Special Functions arising from Elliptic Integrable Systems

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○ **Keywords:**
Representation theory, hypergeometric functions and Painlevé equations

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1 Rational, trigonometric and elliptic

○ Hermite's theorem

If a nonzero entire function $s(z)$ ($z \in \mathbb{C}$) satisfies the functional equation

$$\begin{aligned} s(z+a)s(z-a)s(b+c)s(b-c) + s(z+b)s(z-b)s(c+a)s(c-a) \\ + s(z+c)s(z-c)s(a+b)s(a-b) = 0 \end{aligned} \tag{1.1}$$

for $z, a, b, c \in \mathbb{C}$, then, up to multiplication by $\exp(az^2 + c)$ for some $a, c \in \mathbb{C}$, it belongs to one of the following three classes of functions:

$$\begin{array}{llcll} (0) & \textit{rational} & : & s(z) = z & \Omega = 0 \\ (1) & \textit{trigonometric} & : & s(z) = \sin(\pi z/\omega_1) & \Omega = \mathbb{Z}\,\omega_1 \\ (2) & \textit{elliptic} & : & s(z) = \sigma(z|\Omega) & \Omega = \mathbb{Z}\,\omega_1 \oplus \mathbb{Z}\,\omega_2, \end{array} \tag{1.2}$$

where $\sigma(z|\Omega)$ denotes the Weierstrass sigma function

$$\sigma(z|\Omega) = z \prod_{\omega \in \Omega,\, \omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{z^2/2\omega + z/\omega}. \tag{1.3}$$

○ Hirota equation in dimension one

For a nonzero odd entire function $s(z)$ given, consider the 1-dimensional non-autonomous *Hirota equation*

$$(H) \qquad \tau(z \pm a)\, s(b \pm c) + \tau(z \pm b)\, s(c \pm a) + \tau(z \pm c)\, s(a \pm b) = 0 \tag{1.4}$$

for $\tau(z)$, where $\tau(z \pm a) = \tau(z+a)\tau(z-a)$. One can show that, if $s'(0) \neq 0$, any nonzero holomorphic solution $\tau(z)$ must be a function in the three classes of functions.

Regarding the LHS of (H) as a holomorphic function in (a, b, c), expand it at $(0, 0, 0)$:

$$\tau(z \pm a) = \sum_{i=0}^{\infty} \frac{a^i}{i!}\, D_z^i\, \tau(z)\cdot\tau(z), \quad s(b \pm c) = \sum_{j,k=0}^{\infty} f_{jk}\, \frac{b^j}{j!}\frac{c^k}{k!}, \tag{1.5}$$

where f_{jk} are determined by the Taylor coefficients of $s(z) = s_1 z/1! + s_3 z^3/3! + \cdots$. Then (H) can be regarded as an infinite family of Hirota bilinear differential equations

$$(H_{ijk}) \qquad \left(f_{jk} D_z^i + f_{ki} D_z^j + f_{ij} D_z^k\right) \tau(z)\cdot\tau(z) = 0 \quad (i, j, k \in \mathbb{N}). \tag{1.6}$$

The first nontrivial bilinear differential equation arises when $(i,j,k)=(0,2,4)$:

$$\begin{aligned} (H_{024}) \qquad & \left(f_{24}+f_{40}D_z^2+f_{02}D_z^4\right)\tau(z)\cdot\tau(z)=0, \\ & f_{24}=4(s_3^3-s_1s_5), \quad f_{40}=8s_1s_3, \quad f_{02}=-2s_1^2. \end{aligned} \tag{1.7}$$

If we set $\varphi(z)=-\partial_z^2\log\tau(z)$,

$$D_z^2\,\tau(z)\cdot\tau(z)=-2\tau(z)^2\varphi(z), \quad D_z^4\,\tau(z)\cdot\tau(z)=2\tau(z)^2\big(6\varphi(z)^2-\varphi''(z)\big), \tag{1.8}$$

and hence LHS of (H_{024}) is rewritten as

$$\begin{aligned} & \tau(z)^2\left(f_{24}-2f_{40}\,\varphi(z)+2f_{02}(6\,\varphi(z)^2-\varphi''(z))\right) \\ & =-4\tau(z)^2\left((s_5s_1-s_3^2)+4s_3s_1\varphi(z)+6s_1^2\varphi(z)^2-s_1^2\varphi''(z).\right) \end{aligned} \tag{1.9}$$

This implies

$$\varphi''(z)=6\,\varphi(z)^2+c_1\varphi(z)+\frac{c_2}{2}; \quad c_1=\frac{4s_3}{s_1}, \quad c_2=\frac{2(s_5s_1-s_3^2)}{s_2}. \tag{1.10}$$

Multiplied by $2\varphi'(z)$, this equation is integrated into

$$\begin{aligned} \varphi'(z)^2 &= 4\,\varphi(z)^3+c_1\varphi(z)^2+c_1\varphi(z)+c_3 \\ &= 4\,(\varphi(z)-\alpha_1)(\varphi(z)-\alpha_2)(\varphi(z)-\alpha_3). \end{aligned} \tag{1.11}$$

2 Elliptic hypergeometric functions

○ Elliptic hypergeometric series

Let $[z] = s(z)$ $(z \in \mathbb{C})$ be a function in the three classes of Hermite's Theorem. Fixing a generic nonzero constant $\delta \in \mathbb{C}$, we define the *δ-shifted factorials* for $[z]$ by

$$[z]_k = [z][z+\delta]\cdots[z+(k-1)\delta] \quad (k = 0, 1, 2\ldots), \tag{2.1}$$

very well-posed hypergeometric series

$${}_{r+5}V_{r+4}\big(\alpha_0; \alpha_1, \ldots, \alpha_r; z\big) = \sum_{k=0}^{\infty} \frac{[\alpha_0 + 2k\delta]}{[\alpha_0]} \frac{[\alpha_0]_k}{[\delta]_k} \prod_{i=1}^{r} \frac{[\alpha_i]_k}{[\delta + \alpha_0 - \alpha_i]_k} z^k \tag{2.2}$$

(as a formal power series in z), and

$${}_{r+5}V_{r+4}\big(\alpha_0; \alpha_1, \ldots, \alpha_r\big) = \sum_{k=0}^{N} \frac{[\alpha_0 + 2k\delta]}{[\alpha_0]} \frac{[\alpha_0]_k}{[\delta]_k} \prod_{i=1}^{r} \frac{[\alpha_i]_k}{[\delta + \alpha_0 - \alpha_i]_k} \tag{2.3}$$

for terminating V series, when $\alpha_i \equiv -N\delta \pmod{\Omega}$, $N \in \mathbb{N}$, for some $i \in \{0, 1, \ldots, r\}$.

- *Rational case*: When $[z]=z$ and $\delta=1$, the V series above is expressed as

$$ {}_{r+2}F_{r+1}\left(\begin{matrix}\alpha_0, \frac{\alpha_0}{2}+1, \alpha_1, \ldots, \alpha_r \\ \frac{\alpha_0}{2}+1,\ \beta_1,\ \ldots,\ \beta_r\end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{\alpha+2k}{\alpha_0} \frac{(\alpha_0)_k}{(1)_k} \prod_{i=1}^{r} \frac{(\alpha_i)_k}{(1+\alpha_0-\alpha_i)_k} z^k \tag{2.4} $$

with $\alpha_i+\beta_i=\alpha_0+1$ $(i=1,\ldots,r)$, where $(\alpha)_k=\alpha(\alpha+1)\cdots(\alpha+k-1)$.

- *Trigonometric case*: When $[z]=\sin z$, we also use the multiplicative variables

$$ u=e(z)=\exp(2\pi\sqrt{-1}z), \quad q=e(\delta)\ (|q|<1), \quad a_i=e(\alpha_i)\ \ (i=1,\ldots,r). \tag{2.5} $$

Then the V series defined above gives

$$ {}_{r+3}W_{r+2}\,(a_0;a_1,\ldots.a_r;q,v) = \sum_{k=0}^{\infty} \frac{1-q^{2k}a_0}{1-a_0} \frac{(a_0;q)_k}{(q;q)_k} \prod_{i=1}^{r} \frac{(a_i;q)_k}{(qa_0/a_i;q)_k} v^k \tag{2.6} $$

with $v=(qa_0)^{\frac{r-1}{2}}u/a_1\cdots a_r$, where $(a;q)_k=(1-a)(1-qa)\cdots(1-q^{k-1}a)$.

○ Frenkel-Turaev sum and elliptic Bailey transform (1997)

• *Frenkel-Turaev summation formula*:

When $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\delta+2\alpha_0$ (balancing condition) and $\alpha_5=-N\delta$ $(N\in\mathbb{N})$:

$$
\begin{aligned}
&{}_{10}V_9\big(\alpha_0;\,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\big)\\
&=\frac{[\delta+\alpha_0]_N[\delta+\alpha_0-\alpha_1-\alpha_2]_N[\delta+\alpha_0-\alpha_1-\alpha_3]_N[\delta+\alpha_0-\alpha_2-\alpha_3]_N}{[\delta+\alpha_0-\alpha_1]_N[\delta+\alpha_0-\alpha_2]_N[\delta+\alpha_0-\alpha_3]_N[\delta+\alpha_0-\alpha_1-\alpha_2-\alpha_3]_N}.
\end{aligned}
\tag{2.7}
$$

• *Elliptic Bailey transformation formula*:

When $\alpha_1+\alpha_2+\cdots+\alpha_7=2\delta+3\alpha_0$ and $\alpha_7=-N\delta$ $(N\in\mathbb{N})$:

$$
\begin{aligned}
&{}_{12}V_{11}\big(\alpha_0;\,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7\big)\\
&=\frac{[\delta+\alpha_0]_N[\delta+\alpha_0-\alpha_4-\alpha_5]_N[\delta+\alpha_0-\alpha_4-\alpha_6]_N[\delta+\alpha_0-\alpha_5-\alpha_6]_N}{[\delta+\alpha_0-\alpha_4]_N[\delta+\alpha_0-\alpha_5]_N[\delta+\alpha_0-\alpha_6]_N[\delta+\alpha_0-\alpha_4-\alpha_5-\alpha_6]_N}\\
&\quad\cdot{}_{12}V_{11}\big(\widetilde{\alpha}_0;\,\widetilde{\alpha}_1,\widetilde{\alpha}_2,\widetilde{\alpha}_3,\alpha_4,\alpha_5,\alpha_6,-N\delta\big)
\end{aligned}
\tag{2.8}
$$

$$\widetilde{\alpha}_0=\delta+2\alpha_0-\alpha_1-\alpha_2-\alpha_3;$$

$$\widetilde{\alpha}_1=\delta+\alpha_0-\alpha_2-\alpha_3,\quad \widetilde{\alpha}_2=\delta+\alpha_0-\alpha_1-\alpha_3,\quad \widetilde{\alpha}_3=\delta+\alpha_0-\alpha_1-\alpha_2.$$

○ Rahman's q-hypergeometric integral

- *Askey-Wilson beta integral*:

$$\frac{(q;q)_\infty}{4\pi\sqrt{-1}}\int_C \frac{(z^{\pm 2};q)_\infty}{\prod_{i=1}^{4}(u_i z^{\pm 1};q)_\infty}\frac{dz}{z} = \frac{(u_1u_2u_3u_4;q)_\infty}{\prod_{1\le i<j\le 4}(u_iu_j;q)_\infty} \tag{2.9}$$

where $(z;q)_\infty = \prod_{i=0}^{\infty}(1-q^j z)$ $(|q|<1)$.

