On elliptic Lax pairs and isomonodromic deformation problems for integrable lattice systems

Frank Nijhoff, *University of Leeds*

joint work with Neslihan Delice and (partly) with Sikarin Yoo-Kong

Workshop on Representation Theory, Special Functions and Painlevé Equations,
RIMS, Kyoto University, 4 March 2015

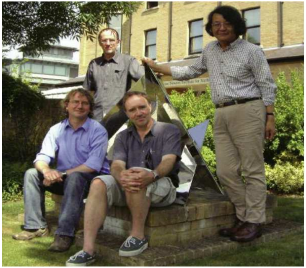

Figure : Organisers of the 2009 DIS programme at the Isaac Newton Institute

In honour of Professor Noumi's 60th birthday

Plan

1. Brief overview of integrable lattice systems (partial difference equations) associated with an elliptic curve;
   - Lattice Landau-Lifschitz (LL) system;

$$\text{Landau} - \text{Lifschitz equation}: \quad \mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times (J\mathbf{S})$$

   - Lattice Krichever-Novikov (KN) equation (aka Adler's equation, Q4)

$$\text{Krichever} - \text{Novikov equation}: \quad u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2 - R(u)}{u_x} \ .$$

2. General elliptic Lax pairs on the lattice[1]
   - "Lattice Landau-Lifschitz" (spin non-zero) type;
   - "Krichever-Novikov" (spin zero) type.
3. Analogous general scheme of elliptic discrete isomonodromic deformation problems[2].

[1] N. Delice, F.W. Nijhoff and S. Yoo-Kong, *On Elliptic Lax Systems on the Lattice and a Compound Theorem for Hyperdeterminants*, J.Phys A: Math. Theor. **48** (2015) 035206.

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for two spin vectors $\mathbf{S}(n,m)$ and $\mathbf{T}(n,m)$, in combination with scalar functions $S_0(n,m)$, $T_0(n,m)$, and (anisotropy) parameters $\mathbf{J}=(J_1,J_2,J_3)$. Here: notation for shifts of functions $f=f(n,m)$ of discrete variables $n,m\in\mathbb{Z}$:

$$\widetilde{f}=f(n+1,m)\ ,\quad \widehat{f}=f(n,m+1)\ .$$

The spin vector $\mathbf{T}$ can be eliminated using the Casimirs $\mathbf{S}^2=1$, $S_0^2+(\mathbf{J}\mathbf{S})^2$, $\mathbf{T}^2$and $T_0^2+(\mathbf{J}\mathbf{T})^2$, leading to a PΔE for $\mathbf{S}$ only, which in a continuum limit goes over into the LL equation.

2. Adler's discretization of KN equation[4], for a function $u(n,m)$, which reads:

$$A\left[(u-b)(\widehat{u}-b)-(a-b)(c-b)\right]\left[(\widetilde{u}-b)(\widehat{\widetilde{u}}-b)-(a-b)(c-b)\right]$$
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where the lattice parameters (a,A), $(b,B)$and (c,C) are points on a Weierstrass elliptic curve

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$$p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-\frac{pQ-qP}{1-p^2q^2}\left[(\widehat{u}\widetilde{u}+u\widehat{\widetilde{u}})-pq(1-u\widetilde{u}\widehat{u}\widehat{\widetilde{u}})\right]=0$$

   $\mathfrak{p}=(p,P)$, $\mathfrak{q}=(q,Q)$ being points on the Jacobi curve $X^2=x^4-(k+1/k)x^2+1$, in [Atkinson & FWN, 2010; Atkinson & Joshi, 2013] ; (For the continuous case of KN, no explicit solution existed; a formal construction for algebro-geometric solutions was given by [D.P. Novikov, 1999]);
3. There are alternative forms of lattice LL (notably by Adler & Yamilov, 1998) but for none of these versions any explicit solutions exist to date.
   NB: There also exist an elliptic lattice KdV system and an elliptic lattice KP system: for those elliptic N-soliton solutions were constructed in explicit (elliptic Cauchy matrix) form [FWN & Puttock, 2003; Jennings & FWN, 2014].

General Elliptic Lax scheme

Consider a lattice Lax pair of the form:

$$\widetilde{\chi}_\kappa = \mathbf{L}_\kappa\, \chi_\kappa \ , \quad \widehat{\chi}_\kappa = \mathbf{M}_\kappa\, \chi_\kappa \ ,$$

defining horizontal and vertical shits of the vector function χ_κ, according to:

with compatibility condition: $\quad \widehat{\mathbf{L}}_\kappa \mathbf{M}_\kappa = \widetilde{\mathbf{M}}_\kappa \mathbf{L}_\kappa$

with vectors χ located at the vertices of the quadrilateral and Lax matrices **L** and **M** attached to the edges. We assume these to be of the form:

$$(\mathbf{L}_\kappa)_{i,j} = \Phi_{N\kappa}(\widetilde{\xi}_i - \xi_j - \alpha) h_j \ ,$$
$$(\mathbf{M}_\kappa)_{i,j} = \Phi_{N\kappa}(\widehat{\xi}_i - \xi_j - \beta) k_j \ , \quad (j = 1, \dots, N)$$

where $\Phi_\kappa(x) = \frac{\sigma(x+\kappa)}{\sigma(x)\,\sigma(\kappa)}$, and in which $\xi = \xi_{n,m}$ are the main dependent variables, while the coefficients h_j, k_j remain to be determined.

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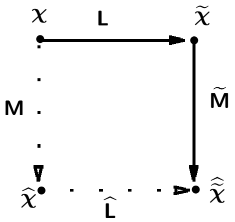

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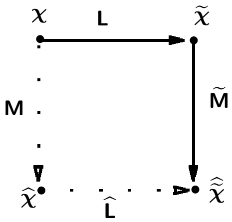

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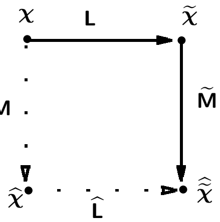

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using the addition formula

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the consistency gives rise to

$$\sum_{l=1}^{N} \widehat{h}_l k_j \left[\zeta(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha) + \zeta(\widehat{\xi}_l - \xi_j - \beta) + \zeta(N\kappa) - \zeta(N\kappa + \widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\right] =$$

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$$(i,j = 1,\dots,N) \ .$$

Due to the dependence on the spectral parameter κ these equations separate into two parts:

$$\left(\sum_{l=1}^{N} \widehat{h}_l\right) k_j = \left(\sum_{l=1}^{N} \widetilde{k}_l\right) h_j \quad , \quad (j=1,\dots,N) \ ,$$

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There are two nontrivial scenarios which we refer to as *"Landau-Lifschitz type"* and *"Krichever-Novikov type"* respectively:

1. Discrete Landau-Lifschitz (LL) type case: $\sum_l h_l \neq 0$, in which case we have that the variables h_j, k_j are proportional to each other, $k_j = \rho h_j$, and after summation we obtain the conservation law:

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Resolution of the compatibility conditions

Under the assumption for the Centre of Mass (CoM) motion:

$$\widetilde{\Xi} + \widehat{\Xi} = \widehat{\widetilde{\Xi}} + \Xi\,, \quad \text{where} \quad \Xi := \sum_{j=1}^{N} \xi_j$$

we can analyse the Lax eqs. by considering the following elliptic function:

$$F(\xi) := \prod_{l=1}^{N} \frac{\sigma(\xi - \widehat{\widetilde{\xi}}_l)\sigma(\xi - \xi_l - \alpha - \beta)}{\sigma(\xi - \widehat{\xi}_l - \alpha)\sigma(\xi - \widetilde{\xi}_l - \beta)}$$

$$= \sum_{l=1}^{N} \left[\zeta(\xi - \widehat{\xi}_l - \alpha) - \zeta(\eta - \widehat{\xi}_l - \alpha)\right] H_l + \sum_{l=1}^{N} \left[\zeta(\xi - \widetilde{\xi}_l - \beta) - \zeta(\eta - \widetilde{\xi}_l - \beta)\right] K_l$$

which holds as an identity for any four sets of variables ξ_l, $\widehat{\xi}_l$, $\widetilde{\xi}_l$, $\widehat{\widetilde{\xi}}_l$ s.t. the above equality for the sum holds. Here η is any one of the zeroes (i.e., $\widehat{\widetilde{\xi}}_i$ or $\xi_i + \alpha + \beta$). The coefficients H_j, K_j are explicitly given by:

$$H_l = \frac{\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widehat{\widetilde{\xi}}_k + \alpha)\sigma(\widehat{\xi}_l - \xi_k - \beta)}{\left[\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widetilde{\xi}_k + \alpha - \beta)\right] \prod_{k\neq l} \sigma(\widehat{\xi}_l - \widehat{\xi}_k)}\,,$$

$$K_l = \frac{\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\widetilde{\xi}}_k + \beta)\sigma(\widetilde{\xi}_l - \xi_k - \alpha)}{\left[\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\xi}_k + \beta - \alpha)\right] \prod_{k\neq l} \sigma(\widetilde{\xi}_l - \widetilde{\xi}_k)}\,.$$

The coefficients obey the identity: $\sum_{l=1}^{N}(H_l + K_l) = 0$.

