed), plus the constants. In fact, the generators themselves are represented by AΔOs that can be regarded as special van Diejen AΔOs. (See E. Rains/S. R., CMP 2013 for these results and other ones.)
- The 4-coupling Heun operator can be tied in with Painlevé VI (via the so-called Painlevé-Calogero correspondence). The conjecture (S. R., 2008) that the 8-coupling 'relativistic' Heun (i. e., van Diejen) operator has a similar relation to the Sakai elliptic difference Painlevé equation is still open.

- Turning finally to 'relativistic' BC_N with $N > 1$, the commuting modular pair $H_\pm$ of defining Hamiltonians is of the form

$$\sum_{j=1}^{N}\Big(\mathcal{V}_{j,\pm}(x)^{1/2}e^{-ia_\mp\partial_{x_j}}\mathcal{V}_{j,\pm}(-x)^{1/2}+(x\to -x)\Big)+\mathcal{V}_\pm(x).$$

Here, we have

$$\mathcal{V}_{j,\delta}(x) := V_\delta(x_j)\prod_{\substack{k\neq j\\ \tau=+,-}}\frac{R_\delta(x_j-\tau x_k-ib+ia_\delta/2)}{R_\delta(x_j-\tau x_k+ia_\delta/2)},$$

with $V_\delta(z)$ the previous BC_1 coefficient, and with $\mathcal{V}_\delta(x)$ an elliptic function whose definition we skip.
- Next, we introduce the Harish-Chandra function

$$C(x) := \prod_{j=1}^{N}c_e(x_j)\cdot\prod_{\substack{1\leq j<k\leq N\\ \tau=+,-}}\frac{G(x_j-\tau x_k-ib+ia)}{G(x_j-\tau x_k+ia)},$$

weight function $W(x) := 1/C(x)C(-x)$ and scattering function $U(x) := C(x)/C(-x)$.

- Then we get again the two H_δ-avatars

$$A_\delta(x) := W(x)^{-1/2} H_\delta(x) W(x)^{1/2},$$

and

$$\mathcal{A}_\delta(x) := U(x)^{-1/2} H_\delta(x) U(x)^{1/2} = C(x)^{-1} A_\delta(x) C(x).$$

- The AΔOs $A_\pm$ and $H_\pm$ are BC_N-invariant, whereas $\mathcal{A}_\pm$ are not invariant under sign changes of x_j (since $C(x)$ is not). The AΔOs $\mathcal{A}_\pm$ and $H_\pm$ are D_8-invariant, whereas $A_\pm$ are not invariant under even sign changes of γ_μ (since $C(x)$ is not).
- This 9-coupling family admits a great many degenerations and limits. In particular, the trigonometric specialization of A_+ is the 5-coupling Koornwinder AΔO, which has Koornwinder-Macdonald polynomials as eigenfunctions, and the 'nonrelativistic' limit of H_+ yields the previous 5-coupling Inozemtsev PDO.

2. Eigenfunctions and kernel functions

Generalities

- Given a set of commuting operators, the obvious first problem is to show or rule out the existence of joint eigenfunctions. In case joint eigenfunctions exist, the next problem is to obtain explicit information about them. Finally, with sufficient information available, the problem of finding a Hilbert space reinterpretation of the commuting operators can be addressed.
- For the Hilbert space joint eigenfunction problem, the spectral theorem is of little use, since it assumes the existence of commuting self-adjoint operators. The PDOs/AΔOs are only formally self-adjoint, however.
- Especially in the AΔO case, there are hardly any 'useful' existence results available. In fact, already for the 1-variable case there are simple examples of commuting AΔOs without joint eigenfunctions.