- *Nassrallah-Rahman*: Under the balancing condition $u_0u_1\cdots u_5 = q$,

$$\frac{(q;q)_\infty}{4\pi\sqrt{-1}}\int_C \frac{(z^{\pm 2};q)_\infty (qu_0^{-1}z^{\pm 1};q)_\infty}{\prod_{i=1}^{5}(u_i z^{\pm 1};q)_\infty}\frac{dz}{z} = \frac{\prod_{i=1}^{5}(qu_i/u_0;q)_\infty}{\prod_{1\le i<j\le 5}(u_iu_j;q)_\infty}. \tag{2.10}$$

- *Rahman* (1986): Under the balancing condition $u_0u_1\cdots u_7 = q^2$,

$$\begin{aligned}
&\prod_{1\le i<j\le 6}(u_iu_j;q)_\infty\cdot\frac{(q;q)_\infty}{4\pi\sqrt{-1}}\int_C \frac{(z^{\pm 2};q)_\infty \prod_{i=0,7}(qu_i^{-1}z^{\pm 1};q)_\infty}{\prod_{i=1}^{6}(u_i z^{\pm 1};q)_\infty}\frac{dz}{z}\\
&= \frac{\prod_{i=1}^{6}(qu_i/u_0;q)_\infty (q/u_iu_7;q)_\infty}{(q^2u_0^2;q)_\infty (u_0/u_7;q)_\infty}\,{}_{10}W_9\big(q/u_0^2;q/u_0u_1,q/u_0u_2,\ldots,q/u_0u_7;q,q\big)\\
&\quad+\frac{\prod_{i=1}^{6}(qu_i/u_7;q)_\infty (q/u_iu_0;q)_\infty}{(q^2u_7^2;q)_\infty (u_7/u_0;q)_\infty}\,{}_{10}W_9\big(q/u_7^2;q/u_1u_7,q/u_2u_7,\ldots,q/u_6u_7;q,q\big)
\end{aligned} \tag{2.11}$$

○ Theta function and elliptic gamma function

Assuming that $\Omega = \mathbb{Z}1 \oplus \mathbb{Z}\tau$, $\mathrm{Im}\tau > 0$, we set $p = e(\tau)$, $|p| < 1$. We also use the multiplicative notation

$$\begin{aligned}&\theta(u;p) = (u;p)_\infty (p/u;p)_\infty, \quad (u;p)_\infty (p/u;p)_\infty (p;p)_\infty = \sum_{k\in\mathbb{Z}} (-1)^k p^{\binom{k}{2}} u^k,\\ &\theta(p/u;p) = \theta(u;p), \quad \theta(pz;p) = -u^{-1}\theta(u;p)\end{aligned} \tag{2.12}$$

for theta functions. Then $[z] = -u^{-\frac{1}{2}}\theta(u;p)$, $u = e(z)$, satisfies the functional equation in Hermite's Theorem. *Ruijsenaars' elliptic gamma function* is defined by

$$\begin{aligned}&\Gamma(u;p,q) = \frac{(pq/u;p,q)_\infty}{(u;p,q)_\infty}, \quad (u;p,q)_\infty = \prod_{i,j=0}^{\infty} (1 - p^i q^j u) \quad (|q| < 1),\\ &\Gamma(pq/u;p,q) = \frac{1}{\Gamma(u;p,q)}, \quad \frac{\Gamma(qu;p,q)}{\Gamma(u;p,q)} = \theta(u;p),\end{aligned} \tag{2.13}$$

and the *triple elliptic gamma function* by

$$\begin{aligned}&\Gamma(u;p,q,r) = (u;p,q,r)_\infty (pqr/u;p,q,r)_\infty,\\ &(u;p,q,r)_\infty = \prod_{i,j,k=0}^{\infty} (1 - p^i q^j r^k u) \quad (|r| < 1),\\ &\Gamma(pqr/u;p,q,r) = \Gamma(u;p,q,r), \quad \frac{\Gamma(ru;p,q,r)}{\Gamma(u;p,q,r)} = \Gamma(u;p,q).\end{aligned} \tag{2.14}$$

○ **Elliptic hypergeometric integrals (van Diejen, Spiridonov, Rains)**

$$I(u_0, u_1, \ldots, u_r; p, q) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{i=0}^{r} \Gamma(u_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z} \tag{2.15}$$

• *Summation formula* : Under the balancing condition $u_0 u_1 \cdots u_5 = pq$,

$$I(u_0, u_1, \ldots, u_5; p, q) = \prod_{0 \le i < j \le 5} \Gamma(u_i u_j; p, q) \tag{2.16}$$

• *Two transformation formulas*: Under the balancing condition $u_0 u_1 \cdots u_7 = p^2 q^2$,

$$\begin{aligned} & I(u_0, u_1, \ldots, u_7; p, q) \\ & = I(\widetilde{u}_0, \widetilde{u}_1, \ldots, \widetilde{u}_7; p, q) \prod_{0 \le i < j \le 3} \Gamma(u_i u_j; p, q) \prod_{4 \le i < j \le 7} \Gamma(u_i u_j; p, q) \end{aligned} \tag{2.17}$$

$$\widetilde{u}_i = u_i \sqrt{pq/u_0 u_1 u_2 u_3} \quad (i = 0, 1, 2, 3), \quad u_i \sqrt{pq/u_4 u_5 u_6 u_7} \quad (i = 4, 5, 6, 7)$$

$$\begin{aligned} & I(u_0, u_1 \ldots, u_7; p, q) \\ & = I(\sqrt{pq}/u_0, \sqrt{pq}/u_1, \ldots, \sqrt{pq}/u_7; p, q) \prod_{0 \le i < j \le 7} \Gamma(u_i u_j; p, q) \end{aligned} \tag{2.18}$$

- *Three term relations*:

$$T_{q,u_i}\Gamma(u_i z^{\pm\pm1}) = \Gamma(qu_i z^{\pm\pm1}) = \Gamma(u_i z^{\pm1};p,q)\theta(u_i z^{\pm1};p) \tag{2.19}$$

From the functional equation

$$\begin{aligned} u_k\theta(u_j u_k^{\pm1};p)\theta(u_i z^{\pm1};p) + u_i\theta(u_k u_i^{\pm1};p)\theta(u_j z^{\pm1};p) \\ + u_j\theta(u_i u_j^{\pm1};p)\theta(u_k z^{\pm1};p) = 0, \end{aligned} \tag{2.20}$$

we obtain the three term relations for $I(u) = I(u_0, \dots, u_7; p, q)$:

$$u_k\theta(u_j u_k^{\pm1};p)T_{q,u_i}I(u) + u_i\theta(u_k u_i^{\pm1};p)T_{q,u_j}I(u) + u_j\theta(u_i u_j^{\pm1};p)T_{q,u_k}I(u) = 0. \tag{2.21}$$

In additive variables $x = (x_0, x_1, \dots, x_7)$ with $u_i = e(x_i)$ $(i = 0, 1, \dots, 7)$,

$$J(x) = e(-Q(x))I(u), \quad Q(x) = \tfrac{1}{2\delta}(x|x) = \tfrac{1}{2\delta}(x_0^2 + \cdots + x_7^2). \tag{2.22}$$

satisfies

$$[x_j \pm x_k]T^{\delta}_{x_i}J(x) + [x_k \pm x_i]T^{\delta}_{x_j}J(x) + [x_i \pm x_j]T^{\delta}_{x_k}J(x) = 0. \tag{2.23}$$

Three term relations + Bailey type transformations

$\Longrightarrow$ System of elliptic hypergeometric difference equations

○ From integrals to series

Suppose that $u_0u_1\cdots u_7 = q^2$ (balancing condition), and that $q/u_0u_i = q^{-N}$ $(N \in \mathbb{N})$ for some $i \in \{1,\ldots,6\}$ or $q/u_0u_7 = pq^{-N}$ $(N \in \mathbb{N})$. Then we have

$$
\begin{aligned}
&I(pu_0, u_1, \ldots, u_6, pu_7) \\
&= \prod_{1\le i<j\le 6} \Gamma(u_iu_j;p,q) \frac{\Gamma(q^2/u_0^2;p,q)\Gamma(u_0/u_7;p,q)}{\prod_{i=1}^{6} \Gamma(qu_i/u_0;p,q)\Gamma(q/u_iu_7;p,q)} \qquad (2.24)\\
&\quad \cdot {}_{12}V_{11}(q/u_0^2; q/u_0u_1, \ldots, q/u_0u_6, q/u_0u_7; , p, q; q)
\end{aligned}
$$

in the multiplicative notation of V series.

$$
\begin{aligned}
&{}_{r+5}V_{r+4}\big(a_0; a_1, \ldots, a_r; p, q; u\big) = \sum_{k=0}^{\infty} \frac{\theta(q^{2k}a_0)}{\theta(a_0)} \frac{\theta(a_0)_k}{\theta(q)_k} \prod_{i=1}^{r} \frac{\theta(a_i)_k}{\theta(qa_0/a_i)_k} u^k \qquad (2.25)\\
&\theta(a)_k = \theta(a;p)\theta(qa;p)\cdots\theta(q^{k-1}a;p) = \theta(a;p,q)_k = \Gamma(q^k a;p,q)/\Gamma(a;p,q).
\end{aligned}
$$

3 Elliptic difference Painlevé equation

○ Sakai's table of discrete Painlevé equations (2001)

Nine-point blowups of $\mathbb{P}^2$ which admits affine Weyl group symmetries.