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$$F(\xi) := \prod_{l=1}^{N} \frac{\sigma(\xi - \widehat{\widetilde{\xi}}_l)\sigma(\xi - \xi_l - \alpha - \beta)}{\sigma(\xi - \widehat{\xi}_l - \alpha)\sigma(\xi - \widetilde{\xi}_l - \beta)}$$

$$= \sum_{l=1}^{N} \left[\zeta(\xi - \widehat{\xi}_l - \alpha) - \zeta(\eta - \widehat{\xi}_l - \alpha)\right] H_l + \sum_{l=1}^{N} \left[\zeta(\xi - \widetilde{\xi}_l - \beta) - \zeta(\eta - \widetilde{\xi}_l - \beta)\right] K_l$$

which holds as an identity for any four sets of variables ξ_l, $\widehat{\xi}_l$, $\widetilde{\xi}_l$, $\widehat{\widetilde{\xi}}_l$ s.t. the above equality for the sum holds. Here η is any one of the zeroes (i.e., $\widehat{\widetilde{\xi}}_i$ or $\xi_i + \alpha + \beta$). The coefficients H_j, K_j are explicitly given by:

$$H_l = \frac{\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widehat{\widetilde{\xi}}_k + \alpha)\sigma(\widehat{\xi}_l - \xi_k - \beta)}{\left[\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widetilde{\xi}_k + \alpha - \beta)\right] \prod_{k \neq l} \sigma(\widehat{\xi}_l - \widehat{\xi}_k)} \,,$$

$$K_l = \frac{\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\widetilde{\xi}}_k + \beta)\sigma(\widetilde{\xi}_l - \xi_k - \alpha)}{\left[\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\xi}_k + \beta - \alpha)\right] \prod_{k \neq l} \sigma(\widetilde{\xi}_l - \widetilde{\xi}_k)} \,.$$

The coefficients obey the identity: $\sum_{l=1}^{N}(H_l + K_l) = 0$.

Resolution of the compatibility conditions

Under the assumption for the Centre of Mass (CoM) motion:

$$\widetilde{\Xi} + \widehat{\Xi} = \widehat{\widetilde{\Xi}} + \Xi\ , \quad \text{where} \quad \Xi := \sum_{j=1}^{N} \xi_j$$

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$$= \sum_{l=1}^{N} \left[\zeta(\xi - \widehat{\xi}_l - \alpha) - \zeta(\eta - \widehat{\xi}_l - \alpha)\right] H_l + \sum_{l=1}^{N} \left[\zeta(\xi - \widetilde{\xi}_l - \beta) - \zeta(\eta - \widetilde{\xi}_l - \beta)\right] K_l$$

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The coefficients H_j, K_j are explicitly given by:

$$H_l = \frac{\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widehat{\widetilde{\xi}}_k + \alpha)\sigma(\widehat{\xi}_l - \xi_k - \beta)}{\left[\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widetilde{\xi}_k + \alpha - \beta)\right] \prod_{k \neq l} \sigma(\widehat{\xi}_l - \widehat{\xi}_k)}\ ,$$

$$K_l = \frac{\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\widetilde{\xi}}_k + \beta)\sigma(\widetilde{\xi}_l - \xi_k - \alpha)}{\left[\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\xi}_k + \beta - \alpha)\right] \prod_{k \neq l} \sigma(\widetilde{\xi}_l - \widetilde{\xi}_k)}\ .$$

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which holds as an identity for any four sets of variables ξ_l, $\widehat{\xi}_l$, $\widetilde{\xi}_l$, $\widehat{\widetilde{\xi}}_l$ s.t. the above equality for the sum holds. Here η is any one of the zeroes (i.e., $\widehat{\widetilde{\xi}}_i$ or $\xi_i + \alpha + \beta$). The coefficients H_j, K_j are explicitly given by:

$$H_l = \frac{\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widehat{\widetilde{\xi}}_k + \alpha)\sigma(\widehat{\xi}_l - \xi_k - \beta)}{\left[\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widetilde{\xi}_k + \alpha - \beta)\right] \prod_{k\neq l} \sigma(\widehat{\xi}_l - \widehat{\xi}_k)} \,,$$

$$K_l = \frac{\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\widetilde{\xi}}_k + \beta)\sigma(\widetilde{\xi}_l - \xi_k - \alpha)}{\left[\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\xi}_k + \beta - \alpha)\right] \prod_{k\neq l} \sigma(\widetilde{\xi}_l - \widetilde{\xi}_k)} \,.$$

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$$\begin{aligned} F(\xi) &:= \prod_{l=1}^{N} \frac{\sigma(\xi - \widehat{\widetilde{\xi}}_l)\sigma(\xi - \xi_l - \alpha - \beta)}{\sigma(\xi - \widehat{\xi}_l - \alpha)\sigma(\xi - \widetilde{\xi}_l - \beta)} \\ &= \sum_{l=1}^{N} \left[\zeta(\xi - \widehat{\xi}_l - \alpha) - \zeta(\eta - \widehat{\xi}_l - \alpha)\right] H_l + \sum_{l=1}^{N} \left[\zeta(\xi - \widetilde{\xi}_l - \beta) - \zeta(\eta - \widetilde{\xi}_l - \beta)\right] K_l \end{aligned}$$

which holds as an identity for any four sets of variables ξ_l, $\widehat{\xi}_l$, $\widetilde{\xi}_l$, $\widehat{\widetilde{\xi}}_l$ s.t. the above equality for the sum holds. Here η is any one of the zeroes (i.e., $\widehat{\widetilde{\xi}}_i$ or $\xi_i + \alpha + \beta$).
The coefficients H_j, K_j are explicitly given by:

$$\begin{aligned} H_l &= \frac{\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widehat{\widetilde{\xi}}_k + \alpha)\sigma(\widehat{\xi}_l - \xi_k - \beta)}{\left[\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widetilde{\xi}_k + \alpha - \beta)\right] \prod_{k\neq l} \sigma(\widehat{\xi}_l - \widehat{\xi}_k)} \,, \\ K_l &= \frac{\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\widetilde{\xi}}_k + \beta)\sigma(\widetilde{\xi}_l - \xi_k - \alpha)}{\left[\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\xi}_k + \beta - \alpha)\right] \prod_{k\neq l} \sigma(\widetilde{\xi}_l - \widetilde{\xi}_k)} \,. \end{aligned}$$

The coefficients obey the identity: $\sum_{l=1}^{N}(H_l + K_l) = 0$.

The generalized LL Class

Using the identities above, taking $\xi = \widehat{\widetilde{\xi}}_i$, $\eta = \xi_j + \alpha + \beta$ in $F(\xi)$, and comparing with the Lax equations, we can identify:

$$tH_l = \rho\widehat{h}_l \quad , \quad -tK_l = \widetilde{k}_l = \widetilde{\rho}\widetilde{h}_l \quad , \quad l = 1, \dots, N \ ,$$

with t an arbitrary proportionality factor. Thus, inserting the explicit expressions for H_l and K_l we obtain a system of $N+2$ equations for the $N+2$ unknowns: $\xi_1, \dots, \xi_N$, ρ, t. This comprises the set of equations

$$\frac{\widetilde{t}}{\widetilde{\rho}}\widetilde{H}_l + \frac{\widehat{t}}{\widehat{\widetilde{\rho}}}\widehat{K}_l = 0 \ , \quad (l = 1, \dots, N) \ , \quad \widetilde{\Xi} + \widehat{\Xi} = \widehat{\widetilde{\Xi}} + \Xi \ .$$

This yields the system of N 7-point equations:

$$\prod_{k=1}^{N} \frac{\sigma(\xi_l - \widetilde{\xi}_k + \alpha)\,\sigma(\xi_l - \underset{\frown}{\xi}_k - \beta)\,\sigma(\xi_l - \underset{\sim}{\widehat{\xi}}_k + \gamma)}{\sigma(\xi_l - \widehat{\xi}_k + \beta)\,\sigma(\xi_l - \underset{\sim}{\xi}_k - \alpha)\,\sigma(\xi_l - \underset{\frown}{\widetilde{\xi}}_k - \gamma)} = p$$

for $N+1$ variables ξ_i $(i = 1, \dots, N)$ and $p = -\underset{\sim}{t}\,\underset{\frown}{\rho}/(\underset{\frown}{t}\,\rho)$, supplemented with the relation $\widehat{\widetilde{\Xi}} + \Xi - \widehat{\Xi} - \widetilde{\Xi} = 0$ which fixes the CoM dynamics. The under-accents $\underset{\sim}{\cdot}$ and $\underset{\frown}{\cdot}$ denote reverse lattice shifts: $\underset{\sim}{\xi}_i(n,m) = \xi_i(n-1,m)$, $\underset{\frown}{\xi}_i(n,m) = \xi_i(n,m-1)$.

The implicit system of P∆Es arises from the following Lagrangian:

$$\mathscr{L} = \sum_{i,j}\left[f(\xi_i - \widetilde{\xi}_j + \alpha) - f(\xi_i - \widehat{\xi}_j + \beta) - f(\widehat{\xi}_i - \widetilde{\xi}_j + \alpha - \beta)\right] - \ln|p|\,\Xi$$

in which the function f is the elliptic dilogarithm $f(x) = \int^x \ln\sigma(\xi)\,d\xi$

The generalized LL Class

Using the identities above, taking $\xi=\widehat{\widetilde{\xi}}_i$, $\eta=\xi_j+\alpha+\beta$ in $F(\xi)$, and comparing with the Lax equations, we can identify:

$$tH_l=\rho\widehat{h}_l \quad , \quad -tK_l=\widetilde{k}_l=\widetilde{\rho}\widetilde{h}_l \quad , \quad l=1,\dots,N\ ,$$

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$$\frac{\widetilde{t}}{\widetilde{\rho}}\widetilde{H}_l+\frac{\widehat{t}}{\widehat{\widetilde{\rho}}}\widehat{K}_l=0\ , \quad (l=1,\dots,N)\ , \quad \widetilde{\Xi}+\widehat{\Xi}=\widehat{\widetilde{\Xi}}+\Xi\ .$$

This yields the system of N 7-point equations:

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for $N+1$ variables ξ_i $(i=1,\dots,N)$ and $p=-\underset{\sim}{t}\,\underset{\frown}{\rho}/(\underset{\frown}{t}\,\rho)$, supplemented with the relation $\widehat{\widetilde{\Xi}}+\Xi-\widehat{\Xi}-\widetilde{\Xi}=0$ which fixes the CoM dynamics. The under-accents $\underset{\sim}{\cdot}$ and $\underset{\frown}{\cdot}$ denote reverse lattice shifts: $\underset{\sim}{\xi}_i(n,m)=\xi_i(n-1,m)$, $\underset{\frown}{\xi}_i(n,m)=\xi_i(n,m-1)$.