2. Eigenfunctions and kernel functions

Some eigenfunction results

- Abundant results on eigenfunctions exist for the Lamé/Heun cases (equivalently, the nonrelativistic A_1/BC_1 cases). Far less is known about their relativistic counterparts.
- For A_{N-1} with $N > 2$ there are results of 'Bethe Ansatz' type. They are restricted to certain discrete couplings and to the defining Hamiltonian (Felder/Varchenko for the PDO case, Billey for the AΔO case).
- Results by Komori/Takemura on the nr/PDO case yield existence of joint Hilbert space eigenfunctions reducing to (basically) the Jack-Sutherland polynomials in the trigonometric limit. Since perturbation theory is used, restrictions on the imaginary period and the coupling are present.

2. Eigenfunctions and kernel functions

Kernel functions: a survey

- Given a pair of operators $H_1(x)$ and $H_2(y)$, a kernel function is a function $\Psi(x, y)$ satisfying

$$H_1(x)\Psi(x, y) = H_2(y)\Psi(x, y).$$

  Here, x and y may vary over spaces of different dimension. Used as kernels of integral operators, the latter can be used to connect eigenfunctions of H_2 to those of H_1.
- For the above elliptic N-variable Hamiltonians, kernel functions with both x and y varying over $\mathbb{C}^N$ are known, imposing one coupling constraint for the BC_N case with $N > 1$. Probably the earliest result (with H_1, H_2 Lamé operators) is due to Whittaker (1915).

- The first multi-variate result has been obtained by Langmann (2000). It pertains to the defining A_{N-1} PDO. Specifically, H_1 and H_2 equal (with $m = \hbar = 1$)

$$H_{nr} = -\frac{1}{2}\sum_{j=1}^{N} \partial_{x_j}^2 + g(g-1) \sum_{1\leq j<k\leq N} \wp(x_j - x_k),$$

and his kernel function amounts to

$$W_{nr}(x)^{1/2} W_{nr}(y)^{1/2} \prod_{j,k=1}^{N} R(x_j - y_k + \xi)^{-g},$$

$$W_{nr}(x) := \Big(\prod_{1\leq j<k\leq N} R(x_j - x_k + i\alpha/2) R(x_j - x_k - i\alpha/2) \Big)^g .$$

He has used this as a starting point to derive perturbative formulas for H_{nr}-eigenfunctions.
- In later work (partly joint with Takemura), he obtains so-called source identities. They can be specialized to obtain various kernel identities for more general elliptic PDOs (more than one mass, e. g.).

- Kernel functions for the $2N$ commuting A_{N-1} AΔOs were first presented at the Kyoto EIS Workshop (S. R., 2004). For $A_{k,\delta}$ one can take in particular

$$\mathcal{S}_\xi(x,y) = \prod_{j,k=1}^{N} \frac{G(x_j - y_k - ib/2 + \xi)}{G(x_j - y_k + ib/2 + \xi)}, \quad \xi \in \mathbb{C}.$$

- Taking the nonrelativistic limit of the $H_{k,\delta}$-kernel function

$$W(x)^{1/2} W(y)^{1/2} \mathcal{S}_\xi(x,y),$$

  we get Langmann's kernel function, together with the kernel function property for the higher-order commuting PDOs.
- Similar kernel functions for the defining BC_N AΔO and PDO also date back to the Kyoto EIS Workshop. (For $N > 1$ one balancing condition is needed.)

3. Kernel functions: HS approach

Preamble

- A long-standing goal is to reinterpret the $2N$ commuting A_{N-1} AΔOs $\mathcal{A}_{k,\delta}(x)$ as commuting self-adjoint operators on the Hilbert space

$$\mathcal{H}_A := L^2(F_A, dx),$$

$$F_A := \{-\pi/2r < x_N < \cdots < x_1 \leq \pi/2r\}.$$

- Likewise, the 2 commuting BC_N AΔOs $\mathcal{A}_\delta(x)$ ought to be promoted to commuting self-adjoint operators on the Hilbert space

$$\mathcal{H}_B := L^2(F_B, dx),$$

$$F_B := \{0 < x_N < \cdots < x_1 \leq \pi/2r\}.$$

- To this end, we need 'only' show existence of an ONB of joint eigenfunctions with real eigenvalues.