- *Rational surfaces (anti-canonical divisors)*:

$$(eP): \quad A_0^{(1)}$$

$$\begin{array}{lllll}
(qP): & A_0^{(1)} \to A_1^{(1)} \to A_2^{(1)} \to A_3^{(1)} \to A_4^{(1)} \to A_5^{(1)} \to A_6^{(1)} & \to & A_7^{(1)} \to A_8^{(1)} \\
 & & \searrow & \\
 & & & A_7'^{(1)}
\end{array}$$

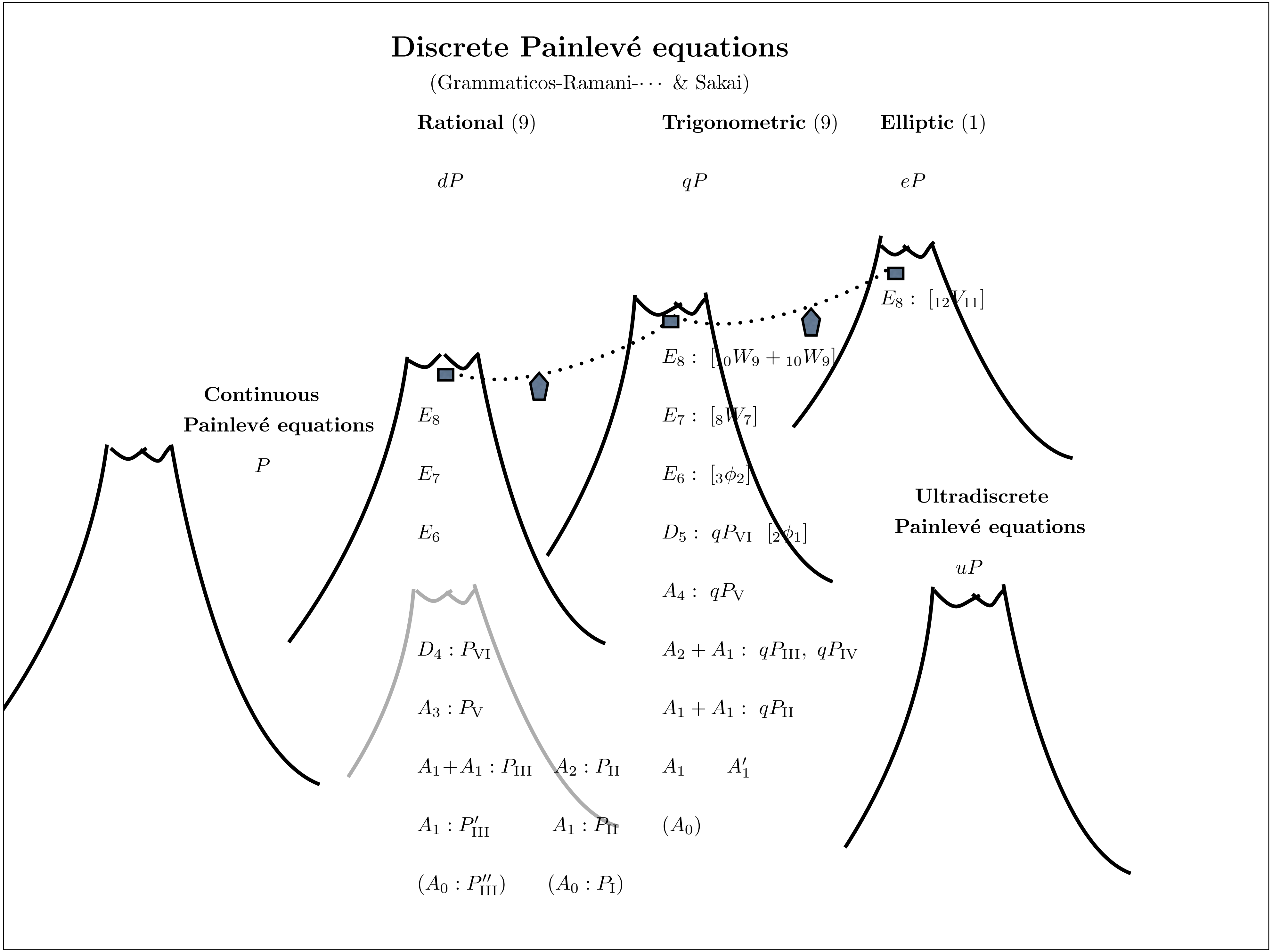
$$\begin{array}{llllll}
(dP): & A_0^{(1)} \to A_1^{(1)} \to A_2^{(1)} \to D_4^{(1)} \to D_5^{(1)} & \to & D_6^{(1)} & \to & D_7^{(1)} \to D_8^{(1)} \\
 & & \searrow & & \searrow & \\
 & & & E_6^{(1)} & \to & E_7^{(1)} \to E_8^{(1)}
\end{array}$$

- *Affine Weyl group symmetry*:

$$(eP): \quad E_8^{(1)}$$

$$\begin{array}{lllll}
(qP): & E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_5^{(1)} \to A_4^{(1)} \to (A_2+A_1)^{(1)} \to (A_1+A_1')^{(1)} & \to & A_1'^{(1)} \to A_0^{(1)} \\
 & & \searrow & \\
 & & & A_1^{(1)}
\end{array}$$

$$\begin{array}{llllll}
(dP): & E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_4^{(1)} \to A_3^{(1)} & \to & (2A_1)^{(1)} & \to & A_1'^{(1)} \to A_0^{(1)} \\
 & & \searrow & & \searrow & \\
 & & & A_2^{(1)} & \to & A_1^{(1)} \to A_0^{(1)}
\end{array}$$


Discrete Painlevé equations
(Grammaticos-Ramani-$\cdots$ & Sakai)
Rational (9)
Trigonometric (9)
Elliptic (1)
dP
qP
eP
Continuous
Painlevé equations
P
E_8
E_7
E_6
$D_4 : P_{\rm VI}$
$A_3 : P_{\rm V}$
$A_1{+}A_1 : P_{\rm III}$
$A_2 : P_{\rm II}$
$A_1 : P'_{\rm III}$
$A_1 : P_{\rm II}$
$(A_0 : P''_{\rm III})$
$(A_0 : P_{\rm I})$
$E_8 :\ [{}_{10}W_9 + {}_{10}W_9]$
$E_7 :\ [{}_8W_7]$
$E_6 :\ [{}_3\phi_2]$
$D_5 :\ qP_{\rm VI}\ \ [{}_2\phi_1]$
$A_4 :\ qP_{\rm V}$
$A_2 + A_1 :\ qP_{\rm III},\ qP_{\rm IV}$
$A_1 + A_1 :\ qP_{\rm II}$
A_1
A'_1
(A_0)
$E_8 :\ [{}_{12}V_{11}]$
Ultradiscrete
Painlevé equations
uP

○ Standard Cremona transformation (quadratic transformation)

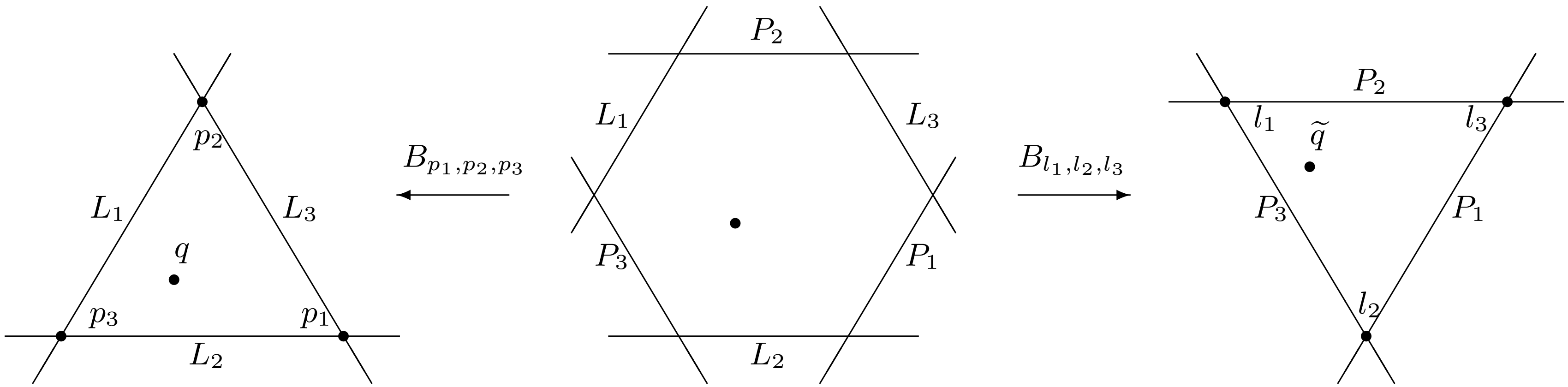
For a triple p_1, p_2, p_3 of points in $\mathbb{P}^2$ which are not collinear, choose homogeneous coordinates $(x_1 : x_2 : x_3)$ such that

$$p_1 = (1:0:0), \quad p_2 = (0:1:0), \quad p_3 = (0:0:1). \tag{3.1}$$

The birational mapping $\mathrm{cr}_{p_1,p_2,p_3} : \mathbb{P}^2 \cdots\to \mathbb{P}^2$ defined by

$$q = (x_1 : x_2 : x_3) \to \widetilde{q} = (x_2x_3 : x_1x_3 : x_1x_2) = \Bigl(\frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}\Bigr) \tag{3.2}$$

is called the *standard Cremona transformation* with respect to (p_1, p_2, p_3).



○ Birational Weyl group action on the configuration space

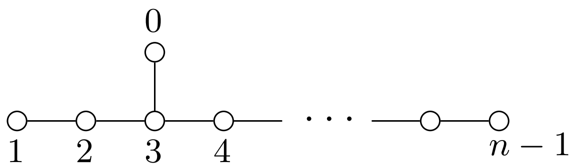
Configuration space $\mathbb{X}_{3,n}$ of n points in general position in $\mathbb{P}^2$ and its transversal $\mathcal{U}_{3,n}$:

$$\mathbb{X}_{3,n} = \mathrm{GL}(3;\mathbb{C})\backslash \mathrm{Mat}^*(3,n;\mathbb{C})/\mathbb{T}^n, \quad \mathbb{T}^n = (\mathbb{C}^*)^n$$

$$\mathcal{U}_{3,n} = \left\{ U = \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 & p_5 & \cdots & p_n \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & u_{15} & \cdots & u_{1n} \\ 0 & 1 & 0 & 1 & u_{25} & \cdots & u_{2n} \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \end{array} \;\middle|\; \det(U)_{j_1,j_2,j_3} \neq 0 \ \text{ for distinct } i,j,k \right\}$$

We denote by $\mathcal{K}(\mathbb{X}_{3,n}) = \mathbb{C}(u)$, $u = (u_{ij})_{ij}$, the field of rational functions on $\mathbb{X}_{3,n}$. Then the Weyl group $W_{3,n} = W(T_{2,3,n-3}) = \langle s_0, s_1, \ldots, s_{n-1}\rangle$ associated with the tree $T_{2,3,n-3}$ acts on $\mathcal{K}(\mathbb{X}_{3,n})$ as a group of automorphisms.