The implicit system of P∆Es arises from the following Lagrangian:

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$$\frac{\widetilde{t}}{\widetilde{\rho}}\widetilde{H}_l+\frac{\widehat{t}}{\widehat{\widetilde{\rho}}}\widehat{K}_l=0\ ,\quad (l=1,\dots,N)\ ,\quad \widetilde{\Xi}+\widehat{\Xi}=\widehat{\widetilde{\Xi}}+\Xi\ .$$

This yields the system of N 7-point equations:

$$\prod_{k=1}^{N}\frac{\sigma(\xi_l-\widetilde{\xi}_k+\alpha)\,\sigma(\xi_l-\underset{\frown}{\xi}_k-\beta)\,\sigma(\xi_l-\underset{\sim}{\widehat{\xi}}_k+\gamma)}{\sigma(\xi_l-\widehat{\xi}_k+\beta)\,\sigma(\xi_l-\underset{\sim}{\xi}_k-\alpha)\,\sigma(\xi_l-\underset{\frown}{\widetilde{\xi}}_k-\gamma)}=p$$

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The implicit system of PΔEs arises from the following Lagrangian:

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in which the function f is the elliptic dilogarithm $f(x)=\int_0^x \ln\sigma(\xi)\,d\xi$

The generalized LL Class

Using the identities above, taking $\xi = \widehat{\widetilde{\xi}}_i$, $\eta = \xi_j + \alpha + \beta$ in $F(\xi)$, and comparing with the Lax equations, we can identify:

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for $N+1$ variables ξ_i $(i = 1, \dots, N)$ and $p = -\underset{\sim}{t}\,\underset{\frown}{\rho}/(\underset{\frown}{t}\,\rho)$, supplemented with the relation $\widehat{\widetilde{\Xi}} + \Xi - \widehat{\Xi} - \widetilde{\Xi} = 0$ which fixes the CoM dynamics. The under-accents $\underset{\sim}{\cdot}$ and $\underset{\frown}{\cdot}$ denote reverse lattice shifts: $\underset{\sim}{\xi}_i(n,m) = \xi_i(n-1,m)$, $\underset{\frown}{\xi}_i(n,m) = \xi_i(n,m-1)$.

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in which the function f is the elliptic dilogarithm $f(x) = \int^x \ln\sigma(\xi)\,d\xi$.

The Adler class of lattice equation; case $N = 2$

For the case $N = 2$ the Lax pair in this class has the form:

$$\widetilde{\chi} = \mathbf{L}_\kappa \chi \quad = \quad \lambda \begin{pmatrix} \Phi_{2\kappa}(\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(\widetilde{\xi} + \xi - \alpha) \\ \Phi_{2\kappa}(-\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(-\widetilde{\xi} + \xi - \alpha) \end{pmatrix} \boldsymbol{\chi}$$

$$\widehat{\chi} = \mathbf{M}_\kappa \chi \quad = \quad \mu \begin{pmatrix} \Phi_{2\kappa}(\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(\widehat{\xi} + \xi - \beta) \\ \Phi_{2\kappa}(-\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(-\widehat{\xi} + \xi - \beta) \end{pmatrix} \boldsymbol{\chi}\ ,$$

in which the coefficients λ and μ are functions $\lambda = \lambda(\xi, \widetilde{\xi}; \alpha)$ and $\mu = \mu(\xi, \widehat{\xi}; \beta)$, respectively. The Lax equations are of the form:

$$\widehat{\lambda}\mu \left[\zeta(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right]$$

$$= \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi + \alpha)\right]$$

$$\widehat{\lambda}\mu \left[\zeta(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi - \beta) - \zeta(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi + \beta)\right]$$

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$$\widehat{\lambda}\mu \left[\zeta(-\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(-\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right]$$

$$= \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi + \alpha)\right]$$

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$$\widehat{\boldsymbol{\chi}} = \mathbf{M}_\kappa \chi = \mu \begin{pmatrix} \Phi_{2\kappa}(\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(\widehat{\xi} + \xi - \beta) \\ \Phi_{2\kappa}(-\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(-\widehat{\xi} + \xi - \beta) \end{pmatrix} \boldsymbol{\chi} ,$$

in which the coefficients λ and μ are functions $\lambda = \lambda(\xi, \widetilde{\xi}; \alpha)$ and $\mu = \mu(\xi, \widehat{\xi}; \beta)$, respectively. The Lax equations are of the form:

$$\widehat{\lambda}\mu \left[\zeta(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right]$$
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$$\widehat{\lambda}\mu \left[\zeta(-\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(-\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right]$$
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$$= \widetilde{\mu}\lambda \left[\zeta(-\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha) - \zeta(-\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi + \alpha)\right]$$

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\begin{aligned}
\widetilde{\boldsymbol{\chi}} = \mathbf{L}_\kappa \chi &= \lambda \begin{pmatrix} \Phi_{2\kappa}(\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(\widetilde{\xi} + \xi - \alpha) \\ \Phi_{2\kappa}(-\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(-\widetilde{\xi} + \xi - \alpha) \end{pmatrix} \boldsymbol{\chi} \\
\widehat{\boldsymbol{\chi}} = \mathbf{M}_\kappa \chi &= \mu \begin{pmatrix} \Phi_{2\kappa}(\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(\widehat{\xi} + \xi - \beta) \\ \Phi_{2\kappa}(-\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(-\widehat{\xi} + \xi - \beta) \end{pmatrix} \boldsymbol{\chi} \, ,
\end{aligned}
$$

in which the coefficients λ and μ are functions $\lambda = \lambda(\xi, \widetilde{\xi}; \alpha)$ and $\mu = \mu(\xi, \widehat{\xi}; \beta)$, respectively. The Lax equations are of the form:

$$
\begin{aligned}
&\widehat{\lambda}\mu \left[\zeta(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right] \\
&\quad = \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi + \alpha)\right] \\
&\widehat{\lambda}\mu \left[\zeta(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi - \beta) - \zeta(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi + \beta)\right] \\
&\quad = \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha)\right] \\
&\widehat{\lambda}\mu \left[\zeta(-\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(-\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right] \\
&\quad = \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi + \alpha)\right] \\
&\widehat{\lambda}\mu \left[\zeta(-\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi - \beta) - \zeta(-\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi + \beta)\right] \\
&\quad = \widetilde{\mu}\lambda \left[\zeta(-\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha) - \zeta(-\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi + \alpha)\right]
\end{aligned}
$$

The Adler class of lattice equation; case $N = 2$

For the case $N = 2$ the Lax pair in this class has the form:

$$\widetilde{\boldsymbol{\chi}} = \mathbf{L}_\kappa \chi = \lambda \begin{pmatrix} \Phi_{2\kappa}(\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(\widetilde{\xi} + \xi - \alpha) \\ \Phi_{2\kappa}(-\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(-\widetilde{\xi} + \xi - \alpha) \end{pmatrix} \boldsymbol{\chi}$$

$$\widehat{\boldsymbol{\chi}} = \mathbf{M}_\kappa \chi = \mu \begin{pmatrix} \Phi_{2\kappa}(\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(\widehat{\xi} + \xi - \beta) \\ \Phi_{2\kappa}(-\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(-\widehat{\xi} + \xi - \beta) \end{pmatrix} \boldsymbol{\chi} ,$$

in which the coefficients λ and μ are functions $\lambda = \lambda(\xi, \widetilde{\xi}; \alpha)$ and $\mu = \mu(\xi, \widehat{\xi}; \beta)$, respectively. The Lax equations are of the form:

$$\widehat{\lambda}\mu \left[\zeta(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right]$$

$$= \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi + \alpha)\right]$$

$$\widehat{\lambda}\mu \left[\zeta(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi - \beta) - \zeta(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi + \beta)\right]$$

$$= \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha)\right]$$

$$\widehat{\lambda}\mu \left[\zeta(-\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(-\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right]$$

$$= \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi + \alpha)\right]$$

$$\widehat{\lambda}\mu \left[\zeta(-\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi - \beta) - \zeta(-\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi + \beta)\right]$$

$$= \widetilde{\mu}\lambda \left[\zeta(-\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha) - \zeta(-\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi + \alpha)\right]$$

which can be rewritten using the elliptic function addition law:

$$\zeta(x)+\zeta(y)+\zeta(z)-\zeta(x+y+z)=\frac{\sigma(x+y)\,\sigma(y+z)\,\sigma(x+z)}{\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(x+y+z)}\ .$$