- Under suitable restrictions on the parameters, the kernel functions give rise to Hilbert-Schmidt (HS) integral operators $\mathcal{I}_\xi$ and $\mathcal{I}$ on $\mathcal{H}_A$ and $\mathcal{H}_B$, resp. Then the operators

$$\mathcal{T}_\xi := \mathcal{I}_\xi \mathcal{I}_\xi^*, \quad \mathcal{T} := \mathcal{I}\mathcal{I}^*,$$

  are self-adjoint trace class operators.
- The spectral theorem now guarantees the existence of an ONB of eigenvectors for these operators, but it yields no further information. In particular, the operators can a priori have an infinite-dimensional zero-eigenvalue eigenspace.
- It follows from recent results (S. R., 2012) that the relevant operators actually have trivial null space and dense range.
- Crux: it can be expected that the $\mathcal{T}_\xi$- and $\mathcal{T}$-eigenvectors extend to meromorphic eigenfunctions of the AΔOs $\mathcal{A}_{k,\delta}$ and $\mathcal{A}_\delta$ with real eigenvalues.

- Reason: the AΔOs are formally self-adjoint and formally satisfy

$$[\mathcal{A}_{k,\delta}, \mathcal{T}_{\xi}] = 0, \quad [\mathcal{A}_{\delta}, \mathcal{T}] = 0,$$

due to the kernel identities. Thus the eigenvector ONB of the trace class operators 'should' yield an ONB of joint eigenfunctions of the commuting AΔOs.

- This approach is easily understood and formally convincing, but a lot of analysis is needed to make it work. This involves in particular complex analysis to prove the meromorphy of the $\mathcal{T}$-eigenfunctions, and functional analysis to control dense domains for the AΔOs. (No general Hilbert space theory for AΔOs exists to date.)

3. Kernel functions: HS approach

The A_{N-1} case

- Work in progress (need $b \in (0, a_+ + a_-)$); There is circumstantial evidence for the conjecture that the ONB can be labelled by

$$n \in \mathbb{Z}^N_{\geq} \equiv \{n \in \mathbb{Z}^N \mid n_1 \geq \cdots \geq n_N\},$$

  in such a way that when the minimum of the gaps $n_j - n_{j+1}$, $j = 1, \ldots, N-1$, tends to ∞ one has asymptotics proportional to

$$\sum_{\sigma \in S_N} \frac{C(x_\sigma)}{C(x)} \exp(2irn \cdot x_\sigma).$$

  (If so, the dual dynamics yield a factorized S-matrix.)
- For the cases $b = a_+$ and $b = a_-$ the joint eigenvector ONB amounts to 'free fermions' ($\sim$ Schur polynomials), the AΔO-eigenvalues are obvious, and the eigenvalues for a modified HS family are explicitly known too (S. R., 2009).

3. Kernel functions: HS approach

The BC_1 case

- Here the 'initial' kernel identity reads

  $$A_\delta(\gamma;x)\mathcal{S}(\sigma(\gamma);x,y) = A_\delta(\gamma';y)\mathcal{S}(\sigma(\gamma);x,y),\ \ \delta = +,-,$$

  where

  $$\gamma' \equiv -J\gamma,$$

  $$\sigma(\gamma) \equiv -\frac{1}{4}\sum_{\mu=0}^{7}\gamma_\mu = -\frac{1}{4}\langle\zeta,\gamma\rangle, \quad \zeta \equiv (1,\ldots,1),$$

  and J can be viewed as the reflection associated with the highest E_8 root $\zeta/2$, i. e.,

  $$J \equiv \mathbf{1}_8 - \frac{1}{4}\zeta\otimes\zeta.$$

- The kernel function is given by

  $$\mathcal{S}(t;x,y) \equiv \prod_{\delta_1,\delta_2=+,-} G(\delta_1 x + \delta_2 y - ia + it),$$

  with $G(z)$ the elliptic gamma function.