$T_{2,3,n-3}$:



$$s_i^2 = 1$$
$$s_i s_j = s_j s_i \quad (i \circ \quad \circ j)$$
$$s_i s_j s_i = s_j s_i s_j \quad (i \circ\!\!-\!\!-\!\!\circ j)$$

$\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1}\rangle$: permutation of points, $s_0 = \mathrm{cr}_{123}$: standard Cremona

n	4	5	6	7	8	9	10	$\cdots$
root system	A_4	D_5	E_6	E_7	E_8	$E_8^{(1)}$	$*$	$\cdots$
$\dim_{\mathbb{C}} \mathbb{X}_{3,n}$	0	2	4	6	8	10	12	$\cdots$

($*$: of indefinite type)

The simple reflection s_k $(k = 0, 1, \ldots, n-1)$ acts on u_{ij} $(i = 1, 2; j = 5, \ldots, n)$ as

$$
\begin{array}{ll}
k = 0: & s_0(u_{ij}) = \dfrac{1}{u_{ij}} \\
k = 1: & s_1(u_{1j}) = u_{2,j}, \quad s_1(u_{2j}) = u_{1j} \\
k = 2: & s_2(u_{1j}) = \dfrac{u_{1j}}{u_{2j}}, \quad s_2(u_{2j}) = \dfrac{1}{u_{2j}} \\
k = 3: & s_3(u_{ij}) = 1 - u_{ij} \\
k = 4: & s_4(u_{i5}) = \dfrac{1}{u_{i5}}, \quad s_4(u_{ij}) = \dfrac{u_{ij}}{u_{i5}} \quad (j = 6, \ldots, n) \\
k = 5, \ldots, n-1: & s_k(u_{ij}) = u_{i, s_k(j)}
\end{array}
\tag{3.3}
$$

For any $w \in W_{3,n}$, the action of w on the coordinates u_{ij} is expressed as

$$
w(u_{ij}) = R_{ij}^{w}(u) \quad (i = 1, 2;\ j = 5, \ldots, n). \tag{3.4}
$$

Since $w_1 w_2(u_{ij}) = w_1(w_2(u))$, these rational functions $R_{ij}^{w}(u)$ satisfy the compatibility condition

$$
R_{ij}^{w_1 w_2}(u) = R_{ij}^{w_2}(R^{w_1}(u)) \quad (w_1, w_2 \in W_{3,8}); \qquad R^{w_1}(u) = (R_{kl}^{w_1}(u))_{kl}. \tag{3.5}
$$

○ Discrete Painlevé equation of type $\boldsymbol{E_8^{(1)}}$

Consider the case of $9+1$ points: $W_{3,9} = W(E_8^{(1)})$. We regard the 9 points $p_1, \ldots, p_9$ as reference points for Cremona transformations, and $q = p_{10}$ as the general point (z_1, z_2) in $\mathbb{P}^2$ to be transformed by $W(E_8^{(1)})$:

$$U = \begin{array}{c} \begin{array}{cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & \cdots & p_9 & q \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & u_{15} & \cdots & u_{19} & z_1 \\ 0 & 1 & 0 & 1 & u_{25} & \cdots & u_{29} & z_2 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \end{array} \in \mathcal{U}_{3,10} \tag{3.6}$$

For each $w \in W(E_8^{(1)})$, the action of w on $u = (u_{ij})_{1\le i\le 2;\, 5\le j\le 9}$ and $z = (z_1, z_2)$ is described as

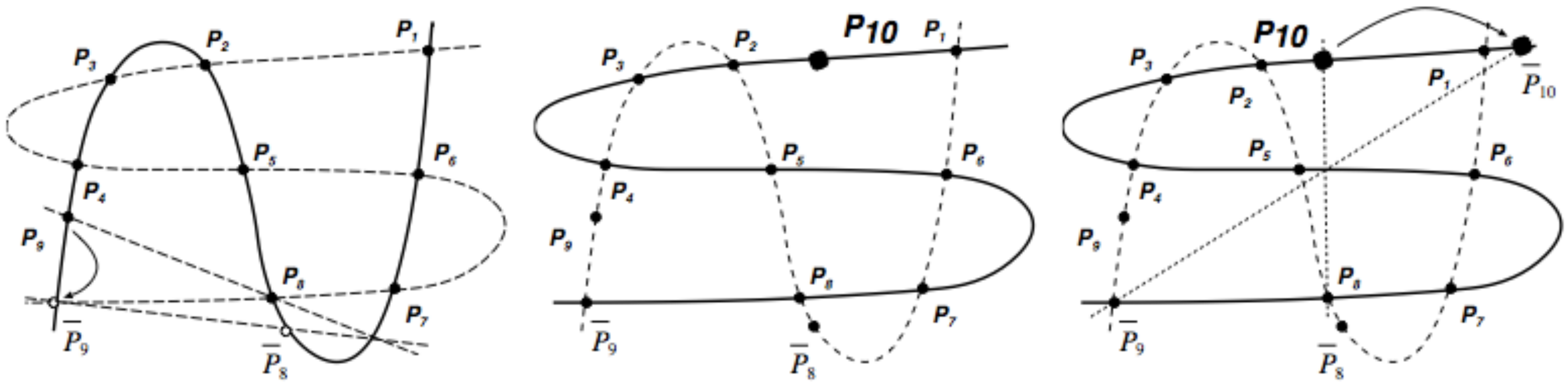
$$w(u_{ij}) = R^w_{ij}(u), \quad w(z_i) = S^w_i(u; z). \tag{3.7}$$

Note that $W(E_8^{(1)}) = T_Q \rtimes W(E_8)$: $T_Q = \{T_\alpha \mid \alpha \in Q(E_8)\}$. With $u = (u_{ij})$ regarded as parameters, the birational transformation

$$T_\alpha(z_1) = S^\alpha_1(u; z), \quad T_\alpha(z_2) = S^\alpha_2(u; z); \quad \alpha \in Q(E_8) \tag{3.8}$$

associated with the translation $T_\alpha \in W(E_8^{(1)})$, $\alpha \in Q(E_8)$, is the *discrete Painlevé equation* of type $E_8^{(1)}$ in the direction α for the unknown functions z_1, z_2.

○ Geometric description of T_{α_8}, $\alpha_8 = \varepsilon_8 - \varepsilon_9$



$$p_j = (u_{1j}, u_{2j}) \quad (j = 1, \ldots, 9), \qquad q = p_{10} = (z_1, z_2)$$

$$T_{\alpha_8}(z_1) = S_1^{\alpha_8}(u; z), \qquad T_{\alpha_8}(z_2) = S_2^{\alpha_8}(u; z)$$

- *Picard lattice*:

A characteristic feature of the birational Weyl group action on $\mathbb{X}_{3,n}$ is that the rational functions $R^w_{ij}(u)$ are controlled by the *Picard lattice*:

$$\begin{aligned} &L_{3,n} = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n; \\ &(e_0|e_0) = -1, \quad (e_j|e_j) = 1 \ \ (j = 1, \ldots, n), \ \ (e_i|e_j) = 0 \ \ (i \neq j). \end{aligned} \tag{3.9}$$

In the algebro-geometric terms, $L_{3,n}$ is the Picard group of the rational surface obtained from $\mathbb{P}^2$ by blowing up at n points $p_1, \ldots, p_n$.

e_0 : class of lines in $\mathbb{P}^2$, $e_1, \ldots, e_n$: exceptional curves corresponding to $p_1, \ldots, p_n$.
$(\Lambda|\Lambda') = -\Lambda \cdot \Lambda'$ (minus of the intersection number of divisor classes)

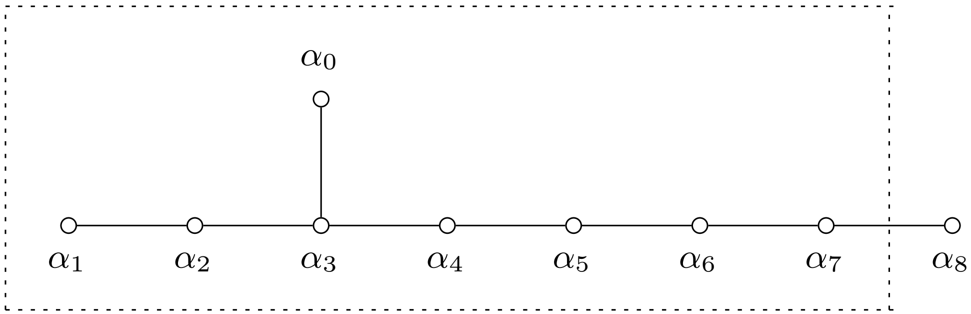
In this lattice the root system of type $T_{2,3,n-3}$ is realized by the simple roots

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \quad \alpha_j = e_j - e_{j+1} \ \ (j = 1, \ldots, n-1). \tag{3.10}$$

The Weyl group $W_{3,n} = \langle s_0, s_1, \ldots, s_{n-1} \rangle$ acts on $L_{3,n}$ as a group of isometries through the simple reflections $s_i = s_{\alpha_i}$ $(i = 0, 1, \ldots, n-1)$:

$$s_i(\Lambda) = \Lambda - (\alpha_i|\Lambda)\alpha_i \qquad (\Lambda \in L_{3,n}). \tag{3.11}$$

- *Case of nine-point configurations*:



$$\alpha_0 = e_0 - e_1 - e_2 - e_3,$$
$$\alpha_j = e_j - e_{j+1} \quad (j = 1, \ldots, 8)$$

$$Q(E_8) \subset \ Q(E_8^{(1)}) = Q(E_8) \oplus \mathbb{Z}\delta \subset L_{3,9} = Q(E_8) \oplus \mathbb{Z}\delta \oplus \mathbb{Z}e_9 \tag{3.12}$$