We, thus, obtain:

$$\widehat{\lambda}\mu\frac{\sigma(2\widehat{\xi})\,\sigma(\widehat{\widetilde{\xi}}+\xi+\beta-\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)\,\sigma(\widehat{\xi}-\xi-\beta)\,\sigma(\widehat{\xi}+\xi+\beta)}$$

$$=\widetilde{\mu}\lambda\frac{\sigma(2\widetilde{\xi})\,\sigma(\widehat{\widetilde{\xi}}+\xi+\alpha-\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)\,\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}$$

$$\widehat{\lambda}\mu\frac{\sigma(2\widehat{\xi})\,\sigma(\widehat{\widetilde{\xi}}-\xi+\beta-\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)\,\sigma(\widehat{\xi}-\xi+\beta)\,\sigma(\widehat{\xi}+\xi-\beta)}$$

$$=\widetilde{\mu}\lambda\frac{\sigma(2\widetilde{\xi})\,\sigma(\widehat{\widetilde{\xi}}-\xi+\alpha-\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)\,\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}$$

$$\widehat{\lambda}\mu\frac{\sigma(2\widehat{\xi})\,\sigma(\widehat{\widetilde{\xi}}-\xi-\beta+\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}+\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}+\alpha)\,\sigma(\widehat{\xi}-\xi-\beta)\,\sigma(\widehat{\xi}+\xi+\beta)}$$

$$=\widetilde{\mu}\lambda\frac{\sigma(2\widetilde{\xi})\,\sigma(\widehat{\widetilde{\xi}}-\xi-\alpha+\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}+\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}+\beta)\,\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}$$

$$\widehat{\lambda}\mu\frac{\sigma(2\widehat{\xi})\,\sigma(\widehat{\widetilde{\xi}}+\xi-\beta+\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}+\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}+\alpha)\,\sigma(\widehat{\xi}-\xi+\beta)\,\sigma(\widehat{\xi}+\xi-\beta)}$$

$$=\widetilde{\mu}\lambda\frac{\sigma(2\widetilde{\xi})\,\sigma(\widehat{\widetilde{\xi}}+\xi-\alpha+\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}+\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}+\beta)\,\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}\ .$$

3-Leg equations

Eliminating the factors $\widehat{\lambda}\mu/(\widetilde{\mu}\lambda)$ we get only two separate equations:

$$\frac{\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}{\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}\,\frac{\sigma(\widehat{\xi}-\xi-\beta)\,\sigma(\widehat{\xi}+\xi+\beta)}{\sigma(\widehat{\xi}-\xi+\beta)\,\sigma(\widehat{\xi}+\xi-\beta)}=\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\gamma)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\gamma)}$$

and

$$\frac{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}+\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}+\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)}\,\frac{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}+\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}+\beta)}=\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\gamma)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\gamma)}\ .$$

Actually, they are one and the same equation, and lead both to the same rational form for $u=\wp(\xi)$, namely Adler's lattice equation in the Weierstrass form.

This is a consequence of the following elliptic *identity*:

$$\begin{aligned}&(X-\wp(\xi+\alpha))(Y-\wp(\xi-\beta))(Z-\wp(\xi-\alpha+\beta))\\&-t^2(X-\wp(\xi-\alpha))(Y-\wp(\xi+\beta))(Z-\wp(\xi+\alpha-\beta))\\&=s\{A\,[(\wp(\xi)-b)(Y-b)-(a-b)(c-b)]\,[(X-b)(Z-b)-(a-b)(c-b)]\\&\quad+B\,[(\wp(\xi)-a)(X-a)-(b-a)(c-a)]\,[(Y-a)(Z-a)-(b-a)(c-a)]\\&\quad-ABC(a-b)\}\ ,\end{aligned}$$

which holds for arbitrary (complex) variables X, Y, and Z. Here:

$$t=\frac{\sigma(\xi-\alpha)\sigma(\xi+\beta)\sigma(\xi+\alpha-\beta)}{\sigma(\xi+\alpha)\sigma(\xi-\beta)\sigma(\xi-\alpha+\beta)},\quad s=\frac{1-t^2}{(A+B)\wp(\xi)-Ab-aB},$$

and where (a,A), (b,B) and (c,C) are given as before.

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$$\frac{\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}{\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}\,\frac{\sigma(\widehat{\xi}-\xi-\beta)\,\sigma(\widehat{\xi}+\xi+\beta)}{\sigma(\widehat{\xi}-\xi+\beta)\,\sigma(\widehat{\xi}+\xi-\beta)}=\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\gamma)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\gamma)}$$

and

$$\frac{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}+\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}+\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)}\,\frac{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}+\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}+\beta)}=\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\gamma)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\gamma)}\,.$$

Actually, they are one and the same equation, and lead both to the same rational form for $u=\wp(\xi)$, namely Adler's lattice equation in the Weierstrass form.
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$$\begin{aligned}&(X-\wp(\xi+\alpha))(Y-\wp(\xi-\beta))(Z-\wp(\xi-\alpha+\beta))\\&-t^2(X-\wp(\xi-\alpha))(Y-\wp(\xi+\beta))(Z-\wp(\xi+\alpha-\beta))\\&=s\{A\,[(\wp(\xi)-b)(Y-b)-(a-b)(c-b)]\,[(X-b)(Z-b)-(a-b)(c-b)]\\&\qquad+B\,[(\wp(\xi)-a)(X-a)-(b-a)(c-a)]\,[(Y-a)(Z-a)-(b-a)(c-a)]\\&\qquad-ABC(a-b)\}\ ,\end{aligned}$$

which holds for arbitrary (complex) variables X, Y, and Z. Here:

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and where (a,A), (b,B) and (c,C) are given as before.

Rank 3 generalization

In the case $\boxed{N=3}$ the Lax pair

$$\widetilde{\chi} = \begin{pmatrix} h_1\Phi_{3\kappa}(\widetilde{\xi}_1-\xi_1-\alpha) & h_2\Phi_{3\kappa}(\widetilde{\xi}_1-\xi_2-\alpha) & h_3\Phi_{3\kappa}(\widetilde{\xi}_1-\xi_3-\alpha) \\ h_1\Phi_{3\kappa}(\widetilde{\xi}_2-\xi_1-\alpha) & h_2\Phi_{3\kappa}(\widetilde{\xi}_2-\xi_2-\alpha) & h_3\Phi_{3\kappa}(\widetilde{\xi}_2-\xi_3-\alpha) \\ h_1\Phi_{3\kappa}(\widetilde{\xi}_3-\xi_1-\alpha) & h_2\Phi_{3\kappa}(\widetilde{\xi}_3-\xi_2-\alpha) & h_3\Phi_{3\kappa}(\widetilde{\xi}_3-\xi_3-\alpha) \end{pmatrix}\chi\,,$$

$$\widehat{\chi} = \begin{pmatrix} k_1\Phi_{3\kappa}(\widehat{\xi}_1-\xi_1-\beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_1-\xi_2-\beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_1-\xi_3-\beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_2-\xi_1-\beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_2-\xi_2-\beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_2-\xi_3-\beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_3-\xi_1-\beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_3-\xi_2-\beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_3-\xi_3-\beta) \end{pmatrix}\chi\,,$$

subject to $\sum_{i=1}^{3} h_i = \sum_{i=1}^{3} k_i = 0$, and where the coefficients h_j, k_j are some functions of the variables ξ_j, and of their shifts.
The compatibility conditions of this Lax pair results a coupled set of Lax equations in terms of any two of the three variables ξ_j. Eliminating $h_3=-h_1-h_2$ and $k_3=-k_1-k_2$ and using addition formulae we get the system of equations:

$$\sum_{l=1}^{2}\widehat{h}_l k_j \frac{\sigma(\widehat{\widetilde{\xi}}_i-\xi_j-\alpha-\beta)\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\widehat{\xi}_3+\xi_j-\alpha+\beta)\sigma(\widehat{\xi}_l-\widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\alpha)\sigma(\widehat{\xi}_l-\xi_j-\beta)\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_3-\alpha)\sigma(-\widehat{\xi}_3+\xi_j+\beta)} =$$

$$= \sum_{l=1}^{2}\widetilde{k}_l h_j \frac{\sigma(\widehat{\widetilde{\xi}}_i-\xi_j-\alpha-\beta)\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\widetilde{\xi}_3+\xi_j+\alpha-\beta)\sigma(\widetilde{\xi}_l-\widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\beta)\sigma(\widetilde{\xi}_l-\xi_j-\alpha)\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_3-\beta)\sigma(-\widetilde{\xi}_3+\xi_j+\alpha)}$$

$\forall\, i,j = 1,2,3.$

we denote the coefficients on the l.h.s. and r.h.s. of the equation as
$A_{ilj} \equiv A_{ilj}(\widehat{\widetilde{\xi}},\widehat{\xi},\xi;\alpha,\beta)$ and $B_{ilj} \equiv B_{ilj}(\widehat{\widetilde{\xi}},\widetilde{\xi},\xi;\alpha,\beta)$ respectively.

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$$\widehat{\chi} = \begin{pmatrix} k_1\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_3 - \beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_3 - \beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_3 - \beta) \end{pmatrix} \chi\,,$$

subject to $\sum_{i=1}^{3} h_i = \sum_{i=1}^{3} k_i = 0$, and where the coefficients h_j, k_j are some functions of the variables ξ_j, and of their shifts.

The compatibility conditions of this Lax pair results a coupled set of Lax equations in terms of any two of the three variables ξ_j. Eliminating $h_3 = -h_1 - h_2$ and $k_3 = -k_1 - k_2$ and using addition formulae we get the system of equations:

$$\sum_{l=1}^{2} \widehat{h}_l k_j \frac{\sigma(\widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\sigma(\widehat{\xi}_l - \xi_j - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\sigma(-\widehat{\xi}_3 + \xi_j + \beta)} =$$

$$= \sum_{l=1}^{2} \widetilde{k}_l h_j \frac{\sigma(\widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\sigma(\widetilde{\xi}_l - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\sigma(\widetilde{\xi}_l - \xi_j - \alpha)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)\sigma(-\widetilde{\xi}_3 + \xi_j + \alpha)}$$

$\forall\ i,j = 1,2,3.$

we denote the coefficients on the l.h.s. and r.h.s. of the equation as $A_{ilj} \equiv A_{ilj}(\widehat{\widetilde{\xi}}, \widehat{\xi}, \xi; \alpha, \beta)$ and $B_{ilj} \equiv B_{ilj}(\widehat{\widetilde{\xi}}, \widetilde{\xi}, \xi; \alpha, \beta)$ respectively.