- For the AΔOs $A_{\pm}(\gamma; x)$ the relevant Hilbert space is the weighted L^2 space

$$\mathcal{H}_w \equiv L^2([0, \pi/2r], w_e(\gamma; x)dx).$$

- It is crucial to switch from this AΔO pair to the D_8-invariant AΔOs

$$\mathcal{A}_\delta(\gamma; x) = c_e(\gamma; x)^{-1} A_\delta(\gamma; x) c_e(\gamma; x),$$

which are formally self-adjoint on

$$\mathcal{H} = L^2([0, \pi/2r], dx),$$

for suitable γ (in particular for $\gamma \in \mathbb{R}^8$).

- They satisfy the kernel identity

$$\mathcal{A}_\delta(\gamma; x)\mathcal{K}(\gamma; x, y) = \mathcal{A}_\delta(\gamma'; -y)\mathcal{K}(\gamma; x, y),$$

with

$$\mathcal{K}(\gamma; x, y) \equiv \frac{\mathcal{S}(\sigma(\gamma); x, y)}{c_e(\gamma; x) c_e(\gamma'; -y)}.$$

- With further restrictions on γ, the kernel function $\mathcal{K}(\gamma; x, y)$ yields a HS integral operator $\mathcal{I}(\gamma)$ on $\mathcal{H}$ with a trivial null space and dense range. Requiring $\gamma \in \mathbb{R}^8$ from now on, the restriction

$$\gamma_\mu,\ \gamma'_\mu \in (-a, a),\ \ \sigma(\gamma) \in (0, a),$$

suffices.
- With this restriction, we can show that the resulting eigenvector $\mathcal{H}$-ONB $f_n(\gamma)$, $n = 0, 1, 2, \ldots$, for the self-adjoint trace class operator $\mathcal{I}(\gamma)\mathcal{I}(\gamma)^*$ has the following features:
  - $f_n(\gamma)$ is the restriction to $[0, \pi/2r]$ of a meromorphic function $f_n(\gamma; x)$ with known pole locations depending only on γ;
  - Setting

$$a_s \equiv \min(a_+, a_-),\ \ a_l \equiv \max(a_+, a_-),$$

and assuming a_l is not a multiple of a_s, the functions $f_n(\gamma; x)$ are joint eigenfunctions of $\mathcal{A}_\pm(\gamma; x)$ with real eigenvalues.

- Consequence: With the above restrictions on $a_{\pm}$ and γ understood, the AΔOs give rise to commuting self-adjoint operators $\hat{\mathcal{A}}_{\pm}(\gamma)$ on $\mathcal{H}$ with discrete spectra.
- Further results include:
  • The definition of $\hat{\mathcal{A}}_{\pm}(\gamma)$ implies that the operators are invariant under D_8-transformations of γ.
  • For γ in the ball $\|\gamma\|_2 < a$ (with the origin deleted), the operators are isospectral under E_8-transformations. Generically, this yields 135 (=$|W(E_8)/W(D_8)|$) distinct isospectral operators.
  • For generic γ, we also get 64 distinct commuting HS operators.
  • The asymptotic behavior as $n \to \infty$ of the eigenfunctions $f_n(\gamma; x)$ is the same as that of an $\mathcal{H}$-ONB of functions $P_n(\gamma; x)/c_P(\gamma; x)$, with $P_n(\gamma; x)$ orthonormal polynomials; this relation also leads to detailed information on eigenvalue asymptotics.

- Recent references re HS approach:
- Hilbert-Schmidt-operators vs. integrable systems of elliptic Calogero-Moser type. II. The A_{N-1} case: First steps, Comm. Math. Phys. **286** (2009), 659–680
- Hilbert-Schmidt-operators vs. integrable systems of elliptic Calogero-Moser type. III. The Heun case, SIGMA **5** (2009), 049, 21 pages
- On positive Hilbert-Schmidt operators, Integr. Equ. Oper. Theory, **75** (2013), 393–407
- Hilbert-Schmidt-operators vs. integrable systems of elliptic Calogero-Moser type. IV. The relativistic Heun (van Diejen) case, SIGMA **11** (2015), 004, 78 pages