We denote the *null root* by

$$\begin{aligned} \delta &= 3e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8 - e_9 \\ &= 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \end{aligned} \tag{3.13}$$

the $W(E_8^{(1)})$-invariant element corresponding to the anti-canonical divisor (the cubic curve passing through the nine points $p_1, \ldots, p_9$); $-e_9$ is the role of the fundamental weight 'Λ_0'. For each $\alpha \in Q(E_8^{(1)})$, the *Kac translation* T_α is defined by

$$T_\alpha(\Lambda) = \Lambda + (\delta|\Lambda)\alpha - \big(\tfrac{1}{2}(\alpha|\alpha) + (\alpha|\Lambda)\big)\delta \quad (\Lambda \in L_{3,9}). \tag{3.14}$$

$$W_{3,8} = W(E_8^{(1)}) = T_Q \rtimes W(E_8); \quad Q(E_8) \xrightarrow{\sim} T_Q : \ \alpha \mapsto T_\alpha \tag{3.15}$$

• *Parametrization of cubic curves and point configurations*:
Cartan subalgebra and its dual of the corresponding Kac-Moody Lie algebra.

$$\begin{aligned}\mathfrak{h}_{3,9} &= L_{3,9} \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_9 \\ \mathfrak{h}^*_{3,9} &= \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{3,9}, \mathbb{C}) = \mathbb{C}\varepsilon_0 \oplus \mathbb{C}\varepsilon_1 \oplus \cdots \oplus \mathbb{C}\varepsilon_9; \quad \varepsilon_j = (e_j|\cdot)\end{aligned} \tag{3.16}$$

Regarding $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_9)$ as a coordinate system of $\mathfrak{h}_{3,9}$, we use this affine space for the parametrization of cubic curves and point configurations. For generic $\varepsilon \in \mathfrak{h}_{3,9}$ we define a holomorphic mapping $p_\varepsilon : \ E_\Omega = \mathbb{C}/\Omega \to \mathbb{P}^2$ by

$$p_\epsilon(t) = \left(\frac{[\varepsilon_0-\varepsilon_2-\varepsilon_3 - t]}{[\varepsilon_1-t]} : \frac{[\varepsilon_0-\varepsilon_1-\varepsilon_3 - t]}{[\varepsilon_2-t]} : \frac{[\varepsilon_0-\varepsilon_1-\varepsilon_2 - t]}{[\varepsilon_3-t]} \right) \quad (t \in \mathbb{C}), \tag{3.17}$$

and set $C_\varepsilon = p_\varepsilon(E_\Omega)$. This p_ε induces a $W(E_8^{(1)})$-equivariant meromorphic mapping

$$\varphi_{3,9} : \ \mathfrak{h}_{3,9} \cdots \to \mathbb{X}_{3,9} : \quad \varphi_{3,9}(\varepsilon) = [p_\varepsilon(\varepsilon_1), \cdots, p_\varepsilon(\varepsilon_9)] \in \mathbb{X}_{3,9} \tag{3.18}$$

$$\varphi_{3,9} : \quad u_{ij} = u_{ij}(\varepsilon) = \frac{[\alpha_0+\varepsilon_{34}][\varepsilon_{i4}]}{[\alpha_0+\varepsilon_{i4}][\varepsilon_{34}]} \frac{[\alpha_0+\varepsilon_{ij}][\varepsilon_{3j}]}{[\alpha_0+\varepsilon_{3j}][\varepsilon_{ij}]} \quad (i = 1, 2; \ j = 5, \ldots, 9) \tag{3.19}$$

where $\alpha_0 = \varepsilon_0-\varepsilon_1-\varepsilon_2-\varepsilon_3$, and $\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$, which transfers the configuration of nine points $\varepsilon_1, \ldots, \varepsilon_9$ on E_Ω to that on the cubic curve $C_\varepsilon \subset \mathbb{P}^2$. In this way the $W_{3,9}$ action of $\mathbb{X}_{3,9}$ is *linearized* (solved) through $\varphi_{3,9} : \ \mathfrak{h}_{3,9} \cdots \to \mathbb{X}_{3,9}$:

$$u_{ij}(w(\varepsilon)) = R^w_{ij}(u(\varepsilon)) \qquad (i = 1, 2; \ j = 5, \ldots, 9; \ w \in W_{3,9}). \tag{3.20}$$

- *Elliptic Painlevé equation* $eP(E_8^{(1)})$:

  Substituting this solution $u_{ij}(\varepsilon)$, we obtain a system of Cremona transformations

$$w(z_1) = S_1^w(\varepsilon; z_1, z_2), \quad w(z_2) = S_2^w(\varepsilon; z_1, z_2) \qquad (w \in W_{3,9}) \tag{3.21}$$

for unknown functions z_1, z_2 with parameters $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_9) \in \mathfrak{h}_{3,9}$. This system can be described by the action of $W_{3,9} = W(E_8^{(1)})$ as a group of automorphisms of the field of rational functions $\mathcal{K} = \mathbb{K}(z_1, z_2)$ with coefficients in $\mathbb{K} = \mathcal{M}(E_\Omega \otimes_{\mathbb{Z}} L_{3,9})$.

On this field $\mathcal{K} = \mathbb{K}(z_1, z_2)$, the actions of $s_0, s_1, \ldots, s_8$ are given by

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
z_1	$\frac{1}{z_1}$	z_2	$\frac{z_1}{z_2}$	$1 - z_1$	$\frac{z_1}{u_{15}(\varepsilon)}$	z_1	z_1	z_1	z_1
z_2	$\frac{1}{z_2}$	z_1	$\frac{1}{z_2}$	$1 - z_2$	$\frac{z_2}{u_{25}(\varepsilon)}$	z_2	z_2	z_2	z_2

$$u_{15}(\varepsilon) = \frac{[\varepsilon_{14}][\varepsilon_{35}][\varepsilon_{124}][\varepsilon_{235}]}{[\varepsilon_{34}][\varepsilon_{15}][\varepsilon_{234}][\varepsilon_{125}]}, \quad u_{25}(\varepsilon) = \frac{[\varepsilon_{24}][\varepsilon_{35}][\varepsilon_{124}][\varepsilon_{135}]}{[\varepsilon_{34}][\varepsilon_{25}][\varepsilon_{234}][\varepsilon_{125}]}. \quad \begin{array}{l} \varepsilon_{ij} = \varepsilon_i - \varepsilon_j \\ \varepsilon_{ijk} = \varepsilon_0 - \varepsilon_i - \varepsilon_j - \varepsilon_k \end{array}.$$

The Kac translations T_α give rise to a commuting family of birational transformations

$$T_\alpha(z_1) = S_1^\alpha(\varepsilon; z_1, z_2), \quad T_\alpha(z_2) = S_2^\alpha(\varepsilon; z_1, z_2) \qquad (\alpha \in Q(E_8)). \tag{3.22}$$

This system of discrete time evolutions is the *elliptic Painelvé equation* $eP(E_8^{(1)})$.

○ **τ-functions for $eP(E_8^{(1)})$**

We now introduce a system of homogeneous coordinates $(f_1 : f_2 : f_3)$ for $\mathbb{P}^2$ such that

$$z_1 = \frac{[\varepsilon_{12}][\varepsilon_{124}]}{[\varepsilon_{34}][\varepsilon_{234}]}\frac{f_1}{f_3}, \qquad z_2 = \frac{[\varepsilon_{24}][\varepsilon_{124}]}{[\varepsilon_{34}][\varepsilon_{134}]}\frac{f_2}{f_3}, \tag{3.23}$$

together with new dependent variables $\tau_1, \ldots, \tau_9$ corresponding to the nine points $p_1, \ldots, p_9$. Then the action of $W_{3,9}$ on $\mathcal{K} = \mathbb{K}(z_1, z_2)$ can be extended to the field $\mathcal{L} = \mathbb{K}(f_1, f_2, f_3; \tau_1, \ldots, \tau_9)$ as follows:

$$\begin{aligned} &s_0(\tau_i) = f_i\,\tau_i && (i = 1,2,3), && s_0(f_i) = \frac{1}{f_i} && (i = 1,2,3), \\ &s_0(\tau_j) = \tau_j && (j = 4,\ldots,9), && s_k(f_i) = f_{(k,k+1)i} && (k = 1,2), \\ &s_k(\tau_j) = \tau_{(k,k+1)j} && (k = 1,\ldots,8) && s_k(f_i) = f_i && (k = 4,\ldots,8). \end{aligned}$$

$$s_3(f_1) = \frac{\tau_3}{\tau_4}\left(\frac{[\varepsilon_{14}][\varepsilon_{124}]}{[\varepsilon_{123}][\varepsilon_{13}]}f_1 - \frac{[\varepsilon_{34}][\varepsilon_{234}]}{[\varepsilon_{123}][\varepsilon_{13}]}f_3\right), \quad s_3(f_2) = \frac{\tau_3}{\tau_4}\left(\frac{[\varepsilon_{24}][\varepsilon_{124}]}{[\varepsilon_{123}][\varepsilon_{23}]}f_2 - \frac{[\varepsilon_{34}][\varepsilon_{134}]}{[\varepsilon_{123}][\varepsilon_{23}]}f_3\right),$$
$$s_3(f_3) = \frac{\tau_3}{\tau_4}f_3.$$

Theorem A: *The automorphisms $s_0, s_1, \ldots, s_8$ of $\mathcal{L} = \mathbb{K}(f_1, f_2, f_3; \tau_1, \ldots, \tau_9)$ defined as above satisfy the fundamental relations for the simple reflections of $W_{3,9} = \langle s_0, s_1, \ldots, s_8\rangle$.*

In this realization we look at the action of s_3 on f_1:

$$s_3(f_1) = \frac{\tau_3}{\tau_4}\left(\frac{[\varepsilon_{14}][\varepsilon_{124}]}{[\varepsilon_{123}][\varepsilon_{13}]}f_1 - \frac{[\varepsilon_{34}][\varepsilon_{234}]}{[\varepsilon_{123}][\varepsilon_{13}]}f_3\right). \tag{3.24}$$

By using the relations $f_i = s_0(\tau_i)/\tau_i$ $(i = 1, 2, 3)$, this formula can be rewritten as bilinear relations for translates of τ-functions:

$$\frac{s_3 s_0(\tau_1)}{\tau_1} = \frac{\tau_3}{\tau_4}\left(\frac{[\varepsilon_{14}][\varepsilon_{124}]}{[\varepsilon_{123}][\varepsilon_{13}]}\frac{s_0(\tau_1)}{\tau_1} - \frac{[\varepsilon_{34}][\varepsilon_{234}]}{[\varepsilon_{123}][\varepsilon_{13}]}\frac{s_0(\tau_3)}{\tau_3}\right), \tag{3.25}$$

$$[\varepsilon_{123}][\varepsilon_{13}]\tau_4\, s_3 s_0(\tau_1) = [\varepsilon_{14}][\varepsilon_{124}]\tau_3 s_0(\tau_1) - [\varepsilon_{34}][\varepsilon_{234}]\tau_1 s_0(\tau_3). \tag{3.26}$$