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$$\widehat{\chi} = \begin{pmatrix} k_1\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_3 - \beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_3 - \beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_3 - \beta) \end{pmatrix} \chi\,,$$

subject to $\sum_{i=1}^{3} h_i = \sum_{i=1}^{3} k_i = 0$, and where the coefficients h_j, k_j are some functions of the variables ξ_j, and of their shifts.

The compatibility conditions of this Lax pair results a coupled set of Lax equations in terms of any two of the three variables ξ_j. Eliminating $h_3 = -h_1 - h_2$ and $k_3 = -k_1 - k_2$ and using addition formulae we get the system of equations:

$$\sum_{l=1}^{2} \widehat{h}_l k_j \frac{\sigma(\widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\sigma(\widehat{\xi}_l - \xi_j - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\sigma(-\widehat{\xi}_3 + \xi_j + \beta)} =$$

$$= \sum_{l=1}^{2} \widetilde{k}_l h_j \frac{\sigma(\widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\sigma(\widetilde{\xi}_l - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\sigma(\widetilde{\xi}_l - \xi_j - \alpha)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)\sigma(-\widetilde{\xi}_3 + \xi_j + \alpha)}$$

$\forall\ i, j = 1, 2, 3.$

we denote the coefficients on the l.h.s. and r.h.s. of the equation as $A_{ilj} \equiv A_{ilj}(\widehat{\widetilde{\xi}}, \widehat{\xi}, \xi; \alpha, \beta)$ and $B_{ilj} \equiv B_{ilj}(\widehat{\widetilde{\xi}}, \widetilde{\xi}, \xi; \alpha, \beta)$ respectively.

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In the case $\boxed{N=3}$ the Lax pair

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$$\widehat{\chi} = \begin{pmatrix} k_1\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_3 - \beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_3 - \beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_3 - \beta) \end{pmatrix} \chi\ ,$$

subject to $\sum_{i=1}^{3} h_i = \sum_{i=1}^{3} k_i = 0$, and where the coefficients h_j, k_j are some functions of the variables ξ_j, and of their shifts.

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$$\sum_{l=1}^{2} \widehat{h}_l k_j \frac{\sigma(\widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\sigma(\widehat{\xi}_l - \xi_j - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\sigma(-\widehat{\xi}_3 + \xi_j + \beta)} =$$

$$= \sum_{l=1}^{2} \widetilde{k}_l h_j \frac{\sigma(\widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\sigma(\widetilde{\xi}_l - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\sigma(\widetilde{\xi}_l - \xi_j - \alpha)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)\sigma(-\widetilde{\xi}_3 + \xi_j + \alpha)}$$

$\forall\ i, j = 1, 2, 3.$

we denote the coefficients on the l.h.s. and r.h.s. of the equation as $A_{ilj} \equiv A_{ilj}(\widehat{\widetilde{\xi}}, \widehat{\xi}, \xi; \alpha, \beta)$ and $B_{ilj} \equiv B_{ilj}(\widehat{\widetilde{\xi}}, \widetilde{\xi}, \xi; \alpha, \beta)$ respectively.

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In the case $\boxed{N=3}$ the Lax pair

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$$\widehat{\chi} = \begin{pmatrix} k_1\Phi_{3\kappa}(\widehat{\xi}_1-\xi_1-\beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_1-\xi_2-\beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_1-\xi_3-\beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_2-\xi_1-\beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_2-\xi_2-\beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_2-\xi_3-\beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_3-\xi_1-\beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_3-\xi_2-\beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_3-\xi_3-\beta) \end{pmatrix} \chi\ ,$$

subject to $\sum_{i=1}^{3} h_i = \sum_{i=1}^{3} k_i = 0$, and where the coefficients h_j, k_j are some functions of the variables ξ_j, and of their shifts.

The compatibility conditions of this Lax pair results a coupled set of Lax equations in terms of any two of the three variables ξ_j. Eliminating $h_3 = -h_1 - h_2$ and $k_3 = -k_1 - k_2$ and using addition formulae we get the system of equations:

$$\sum_{l=1}^{2} \widehat{h}_l k_j \frac{\sigma(\widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\sigma(\widehat{\xi}_l - \xi_j - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\sigma(-\widehat{\xi}_3 + \xi_j + \beta)} =$$

$$= \sum_{l=1}^{2} \widetilde{k}_l h_j \frac{\sigma(\widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\sigma(\widetilde{\xi}_l - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\sigma(\widetilde{\xi}_l - \xi_j - \alpha)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)\sigma(-\widetilde{\xi}_3 + \xi_j + \alpha)}$$

$$\forall\ i,j = 1,2,3.$$

we denote the coefficients on the l.h.s. and r.h.s. of the equation as

$A_{ilj} \equiv A_{ilj}(\widehat{\widetilde{\xi}}, \widehat{\xi}, \xi; \alpha, \beta)$ and $B_{ilj} \equiv B_{ilj}(\widehat{\widetilde{\xi}}, \widetilde{\xi}, \xi; \alpha, \beta)$ respectively.

Analysis of the compatibility conditions

From the latter conditions we get the following expressions for the common factor h_j/k_j:

$$\frac{h_j}{k_j} = \frac{A_{11j}\widehat{h}_1 + A_{12j}\widehat{h}_2}{B_{11j}\widetilde{k}_1 + B_{12j}\widetilde{k}_2} = \frac{A_{21j}\widehat{h}_1 + A_{22j}\widehat{h}_2}{B_{21j}\widetilde{k}_1 + B_{22j}\widetilde{k}_2} = \frac{A_{31j}\widehat{h}_1 + A_{32j}\widehat{h}_2}{B_{31j}\widetilde{k}_1 + B_{32j}\widetilde{k}_2} \qquad (j = 1, 2, 3) \; .$$

We can rewrite the resulting set of relations as

$$\begin{aligned}
&(A_{11j}B_{21j} - A_{21j}B_{11j})\widehat{h}_1\widetilde{k}_1 + (A_{11j}B_{22j} - A_{21j}B_{12j})\widehat{h}_1\widetilde{k}_2 \\
&\qquad + (A_{12j}B_{21j} - A_{22j}B_{11j})\widehat{h}_2\widetilde{k}_1 + (A_{12j}B_{22j} - A_{22j}B_{12j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&(A_{11j}B_{31j} - A_{31j}B_{11j})\widehat{h}_1\widetilde{k}_1 + (A_{11j}B_{32j} - A_{31j}B_{12j})\widehat{h}_1\widetilde{k}_2 \\
&\qquad + (A_{12j}B_{31j} - A_{32j}B_{11j})\widehat{h}_2\widetilde{k}_1 + (A_{12j}B_{32j} - A_{32j}B_{12j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&(A_{21j}B_{31j} - A_{31j}B_{21j})\widehat{h}_1\widetilde{k}_1 + (A_{21j}B_{32j} - A_{31j}B_{22j})\widehat{h}_1\widetilde{k}_2 \\
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&\qquad (j = 1, 2, 3) \; ,
\end{aligned}$$

where

$$\begin{aligned}
A_{ilj} &= \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)} \; , \\
B_{ilj} &= \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widetilde{\xi}_l - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\,\sigma(\widetilde{\xi}_l - \xi_j - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)\,\sigma(\widetilde{\xi}_3 - \xi_j - \alpha)} \; .
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&(A_{11j}B_{31j} - A_{31j}B_{11j})\widehat{h}_1\widetilde{k}_1 + (A_{11j}B_{32j} - A_{31j}B_{12j})\widehat{h}_1\widetilde{k}_2 \\
&\qquad + (A_{12j}B_{31j} - A_{32j}B_{11j})\widehat{h}_2\widetilde{k}_1 + (A_{12j}B_{32j} - A_{32j}B_{12j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&(A_{21j}B_{31j} - A_{31j}B_{21j})\widehat{h}_1\widetilde{k}_1 + (A_{21j}B_{32j} - A_{31j}B_{22j})\widehat{h}_1\widetilde{k}_2 \\
&\qquad + (A_{22j}B_{31j} - A_{32j}B_{21j})\widehat{h}_2\widetilde{k}_1 + (A_{22j}B_{32j} - A_{32j}B_{22j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&\qquad (j = 1, 2, 3)\ ,
\end{aligned}$$

where

$$\begin{aligned}
A_{ilj} &= \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)}\ , \\
B_{ilj} &= \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widetilde{\xi}_l - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\,\sigma(\widetilde{\xi}_l - \xi_j - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)\,\sigma(\widetilde{\xi}_3 - \xi_j - \alpha)}\ .
\end{aligned}$$

Analysis of the compatibility conditions

From the latter conditions we get the following expressions for the common factor h_j/k_j:

$$\frac{h_j}{k_j} = \frac{A_{11j}\widehat{h}_1 + A_{12j}\widehat{h}_2}{B_{11j}\widetilde{k}_1 + B_{12j}\widetilde{k}_2} = \frac{A_{21j}\widehat{h}_1 + A_{22j}\widehat{h}_2}{B_{21j}\widetilde{k}_1 + B_{22j}\widetilde{k}_2} = \frac{A_{31j}\widehat{h}_1 + A_{32j}\widehat{h}_2}{B_{31j}\widetilde{k}_1 + B_{32j}\widetilde{k}_2} \qquad (j = 1, 2, 3)\ .$$

We can rewrite the resulting set of relations as

$$\begin{aligned}
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&\qquad + (A_{12j}B_{21j} - A_{22j}B_{11j})\widehat{h}_2\widetilde{k}_1 + (A_{12j}B_{22j} - A_{22j}B_{12j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&(A_{11j}B_{31j} - A_{31j}B_{11j})\widehat{h}_1\widetilde{k}_1 + (A_{11j}B_{32j} - A_{31j}B_{12j})\widehat{h}_1\widetilde{k}_2 \\
&\qquad + (A_{12j}B_{31j} - A_{32j}B_{11j})\widehat{h}_2\widetilde{k}_1 + (A_{12j}B_{32j} - A_{32j}B_{12j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&(A_{21j}B_{31j} - A_{31j}B_{21j})\widehat{h}_1\widetilde{k}_1 + (A_{21j}B_{32j} - A_{31j}B_{22j})\widehat{h}_1\widetilde{k}_2 \\
&\qquad + (A_{22j}B_{31j} - A_{32j}B_{21j})\widehat{h}_2\widetilde{k}_1 + (A_{22j}B_{32j} - A_{32j}B_{22j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&\qquad (j = 1, 2, 3)\ ,
\end{aligned}$$

where

$$A_{ilj} = \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)}\ ,$$

$$B_{ilj} = \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widetilde{\xi}_l - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\,\sigma(\widetilde{\xi}_l - \xi_j - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)\,\sigma(\widetilde{\xi}_3 - \xi_j - \alpha)}\ .$$

Analysis of the compatibility conditions

From the latter conditions we get the following expressions for the common factor h_j/k_j:

$$\frac{h_j}{k_j} = \frac{A_{11j}\widehat{h}_1 + A_{12j}\widehat{h}_2}{B_{11j}\widetilde{k}_1 + B_{12j}\widetilde{k}_2} = \frac{A_{21j}\widehat{h}_1 + A_{22j}\widehat{h}_2}{B_{21j}\widetilde{k}_1 + B_{22j}\widetilde{k}_2} = \frac{A_{31j}\widehat{h}_1 + A_{32j}\widehat{h}_2}{B_{31j}\widetilde{k}_1 + B_{32j}\widetilde{k}_2} \qquad (j = 1, 2, 3) \ .$$

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&\qquad + (A_{12j}B_{31j} - A_{32j}B_{11j})\widehat{h}_2\widetilde{k}_1 + (A_{12j}B_{32j} - A_{32j}B_{12j})\widehat{h}_2\widetilde{k}_2 = 0 \\
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where

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$$B_{ilj} = \frac{\sigma(\widetilde{\widetilde{\xi}}_i - \widetilde{\xi}_l - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widetilde{\xi}_l - \widetilde{\xi}_3)}{\sigma(\widetilde{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\,\sigma(\widetilde{\xi}_l - \xi_j - \alpha)\,\sigma(\widetilde{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)\,\sigma(\widetilde{\xi}_3 - \xi_j - \alpha)} \ .$$

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From the latter conditions we get the following expressions for the common factor h_j/k_j:

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\end{aligned}$$

Cayley's hyper-determinant

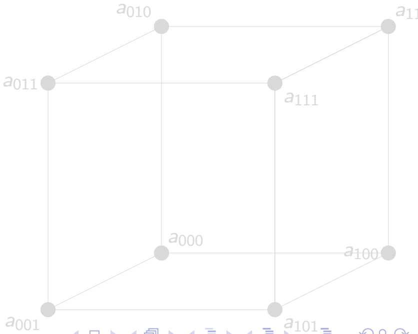
Hyper-determinants were first considered in the 19th century by A. Cayley and L. Schläfli in the *"theory of elimination"*. In the simplest case of format $2 \times 2 \times 2$ cubic matrix, an explicit formula was given by Cayley in 1845.
The hyperdeterminant of hypermatrix $A = (a_{ijk})$ $(i, j, k = 0, 1)$

$$\mathrm{Det}(A) = \left[\det \begin{pmatrix} a_{000} & a_{001} \\ a_{110} & a_{111} \end{pmatrix} + \det \begin{pmatrix} a_{100} & a_{010} \\ a_{101} & a_{011} \end{pmatrix}\right]^2 - 4 \det \begin{pmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{pmatrix} \det \begin{pmatrix} a_{100} & a_{101} \\ a_{110} & a_{111} \end{pmatrix},$$

vanishes identically iff the set of homogeneous bilinear equations in six unknowns

$$\begin{cases} a_{001}x_0y_0 + a_{011}x_0y_1 + a_{101}x_1y_0 + a_{111}x_1y_1 = 0, \\ a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 = 0, \\ a_{010}x_0z_0 + a_{011}x_0z_1 + a_{110}x_1z_0 + a_{111}x_1z_1 = 0, \\ a_{000}x_0z_0 + a_{001}x_0z_1 + a_{100}x_1z_0 + a_{101}x_1z_1 = 0, \\ a_{100}y_0z_0 + a_{101}y_0z_1 + a_{110}y_1z_0 + a_{111}y_1z_1 = 0, \\ a_{000}y_0z_0 + a_{001}x_0z_1 + a_{010}y_1z_0 + a_{011}y_1z_1 = 0, \end{cases}$$

has a non-trivial solution in terms of x_0, x_1, y_0, y_1 and z_0, z_1.



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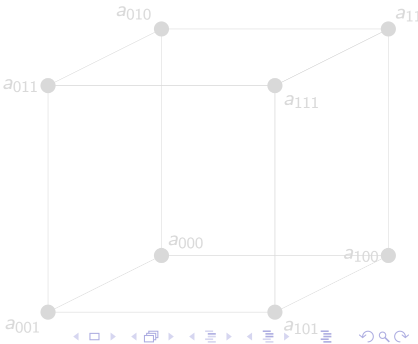

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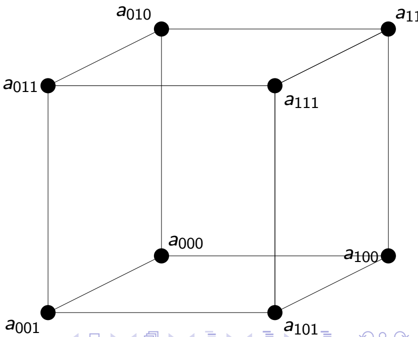

A compound theorem for hyper-determinants

In the case at hand, we have a Cayley type homogeneous system with the variables x_i, y_j identified as the quantities $\widehat{h}_i$ and $\widetilde{k}_j$ respectively and with coefficients a_{ijk} all being 2×2 determinants of the form $\begin{vmatrix} A_{ilj} & A_{i'lj} \\ B_{il'j} & B_{i'l'j} \end{vmatrix}$. Noting the particular structure of this homogeneous system the following *compound theorem for hyper-determinants* is directly applicable.

Lemma (Compound Theorem for $2 \times 2 \times 2$ hyper-determinants)

The following general identity holds for a compound hyper-determinants of format $2 \times 2 \times 2$:

$$\left(\begin{vmatrix} \begin{vmatrix} a & a'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} a' & a'' \\ d' & d'' \end{vmatrix} \\ \\ \begin{vmatrix} c & c'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} c' & c'' \\ d' & d'' \end{vmatrix} \end{vmatrix} + \begin{vmatrix} \begin{vmatrix} a' & a'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} a & a'' \\ d & d'' \end{vmatrix} \\ \\ \begin{vmatrix} c' & c'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} c & c'' \\ d & d'' \end{vmatrix} \end{vmatrix} \right)^2$$

$$-4 \begin{vmatrix} \begin{vmatrix} a & a'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} a & a'' \\ d & d'' \end{vmatrix} \\ \\ \begin{vmatrix} c & c'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} c & c'' \\ d & d'' \end{vmatrix} \end{vmatrix} \cdot \begin{vmatrix} \begin{vmatrix} a' & a'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} a' & a'' \\ d' & d'' \end{vmatrix} \\ \\ \begin{vmatrix} c' & c'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} c' & c'' \\ d' & d'' \end{vmatrix} \end{vmatrix}$$

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A compound theorem for hyper-determinants

In the case at hand, we have a Cayley type homogeneous system with the variables x_i, y_j identified as the quantities $\widehat{h}_i$ and $\widetilde{k}_j$ respectively and with coefficients a_{ijk} all being 2×2 determinants of the form $\begin{vmatrix} A_{ilj} & A_{i'lj} \\ B_{il'j} & B_{i'l'j} \end{vmatrix}$. Noting the particular structure of this homogeneous system the following *compound theorem for hyper-determinants* is directly applicable.

Lemma (Compound Theorem for $2 \times 2 \times 2$ hyper-determinants)

The following general identity holds for a compound hyper-determinants of format $2 \times 2 \times 2$*:*

$$\left(\begin{vmatrix} \begin{vmatrix} a & a'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} a' & a'' \\ d' & d'' \end{vmatrix} \\ \begin{vmatrix} c & c'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} c' & c'' \\ d' & d'' \end{vmatrix} \end{vmatrix} + \begin{vmatrix} \begin{vmatrix} a' & a'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} a & a'' \\ d & d'' \end{vmatrix} \\ \begin{vmatrix} c' & c'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} c & c'' \\ d & d'' \end{vmatrix} \end{vmatrix} \right)^2$$

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$$= \begin{vmatrix} \begin{vmatrix} a & a'' \\ c & c'' \end{vmatrix} & \begin{vmatrix} b & b'' \\ d & d'' \end{vmatrix} \\ \begin{vmatrix} a' & a'' \\ c' & c'' \end{vmatrix} & \begin{vmatrix} b' & b'' \\ d' & d'' \end{vmatrix} \end{vmatrix}^2 .$$

Resolution of the $N = 3$ system

Identifying the coefficients of the system of homogeneous Lax equations as entries of a $2 \times 2 \times 2$ hyper-determinant, we observe that the structure of this hyper-determinant is exactly of the form as given in the Lemma, and hence we have the following immediate corollary.