○ Lattice $\boldsymbol{\tau}$-functions for $\boldsymbol{eP(E_8^{(1)})}$

In order to analyze the action of $W_{3,9}$ on the τ-functions, we consider the $W_{3,9}$-orbit of e_9 in the Picard lattice $L_{3,9}$: $M_{3,9} = W_{3,9}\, e_9 \subset L_{3,9}$. This orbit can also be described intrinsically as

$$M_{3,9} = \{\ \Lambda \in L_{3,9} \mid (\Lambda|\Lambda) = 1,\ (\delta|\Lambda) = -1\ \}; \quad Q(E_8) \xrightarrow{\sim} M_{3,9} : \quad \alpha \mapsto T_\alpha(e_9).$$

Theorem B: *There exists a unique family of elements* $\tau(\Lambda) \in \mathcal{L}$ *(*$\Lambda \in M_{3,9}$*) such that*

$$\tau(e_j) = \tau_j \quad (j = 1, \ldots, 9); \quad w(\tau(\Lambda)) = \tau(w.\Lambda) \quad (\Lambda \in M_{3,9};\ w \in W_{3,9}). \tag{3.27}$$

Furthermore, this family of τ*-functions is characterized by the following non-autonomous Hirota equations: For any distinct* $i, j, k, l \in \{1, \ldots, 9\}$,

$$\begin{aligned}[][\varepsilon_{jkl}][\varepsilon_{jk}]\tau(e_i)\tau(e_0 - e_l - e_i) + [\varepsilon_{kil}][\varepsilon_{ki}]\tau(e_j)\tau(e_0 - e_l - e_j) \\ + [\varepsilon_{ijl}][\varepsilon_{ij}]\tau(e_k)\tau(e_0 - e_l - e_k) = 0.\end{aligned} \tag{3.28}$$

Furthermore, the f *variables are recovered from* $\tau(\Lambda)$ *(*$\Lambda \in M_{3,9}$*) by*

$$f_1 = \frac{\tau(e_0 - e_2 - e_3)}{\tau(e_1)}, \quad f_2 = \frac{\tau(e_0 - e_1 - e_3)}{\tau(e_2)}, \quad f_3 = \frac{\tau(e_0 - e_1 - e_2)}{\tau(e_3)}. \tag{3.29}$$

For each $\Lambda \in M_{3,9}$ we define $\tau(\Lambda) = w(\tau_9) \in \mathcal{L}$ by taking a $w \in W_{3,9}$ such that $\Lambda = w.e_9$; this definition does not depend on the choice of w since τ_9 is invariant under the action of the isotropy subgroup $W_{3.8}$ of e_9. With this definition, the bilinear relation

$$[\varepsilon_{123}][\varepsilon_{13}]\tau_4\, s_3 s_0(\tau_1) = [\varepsilon_{14}][\varepsilon_{124}]\tau_3 s_0(\tau_1) - [\varepsilon_{34}][\varepsilon_{234}]\tau_1 s_0(\tau_3). \tag{3.30}$$

is rewritten in the form

$$\begin{aligned}&[\varepsilon_{123}][\varepsilon_{13}]\tau(e_4)\,\tau(e_0 - e_2 - e_4)\\ &= [\varepsilon_{14}][\varepsilon_{124}]\tau(e_3)\tau(e_0 - e_2 - e_3) - [\varepsilon_{34}][\varepsilon_{234}]\tau(e_1)\tau(e_0 - e_1 - e_2).\end{aligned} \tag{3.31}$$

Then by the action of $\mathfrak{S}_9$ we obtain the bilinear equations as described in Theorem B.

Conversely, suppose that the family $\tau(\Lambda)$ $(\Lambda \in M_{3,9})$ satisfies the property as stated in Theorem B. Then the variables f_i $(i = 1, 2, 3)$ are recovered by $f_i = s_i(\tau_i)/\tau_i$. The non-autonomous Hirota equations mentioned above guarantee the validity of relations to be satisfied under the action of s_3.

Remark: From the expression

$$f_1 = \frac{\tau(e_0 - e_2 - e_3)}{\tau(e_1)}, \quad f_2 = \frac{\tau(e_0 - e_1 - e_3)}{\tau(e_2)}, \quad f_3 = \frac{\tau(e_0 - e_1 - e_2)}{\tau(e_3)}$$

of f variables in terms of τ-functions, we obtain

$$w(f_1) = \frac{\tau(w.(e_0-e_2-e_3))}{\tau(w.e_1)}, \; w(f_2) = \frac{\tau(w.(e_0-e_1-e_3))}{\tau(w.e_2)}, \; w(f_3) = \frac{\tau(w.(e_0-e_1-e_2))}{\tau(w.e_3)}$$

for any $w \in W_{3,9}$.

Remark: $\tau_9 = \tau(e_9)$ is a distinguished τ-function. It is $W(E_8)$-invariant, and all the τ-functions $\tau(\Lambda)$ ($\Lambda \in M_{3,9}$) are expressible as the translates

$$\tau(\Lambda) = T_{e_9-\Lambda}(\tau_9) \quad (\Lambda \in M_{3,9}); \qquad M_{3,9} = T_Q(e_9). \tag{3.32}$$

The system of non-autonomous Hirota equations for $\{\tau(\Lambda)\}_{\Lambda\in M_{3,9}}$ is thus translated into a $W(E_8)$-invariant system of *difference equations* for a single τ-function $\tau = \tau_9$, which we formulate in terms of *ORG τ-functions* in the next section.

In working with difference equations, it is more convenient to use the variables $x = (x_0, x_1, \ldots, x_7) \in V = \mathbb{C}^8$ defined by

$$x_j = \varepsilon_j - \tfrac{1}{2}(\varepsilon_0 - \varepsilon_9) + \tfrac{1}{2}\delta \quad (i = 1, \ldots, 8); \quad x_0 = -x_8 \tag{3.33}$$

instead of the coordinates $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_9)$ of $\mathfrak{h}_{3,9}$.

4 $eP(E_8^{(1)})$ as a system of non-autonomous Hirota equations

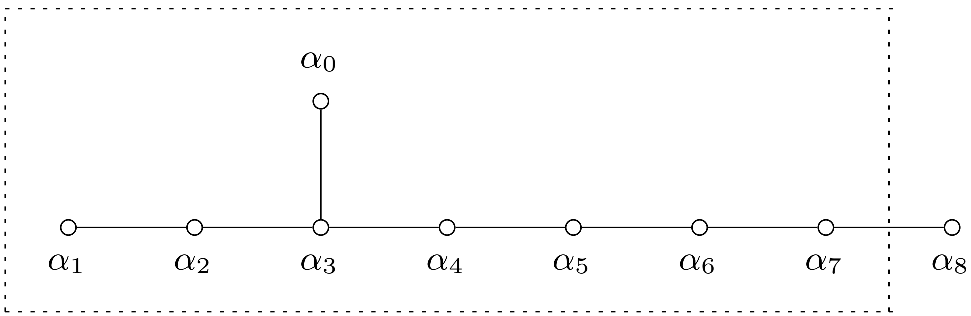
○ A standard realization of the root lattice $P = Q(E_8)$

$$V = \mathbb{C}^8 = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_7; \quad (v_i|v_j) = \delta_{ij} \quad (i, j \in \{0, 1, \ldots, 7\}). \tag{4.1}$$

$$P = \left\{ a \in \mathbb{Z}^8 \cup (\phi + \mathbb{Z}^8) \mid (\phi|a) \in \mathbb{Z} \right\}$$
$$\phi = \tfrac{1}{2}(1,1,1,1,1,1,1,1) = \tfrac{1}{2}(v_0 + v_1 + \cdots + v_7) \tag{4.2}$$

$$\Delta(E_8) = \{ \alpha \in P \mid (\alpha|\alpha) = 2 \}, \quad |\Delta(E_8)| = 240.$$
$$(1):\ \pm v_i \pm v_j \quad (0 \le i < j \le 7) \qquad \cdots \ \binom{8}{2} \cdot 4 = 112 \tag{4.3}$$
$$(2):\ \tfrac{1}{2}(\pm v_0 \pm \cdots \pm v_7) \quad (\text{even number of} - \text{signs}) \qquad \cdots \ 2^7 = 128$$

$$\sum_{a \in P} q^{(a|a)} = 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + \cdots \tag{4.4}$$



$$\alpha_0 = \phi - v_0 - v_1 - v_2 - v_3,$$
$$\alpha_j = v_j - v_{j+1} \quad (j = 1, \ldots, 6)$$
$$\alpha_7 = v_7 + v_0$$
$$\alpha_8 = \delta - \phi$$

○ ORG τ-function (Ohta-Ramani-Grammaticos)

Definition A set of $2l$ vectors $\{\pm a_1, \ldots, \pm a_l\}$ in V is called a C_l*-frame* if

$$\begin{aligned} &(1) \quad (a_i|a_j) = \delta_{ij} \quad (i,j \in \{1,\ldots,l\}), \\ &(2) \quad \{\, \pm a_i \pm a_j \mid 1 \le i < j \le l \,\} \cup \{\, \pm 2a_i \mid 1 \le i \le l \,\} \subset P. \end{aligned} \tag{4.5}$$

There are 2160 vectors $a \in \frac{1}{2}P$ with $(a|a) = 1$. Let $\mathcal{C}_l$ be the set of all C_l frames in P:

$$(\tfrac{1}{2}P)_1 = \bigsqcup_{A \in \mathcal{C}_8} A; \quad |\mathcal{C}_8| = 135, \quad |\mathcal{C}_3| = 135 \cdot \binom{8}{3} = 7560 \tag{4.6}$$