Proposition

Identifying the 8 entries $(a_{ijk})_{i,j,k=0,1}$ the hyper-determinant for the Lax system takes the form as given by the compound theorem, and hence reduces to a perfect square:

$$\left| \begin{array}{cc} \left| \begin{array}{cc} A_{ilj} & A_{il'j} \\ A_{i''lj} & A_{i''l'j} \end{array} \right| & \left| \begin{array}{cc} A_{i'lj} & A_{i'l'j} \\ A_{i''lj} & A_{i''l'j} \end{array} \right| \\ & \\ \left| \begin{array}{cc} B_{ilj} & B_{il'j} \\ B_{i''lj} & B_{i''l'j} \end{array} \right| & \left| \begin{array}{cc} B_{i'lj} & B_{i'l'j} \\ B_{i''lj} & B_{i''l'j} \end{array} \right| \end{array} \right|^2 \qquad (j = 1, 2, 3),$$

where

$$\left| \begin{array}{cc} A_{ilj} & A_{il'j} \\ A_{i''lj} & A_{i''l'j} \end{array} \right| = \frac{\sigma(\widehat{\xi}_l - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_{l'} - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_{l'})}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_{l'} - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_{l'} - \alpha)}$$
$$\times \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\widetilde{\xi}}_{i''})\,\sigma(\widehat{\widetilde{\xi}}_i + \widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \widehat{\xi}_{l'} - \widehat{\xi}_3 + \xi_j - 2\alpha + \beta)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\xi}_{l'} - \xi_j - \beta)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)},$$

in which we can set $i, i' = 1, 2$, $l, l' = 1, 2 \neq 3$, and where we naturally should take $i'' = 3$. A similar expression for the corresponding determinants in terms of the B_{ilj} is obtained by interchanging α and β and the shifts $\widetilde{\ }$ and $\widehat{\ }$.

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where

$$\begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{i''lj} & A_{i''l'j} \end{vmatrix} = \frac{\sigma(\widehat{\xi}_l - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_{l'} - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_{l'})}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_{l'} - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_{l'} - \alpha)}$$
$$\times \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\widetilde{\xi}}_{i''})\,\sigma(\widehat{\widetilde{\xi}}_i + \widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \widehat{\xi}_{l'} - \widehat{\xi}_3 + \xi_j - 2\alpha + \beta)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\xi}_{l'} - \xi_j - \beta)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)},$$

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As the hyper-determinant in the case at hand is a perfect square, the homogeneous Lax system leads to a simple quadratic equation for either the ratios $\widehat{h}_i/\widehat{h}_j$ or $\widetilde{k}_i/\widetilde{k}_j$, $(i,j=1,2)$. Thus, their solutions lead to rational expressions in terms of the quantities A_{ilj} and B_{ilj}. The result of this computation is the following:

Proposition

If the hyperdeterminant of the system is non-vanishing, we have the following two solutions of the system of homogeneous equations, in terms of the ratios:

$$\text{i)}\qquad \frac{\widehat{h}_1}{\widehat{h}_2}=-\frac{A_{32j}}{A_{31j}}\quad\text{with}\quad \frac{\widetilde{k}_1}{\widetilde{k}_2}=-\frac{B_{32j}}{B_{31j}}\ ,$$

$$\text{ii)}\qquad \frac{\widehat{h}_1}{\widehat{h}_2}=-\frac{\begin{vmatrix} B_{11j} & A_{12j} & B_{12j}\\ B_{21j} & A_{22j} & B_{22j}\\ B_{31j} & A_{32j} & B_{32j}\end{vmatrix}}{\begin{vmatrix} B_{11j} & A_{11j} & B_{12j}\\ B_{21j} & A_{21j} & B_{22j}\\ B_{31j} & A_{31j} & B_{32j}\end{vmatrix}}\quad\text{with}\quad \frac{\widetilde{k}_1}{\widetilde{k}_2}=-\frac{\begin{vmatrix} A_{11j} & A_{12j} & B_{12j}\\ A_{21j} & A_{22j} & B_{22j}\\ A_{31j} & A_{32j} & B_{32j}\end{vmatrix}}{\begin{vmatrix} A_{11j} & A_{12j} & B_{11j}\\ A_{21j} & A_{22j} & B_{21j}\\ A_{31j} & A_{32j} & B_{31j}\end{vmatrix}}\ .$$

$(j=1,2,3)$

Remark: The system of equations resulting from solution i) reads as follows

$$\frac{\widehat{h}_1}{\widehat{h}_2}=-\frac{\sigma(\widehat{\widehat{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_j-\alpha+\beta)\,\sigma(\widehat{\widehat{\xi}}_3-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\xi}_1-\xi_j-\beta)\,\sigma(\widehat{\xi}_2-\widehat{\xi}_3)}{\sigma(\widehat{\widehat{\xi}}_3-\widehat{\xi}_1-\widehat{\xi}_3+\xi_j-\alpha+\beta)\,\sigma(\widehat{\widehat{\xi}}_3-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\xi}_2-\xi_j-\beta)\,\sigma(\widehat{\xi}_1-\widehat{\xi}_3)},$$

$$\frac{\widetilde{k}_1}{\widetilde{k}_2}=-\frac{\sigma(\widetilde{\widetilde{\xi}}_3-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_j+\alpha-\beta)\,\sigma(\widetilde{\widetilde{\xi}}_3-\widetilde{\xi}_1-\beta)\,\sigma(\widetilde{\xi}_1-\xi_j-\alpha)\,\sigma(\widetilde{\xi}_2-\widetilde{\xi}_3)}{\sigma(\widetilde{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_3+\xi_j+\alpha-\beta)\,\sigma(\widetilde{\widetilde{\xi}}_3-\widetilde{\xi}_2-\beta)\,\sigma(\widetilde{\xi}_2-\xi_j-\alpha)\,\sigma(\widetilde{\xi}_1-\widetilde{\xi}_3)},$$

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$$\frac{\widetilde{k}_1}{\widetilde{k}_2} = -\frac{\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_2 - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_1 - \beta)\,\sigma(\widetilde{\xi}_1 - \xi_j - \alpha)\,\sigma(\widetilde{\xi}_2 - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_1 - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_2 - \beta)\,\sigma(\widetilde{\xi}_2 - \xi_j - \alpha)\,\sigma(\widetilde{\xi}_1 - \widetilde{\xi}_3)},$$

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$(j=1,2,3)$.

The elliptic rank 3 system

We postulate the coupled system of (implicit) lattice equations for ξ_1, ξ_2, ξ_3 given by:

$$\frac{\begin{vmatrix} B_{111} & A_{121} & B_{121} \\ B_{211} & A_{221} & B_{221} \\ B_{311} & A_{321} & B_{321} \end{vmatrix}}{\begin{vmatrix} B_{111} & A_{111} & B_{121} \\ B_{211} & A_{211} & B_{221} \\ B_{311} & A_{311} & B_{321} \end{vmatrix}} = \frac{\begin{vmatrix} B_{112} & A_{122} & B_{122} \\ B_{212} & A_{222} & B_{222} \\ B_{312} & A_{322} & B_{322} \end{vmatrix}}{\begin{vmatrix} B_{112} & A_{112} & B_{122} \\ B_{212} & A_{212} & B_{222} \\ B_{312} & A_{312} & B_{322} \end{vmatrix}} = \frac{\begin{vmatrix} B_{113} & A_{123} & B_{123} \\ B_{213} & A_{223} & B_{223} \\ B_{313} & A_{323} & B_{323} \end{vmatrix}}{\begin{vmatrix} B_{113} & A_{113} & B_{123} \\ B_{213} & A_{213} & B_{223} \\ B_{313} & A_{313} & B_{323} \end{vmatrix}} ,$$

$$\frac{\begin{vmatrix} A_{111} & A_{121} & B_{121} \\ A_{211} & A_{221} & B_{221} \\ A_{311} & A_{321} & B_{321} \end{vmatrix}}{\begin{vmatrix} A_{111} & A_{121} & B_{111} \\ A_{211} & A_{221} & B_{211} \\ A_{311} & A_{321} & B_{311} \end{vmatrix}} = \frac{\begin{vmatrix} A_{112} & A_{122} & B_{122} \\ A_{212} & A_{222} & B_{222} \\ A_{312} & A_{322} & B_{322} \end{vmatrix}}{\begin{vmatrix} A_{112} & A_{122} & B_{112} \\ A_{212} & A_{222} & B_{212} \\ A_{312} & A_{322} & B_{312} \end{vmatrix}} = \frac{\begin{vmatrix} A_{113} & A_{123} & B_{123} \\ A_{213} & A_{223} & B_{223} \\ A_{313} & A_{323} & B_{323} \end{vmatrix}}{\begin{vmatrix} A_{113} & A_{123} & B_{113} \\ A_{213} & A_{223} & B_{213} \\ A_{313} & A_{323} & B_{313} \end{vmatrix}} ,$$

with the determinants expanded by means of the formulae:

$$A_{ilj} = \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)} ,$$

and

$$\begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{klj} & A_{kl'j} \end{vmatrix} = \frac{\sigma(\widehat{\xi}_l - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_{l'} - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_{l'})}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_{l'} - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_k - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widetilde{\xi}}_k - \widehat{\xi}_{l'} - \alpha)}$$
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The elliptic rank 3 system