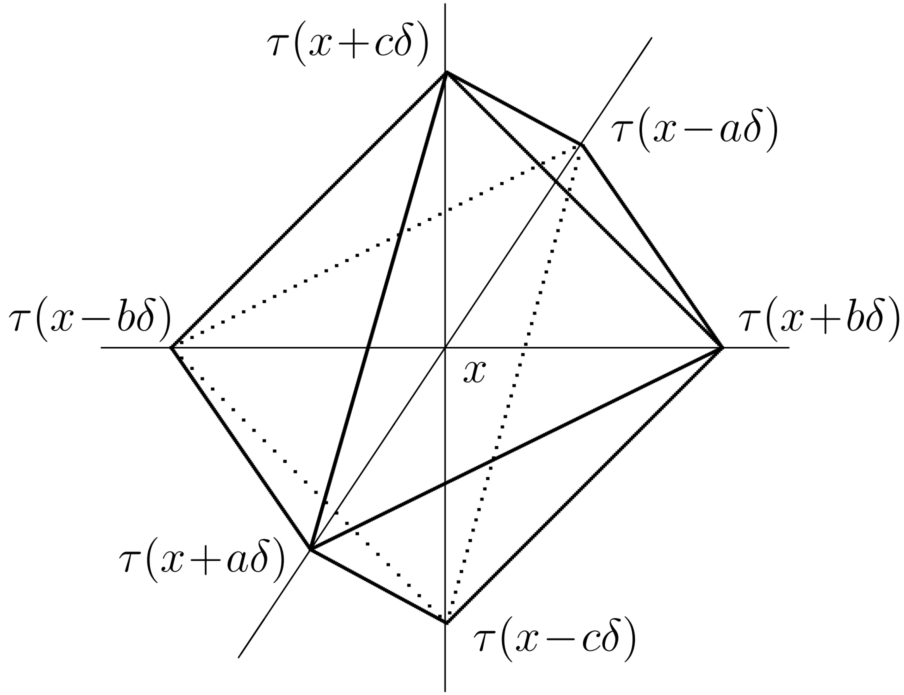
Fix a nonzero constant δ. Let D be a subset of $V = \mathbb{C}^8$ such that $D + P\delta = D$.

Definition A function $\tau(x)$ defined over D is called an *ORG* τ*-function* if it satisfies the non-autonomous Hirota equation

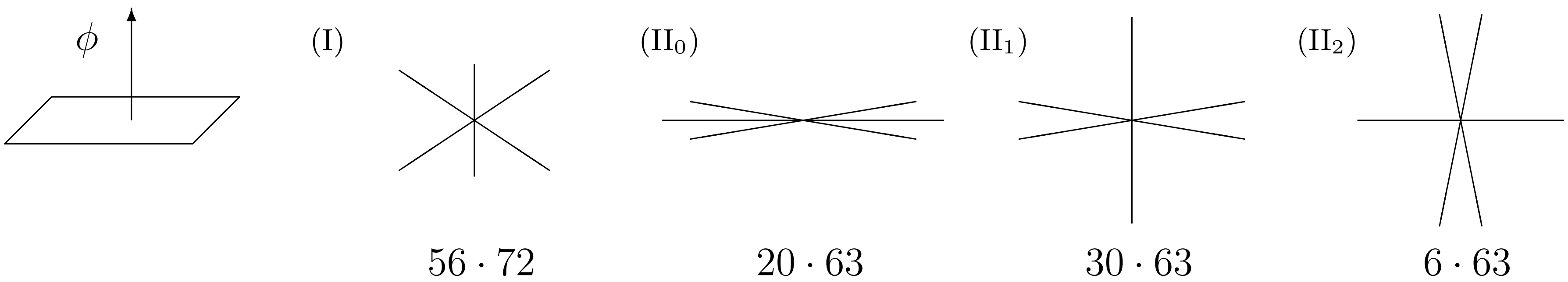
$$[(b \pm c|x)]\,\tau(x \pm a\delta) + [(c \pm a|x)]\,\tau(x \pm b\delta) + [(a \pm b|x)]\,\tau(x \pm c\delta) = 0 \tag{4.7}$$

for any C_3-frame $\{\pm a, \pm b, \pm c\}$ in $P = Q(E_8)$.

Each of the six points $x \pm a\delta$, $x \pm b\delta$, $x \pm c\delta$ belongs to D if and only if the others do. In this formulation $eP(E_8)$ is a $W(E_8)$-invariant system of 7560 non-autonomous Hirota equations.



$$[(b \pm c|x)]\tau(x \pm a\delta) + [(c \pm a|x)]\tau(x \pm b\delta) + [(a \pm b|x)]\tau(x \pm c\delta) = 0$$



Four types of 7560 C_3-frames

○ $eP(E_8)$ τ-function as an infinite chain of $eP(E_7)$ τ-functions

In the E_8 root lattice $P = Q(E_8)$, the E_7 root lattice is realized as

$$Q(E_7) = \{a \in P \mid (\phi|a) = 0\ \} \subset P = Q(E_8); \quad \Delta(E_7) = \Delta(E_8)^{\perp\phi}. \tag{4.8}$$

Fixing a constant $c \in \mathbb{C}$, we consider the union of parallel hyperplanes

$$D_c = \bigsqcup_{n\in\mathbb{Z}} H_{c+n\delta}; \quad H_{c+n\delta} = \{x \in V \mid (\phi|x) = c + n\delta\} \quad (n \in \mathbb{Z}). \tag{4.9}$$

Then an ORG τ-function $\tau(x)$ on D_c can be regarded as a chain $\{\tau^{(n)}(x)\}_{n\in\mathbb{Z}}$ of $eP(E_7)$ τ-functions on parallel hyperplanes by setting $\tau^{(n)} = \tau|_{H_{c+n\delta}}$ $(n \in \mathbb{Z})$.

Four types of bilinear equations corresponding to the types I, II_0, II_1, II_2 of C_3-frames:

$$\begin{aligned}
(\mathrm{I})_{n+\frac{1}{2}} :&\quad [(a_1 \pm a_2|x)]\tau^{(n)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta) + \cdots = 0 \\
(\mathrm{II}_0)_n :&\quad [(a_1 \pm a_2|x)]\tau^{(n)}(x - a_0\delta)\tau^{(n)}(x + a_0\delta) + \cdots = 0 \\
(\mathrm{II}_1)_n :&\quad [(a_1 \pm a_2|x)]\tau^{(n-1)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta) \\
&\quad = [(a_0 \pm a_2|x)]\tau^{(n)}(x \pm a_1\delta) - [(a_0 \pm a_1|x)]\tau^{(n)}(x \pm a_2\delta) \\
(\mathrm{II}_2)_n :&\quad [(a_1 \pm a_2|x)]\tau^{(n)}(x \pm a_0\delta) \\
&\quad = [(a_0 \pm a_2|x)]\tau^{(n-1)}(x - a_1\delta)\tau^{(n+1)}(x + a_1\delta) - \cdots
\end{aligned} \tag{4.10}$$

Definition A meromorphic ORG τ function $\tau(x)$ on $D_c = \bigsqcup_{n\in\mathbb{Z}} H_{c+n\delta}$ is called a *hypergeometric τ-function* if

$$\tau^{(n)}(x) = 0 \quad (n < 0), \qquad \tau^{(0)}(x) \neq 0. \tag{4.11}$$

Theorem C: *Let $\tau^{(0)}(x)$, $\tau^{(1)}(x)$ be nonzero meromorphic functions on H_c, $H_{c+\delta}$ respectively. Suppose that they satisfy*

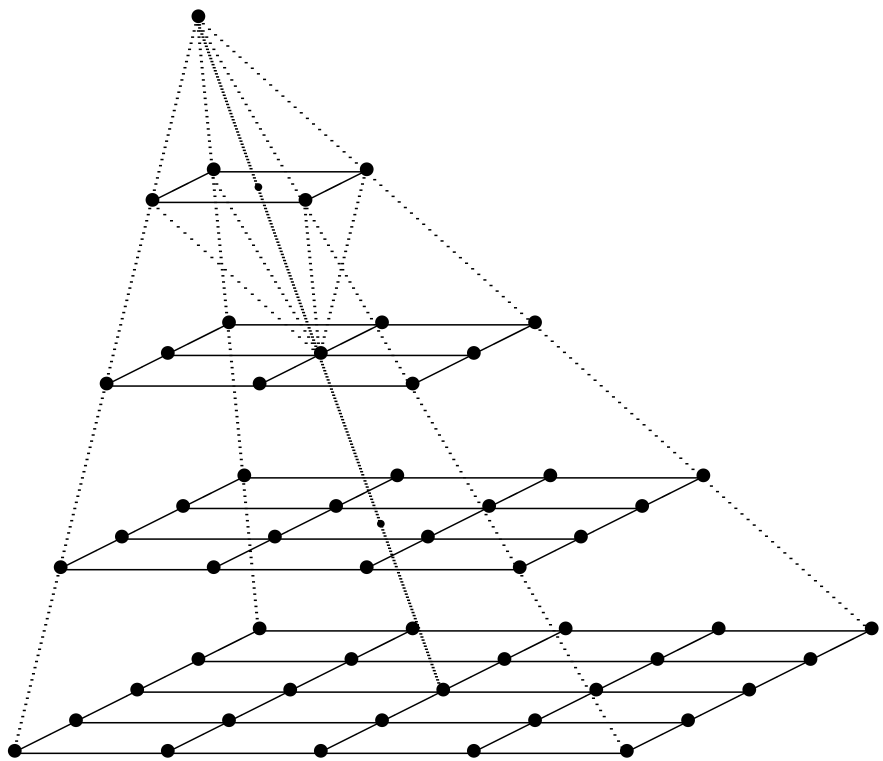
$$[(a_0 \pm a_2|x)]\tau^{(0)}(x \pm a_1\delta) = [(a_0 \pm a_1|x)]\tau^{(0)}(x \pm a_2\delta) \tag{4.12}$$

for any C_3-frame of type II_1, *and*

$$[(a_1 \pm a_2|x)]\tau^{(0)}(x - a_0\delta)\tau^{(1)}(x + a_0\delta) + \cdots = 0 \tag{4.13}$$

for any C_3-frame of type I. *Then these exists a unique a hypergeometric τ-function $\tau(x)$ on D_c such that $\tau^{(0)} = \tau|_{H_c}$ and $\tau^{(1)} = \tau|_{H_{c+\delta}}$.*

Toda equations produce 2-directional Casorati determinants



$$(\mathrm{II}_1)_n: \ [(a_1 \pm a_2|x)]\tau^{(n-1)}(x-a_0\delta)\tau^{(n+1)}(x+a_0\delta)$$
$$= [(a_0 \pm a_2|x)]\tau^{(n)}(x \pm a_1\delta) - [(a_0 \pm a_1|x)]\tau^{(n)}(x \pm a_2\delta)$$

○ Determinant representation of hypergeometric $\boldsymbol{\tau}$-functions

Theorem D: *Under the assumption of Theorem C, suppose that* $\tau^{(1)}(x)$ *is expressed as* $\tau^{(1)}(x) = \gamma^{(1)}(x)\,\varphi(x)$ *with a nonzero meromorphic function* $\gamma^{(1)}(x)$ *satisfying*

$$[(a_0 + a_2|x)]\gamma^{(1)}(x \pm a_1\delta) = [(a_0 + a_1|x)]\gamma^{(1)}(x \pm a_2\delta) \tag{4.14}$$

for a C_3*-frame of type* II_1 *with* $(\phi|a_0) = 1$, $(\phi|a_1) = (\phi|a_2) = 0$. *Then the components* $\tau^{(n)}(x)$ *of the hypergeometric* τ*-function* $\tau(x)$ *are expressed as follows in terms of 2-directional Casorati determinants:*

$$\begin{aligned} &\tau^{(n)}(x) = \gamma^{(n)}(x)K^{(n)}(x) \quad (x \in H_{c+n\delta};\ n = 0, 1, 2, \ldots) \\ &K^{(n)}(x) = \det\left(\varphi_{ij}^{(n)}(x)\right)_{i,j=1}^{n} \\ &\varphi_{ij}(x) = \varphi^{(n)}(x-(n-1)a_0\delta+(n+1-i-j)a_1\delta+(j-i)a_1\delta) \quad (1 \le i, j \le n). \end{aligned} \tag{4.15}$$

The gauge factors $\gamma^{(n)}(x)$ *are determined inductively from* $\gamma^{(0)}(x) = \tau^{(0)}(x)$, $\gamma^{(1)}(x)$ *by*

$$[(a_0 \pm a_2|x)]\gamma^{(n-1)}(x - a_0\delta)\gamma^{(n+1)}(x + a_0\delta) = [(a_1 \pm a_2|x)]\gamma^{(n)}(x \pm a_1\delta). \tag{4.16}$$

The Toda equation $(\mathrm{II}_1)_n$ corresponds to the *Lewis-Carroll formula* for determinants.

○ $W(E_7)$-invariant hypergeometric τ-function

An example of hypergeometric τ-function for $eP(E_8)$ is given by multiple elliptic hypergeometric integrals due to Rains:

$$I^{(n)}(u;p,q) = I^{(n)}(u_0,\ldots,u_7;p,q)$$

$$= \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n!(2\pi\sqrt{-1})^n} \int_{C^n} \prod_{k=1}^{n} \frac{\prod_{i=0}^{7}\Gamma(u_i z_k^{\pm 1};p,q)}{\Gamma(z_k^{\pm 2};p,q)} \prod_{1\le k<l\le n} \theta(z_k^{\pm 1} z_l^{\pm 1};p) \frac{dz_1\cdots dz_n}{z_1\cdots z_n}. \tag{4.17}$$

We consider to construct a hypergeometric τ-function on

$$D_\tau = \bigsqcup_{n\in\mathbb{Z}} H_{\tau+n\delta} \quad \text{with} \quad p = e(\tau), \quad q = e(\delta). \tag{4.18}$$

• $\tau^{(0)}(x)$: The system of first order difference equations for $\tau^{(0)}(x)$ $(x \in H_\tau)$ is solved by a product of triple elliptic gamma functions:

$$\tau^{(0)}(x) = \prod_{0\le i<j\le 7} \Gamma(qu_iu_j;p,q,q) \quad (x \in H_\tau) \tag{4.19}$$

in multiplicative variables $u_i = e(x_i)$ $(i = 0,1,\ldots,7)$, where

$$\Gamma(u;p,q,r) = (u;p,q,r)_\infty (pqr/u;p,q,r)_\infty,$$
$$(u;p,q,r)_\infty = \prod_{i,j,k=0}^{\infty} (1 - p^i q^j r^k u) \quad (|p|,|q|,|r| < 1). \tag{4.20}$$

• $\tau^{(1)}(x)$: Then, the system of Hirota equations between $\tau^{(0)}(x)$ and $\tau^{(1)}(x)$ is solved by the elliptic hypergeometric integral:

$$\begin{aligned}
&\tau^{(1)}(x) = \prod_{0\le i<j\le 7} \Gamma(u_i u_j; p, q, q) e(-Q(x)) I(u; p, q) \quad (x \in H_{\tau+\delta}),\\
&Q(x) = \tfrac{1}{2\delta}(x|x) - (\phi|x),\\
&I(u; p, q) = \frac{(p; p)_\infty (q; q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{i=0}^{r} \Gamma(u_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}.
\end{aligned} \tag{4.21}$$

Note that the condition $x \in H_{\tau+\delta}$ corresponds to the balancing condition $u_0 u_1 \cdots u_7 = p^2 q^2$ in multiplicative variables. In fact, the system of linear difference equations for $\tau^{(1)}(x)$ reduces to the three term relations

$$[x_j \pm x_k] T^{\delta}_{x_i} J(x) + [x_k \pm x_i] T^{\delta}_{x_j} J(x) + [x_i \pm x_j] T^{\delta}_{x_k} J(x) = 0. \tag{4.22}$$

for $J(x) = e(-Q(x)) I(u; p, q)$.

• *Determinant formula for* $\tau^{(n)}(x)$: Using the decomposition $\tau^{(1)}(x) = \gamma^{(1)}(x)\varphi(x)$ with $\varphi(x) = J(x)$, by Theorem D we know that $\tau^{(n)}(x)$ has the determinant formula

$$\begin{aligned}
&\tau^{(n)}(x) = \gamma^{(n)}(x) \det \left(\varphi^{(n)}_{ij}(x)\right)_{i,j=1}^{n}\\
&\varphi^{(n)}_{ij}(x) = \varphi(x - (n-1)a_0 + (n+1-i-j)a_1 + (j-i)a_2)
\end{aligned} \tag{4.23}$$

for any C_3-frame $\{\pm a_0, \pm a_1, \pm a_2\}$ of type II_1 with $(\phi|a_0) = 1$.

• $\tau^{(n)}(x)$ *as a multiple elliptic hypergeometric integral*: This 2-directional Casorati determinant can be rewritten into multiple integrals. By Warnaar's elliptic extension of the Krattenthaler determinant, we finally obtain the expression of $\tau^{(n)}(x)$ in terms of the multiple elliptic hypergeometric integral of Rains:

$$\begin{aligned} &\tau^{(n)}(x+(n-1)\phi) = p^{-\binom{n}{2}} \prod_{0\le i<j\le 7} \Gamma(u_i u_j; p, q, q)\ e(-nQ(x))\, I^{(n)}(u; p, q), \\ &I^{(n)}(u;p,q) \\ &= \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n! (2\pi\sqrt{-1})^n} \int_{C^n} \prod_{k=1}^{n} \frac{\prod_{i=0}^{7} \Gamma(u_i z_k^{\pm 1}; p, q)}{\Gamma(z_k^{\pm 2}; p, q)} \prod_{1\le k<l\le n} \theta(z_k^{\pm 1} z_l^{\pm 1}; p) \frac{dz_1\cdots dz_n}{z_1\cdots z_n}. \end{aligned} \tag{4.24}$$

The sequence $\tau^{(n)}(x)$ $(n=0,1,2,\ldots)$ determined as above provides a $W(E_7)$-invariant hypergeometric τ-function. This fact follows from the $W(E_7)$-invariance of $\tau^{(0)}(x)$, $\tau^{(1)}(x)$ and the uniqueness of extension to $\tau^{(n)}(x)$.

○ Relation to elliptic Askey-Wilson functions

• *Elliptic extension of Askey-Wilson polynomials*: (Spiridonov-Zhedanov)

With parameters $\boldsymbol{a} = (a_0, a_1, a_2, a_4)$ and b, for $l = 0, 1, 2, \ldots$ consider the sequence of terminating elliptic hypergeometric series

$$\Phi_l(z; \boldsymbol{a}, b) = {}_{12}V_{11}(a_0 + b - \delta; a_0 \pm z, 2\alpha_0 + l\delta, -l\delta, b - a_1, b - a_2, b - a_3), \tag{4.25}$$

where $\alpha_0 = \frac{1}{2}(a_0 + a_1 + a_2 + a_3 - \delta)$. These functions satisfy the difference equation

$$\begin{aligned} &L(z, T_z, \boldsymbol{a}; u)\Phi_l(z; \boldsymbol{a}, b) = \Phi_l(z; \boldsymbol{a}, b + \delta)\Lambda_l(\boldsymbol{a}, b; u) \quad (l = 0, 1, 2, \ldots) \\ &\Lambda_l(\boldsymbol{a}, b; u) = \frac{[\alpha_0 + \delta \pm u]}{[\alpha_0 + l\delta \pm \beta]}[\alpha_0 \pm \beta] \prod_{i=0}^{3}[b - a_i], \quad \beta = a_0 + b - \alpha_0 \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} &L(z, T_z, \boldsymbol{a}; u) = A(z; \boldsymbol{a}, b, u))(T_z^{\delta} - 1) + A(-z; \boldsymbol{a}, b, u))(T_z^{-\delta} - 1) + \Lambda_0(\boldsymbol{a}, b; u) \\ &A(z; \boldsymbol{a}, b, u) = \frac{[z - b][z - b + \delta][z + b - \alpha_0 \pm u] \prod_{i=0}^{3}[z + a_i]}{[2z][2z + \delta]} \end{aligned} \tag{4.27}$$

is a six-parameter subfamily of Ruijsenaars-van Diejen operator of type BC_1.

We modify the elliptic Askey-Wilson functions as

$$\Psi_l(z;\boldsymbol{a},b)=\frac{[b\pm z]_l}{[b+2\alpha_0\pm z]_l}\Phi_l(z;\boldsymbol{a},b)\quad (l=0,1,2\ldots). \tag{4.28}$$

Returning to the $W(E_7)$-invariant hypergeometric τ-function $\tau(x)$ previously introduced, assume that $q/u_0u_i=q^{-N}$, $N\in\mathbb{N}$ for some $i\in\{1,\ldots,6\}$. Then, up to a gauge factor, the nth τ-function $\tau^{(n)}(x)$ on $H_{\tau+n\delta}$ coincides with the determinant

$$\det\left(\Psi_{\lambda_j+n-j}(z_i;\boldsymbol{a},b)\right)_{i,j=1}^{n} \tag{4.29}$$

of Schur type specialized as

$$\begin{aligned}(\lambda_1,\ldots,\lambda_n)&=(N,\ldots,N),\\ (z_1,\ldots,z_n)&=(c+(n-1)\delta,c+(n-2)\delta,\ldots,c)\end{aligned} \tag{4.30}$$

under a change of parameters.