We postulate the coupled system of (implicit) lattice equations for ξ_1, ξ_2, ξ_3 given by:

$$\frac{\begin{vmatrix} B_{111} & A_{121} & B_{121} \\ B_{211} & A_{221} & B_{221} \\ B_{311} & A_{321} & B_{321} \end{vmatrix}}{\begin{vmatrix} B_{111} & A_{111} & B_{121} \\ B_{211} & A_{211} & B_{221} \\ B_{311} & A_{311} & B_{321} \end{vmatrix}} = \frac{\begin{vmatrix} B_{112} & A_{122} & B_{122} \\ B_{212} & A_{222} & B_{222} \\ B_{312} & A_{322} & B_{322} \end{vmatrix}}{\begin{vmatrix} B_{112} & A_{112} & B_{122} \\ B_{212} & A_{212} & B_{222} \\ B_{312} & A_{312} & B_{322} \end{vmatrix}} = \frac{\begin{vmatrix} B_{113} & A_{123} & B_{123} \\ B_{213} & A_{223} & B_{223} \\ B_{313} & A_{323} & B_{323} \end{vmatrix}}{\begin{vmatrix} B_{113} & A_{113} & B_{123} \\ B_{213} & A_{213} & B_{223} \\ B_{313} & A_{313} & B_{323} \end{vmatrix}} ,$$

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$$A_{ilj} = \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)} ,$$

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Reductions

General question: how to obtain reductions of integrable lattice equations (PΔEs)?
Lattice systems typically admit several types of reductions, e.g.:

1. Periodic reductions (stationary solutions);
2. Non-autonomous scaling-type reductions (often yielding discrete Painlevé equations).

So far very little work exists for doing this for elliptic lattice systems!
The simplest periodic reduction of the elliptic lattice system is the 1-step period one obtained by imposing

$$\widetilde{\boldsymbol{\chi}}_\kappa = \lambda \boldsymbol{\chi}_\kappa \ ,$$

for which we get an isospectral problem of the form

$$\mathsf{L}_\kappa \boldsymbol{\chi}_\kappa = \lambda \boldsymbol{\chi}_\kappa \ , \quad \widehat{\boldsymbol{\chi}}_\kappa = \mathsf{M}_\kappa \boldsymbol{\chi}_\kappa \ ,$$

and this is precisely the Lax pair for the *discrete-time Ruijsenaars model*[7].
The corresponding non-autonomous analogue is obtained by *de-autonomization*, i.e. the replacement

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i.e. by going over to a non-isospectral problem which in the elliptic case corresponds to a linear difference equation on the torus and the corresponding discrete *isomonodromic deformations*[8].

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Isomonodromic deformation problem

We now generalize the elliptic lattice Lax system to

$$\widetilde{\chi}_\kappa = \mathbf{L}_\kappa\, \chi_\kappa \ , \quad \widehat{\chi}_\kappa = \mathbf{M}_\kappa\, \chi_\kappa \ ,$$

and supplement it with a difference equation on the torus:

$$\chi_{\kappa+\tau} = \mathbf{T}_\kappa\, \chi_\kappa \ ,$$

with Lax matrices

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(\mathbf{L}_\kappa)_{i,j} &= H_{i,j}\, \sigma(\kappa)\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha) \ , \\
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& \qquad\qquad\qquad (i,j = 1, \dots, N)
\end{aligned}$$

in which $H_{i,j}$, $K_{i,j}$ and $S_{i,j}$ *do not depend on* κ and remain to be determined. As it turn out γ, and perhaps α and β, will depend on the discrete variables n, m, while $\xi_i = \xi(n, m)$ are the main independent variables.
In addition to the usual compatibility on the lattice:

$$\widehat{\mathbf{L}}_\kappa\, \mathbf{M}_\kappa = \widetilde{\mathbf{M}}_\kappa\, \mathbf{L}_\kappa \ ,$$

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We will now analyse the first of these latter two Lax equations.

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with Lax matrices

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in which $H_{i,j}$, $K_{i,j}$ and $S_{i,j}$ *do not depend on* κ and remain to be determined. As it turn out γ, and perhaps α and β, will depend on the discrete variables n, m, while $\xi_i = \xi(n,m)$ are the main independent variables.
In addition to the usual compatibility on the lattice:

$$\widehat{\mathbf{L}}_\kappa \, \mathbf{M}_\kappa = \widetilde{\mathbf{M}}_\kappa \, \mathbf{L}_\kappa \ ,$$

we have also:

$$\mathbf{L}_{\kappa+\tau} \, \mathbf{T}_\kappa = \widetilde{\mathbf{T}}_\kappa \, \mathbf{L}_\kappa \ , \quad \mathbf{M}_{\kappa+\tau} \, \mathbf{T}_\kappa = \widehat{\mathbf{T}}_\kappa \, \mathbf{M}_\kappa \ .$$

We will now analyse the first of these latter two Lax equations.

Compatibility

Let us now consider first the compatibility $\mathbf{L}_{\kappa+\tau}\,\mathbf{T}_\kappa = \widetilde{\mathbf{T}}_\kappa\,\mathbf{L}_\kappa$. We use the addition formula

$$\Phi_\kappa(x)\Phi_\lambda(y) = \Phi_\kappa(x-y)\;\Phi_{\kappa+\lambda}(y) + \Phi_{\kappa+\lambda}(x)\;\Phi_\lambda(y-x)\,.$$

and also using the properties : $\Phi_\kappa(\tau)\Phi_{\kappa+\tau}(x) = \Phi_\kappa(x+\tau)\Phi_\tau(x)$, and $\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y)\left[\zeta(\kappa)+\zeta(x)+\zeta(y)-\zeta(\kappa+x+y)\right]$. We compute:

$$\sigma(\tau)\Phi_\kappa(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\,\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_l-\alpha)\,\Phi_\kappa(\xi_l-\xi_j-\gamma)$$

$$=\sigma(\tau)\Phi_\kappa(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\left[\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Phi_{-\tau}(\xi_l-\xi_j-\gamma)\right.$$

$$\left.+\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\right]$$

$$=\sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})\Phi_\kappa(\widetilde{\xi}_l-\xi_j-\alpha),$$

which subsequently yields:

$$\sigma(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\Big[\Phi_\kappa(\tau)\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Phi_{-\tau}(\xi_l-\xi_j-\gamma)+$$

$$+\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)\big(\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)+$$

$$+\zeta(\tau)+\zeta(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\big)\Big]=$$

$$=\sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})\big[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})+$$

$$\forall i,j=1,\ldots,N. \qquad +\zeta(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\big]$$

Compatibility

Let us now consider first the compatibility $\mathbf{L}_{\kappa+\tau}\,\mathbf{T}_\kappa = \widetilde{\mathbf{T}}_\kappa\,\mathbf{L}_\kappa$. We use the addition formula

$$\Phi_\kappa(x)\Phi_\lambda(y) = \Phi_\kappa(x-y)\ \Phi_{\kappa+\lambda}(y) + \Phi_{\kappa+\lambda}(x)\ \Phi_\lambda(y-x)\,.$$

and also using the properties : $\Phi_\kappa(\tau)\Phi_{\kappa+\tau}(x) = \Phi_\kappa(x+\tau)\Phi_\tau(x)$, and $\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y)\left[\zeta(\kappa)+\zeta(x)+\zeta(y)-\zeta(\kappa+x+y)\right]$. We compute:

$$\sigma(\tau)\Phi_\kappa(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\,\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_l-\alpha)\,\Phi_\kappa(\xi_l-\xi_j-\gamma)$$

$$=\sigma(\tau)\Phi_\kappa(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\,\big[\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Phi_{-\tau}(\xi_l-\xi_j-\gamma)$$

$$+\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\big]$$

$$=\sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})\Phi_\kappa(\widetilde{\xi}_l-\xi_j-\alpha),$$

which subsequently yields:

$$\sigma(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\,\Big[\Phi_\kappa(\tau)\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Phi_{-\tau}(\xi_l-\xi_j-\gamma)+$$

$$+\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)\big(\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)+$$

$$+\zeta(\tau)+\zeta(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\big)\Big]=$$

$$=\sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})\big[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})+$$

$$\forall i,j=1,\ldots,N. \qquad +\zeta(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\big]$$

Compatibility

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$$\sigma(\tau)\Phi_\kappa(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\,\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_l-\alpha)\,\Phi_\kappa(\xi_l-\xi_j-\gamma)$$

$$=\sigma(\tau)\Phi_\kappa(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\,\big[\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Phi_{-\tau}(\xi_l-\xi_j-\gamma)$$

$$+\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\big]$$

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$$+\zeta(\tau)+\zeta(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\big)\Big]=$$

$$=\sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})\big[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})+$$

$$\forall i,j=1,\dots,N. \qquad +\zeta(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\big]$$

Compatibility

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$$=\sigma(\tau)\Phi_\kappa(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\,\Big[\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Phi_{-\tau}(\xi_l-\xi_j-\gamma)$$

$$+\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Big]$$

$$=\sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})\Phi_\kappa(\widetilde{\xi}_l-\xi_j-\alpha),$$

which subsequently yields:

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$$+\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)\big(\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)+$$

$$+\zeta(\tau)+\zeta(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\big)\Big]=$$

$$=\sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})\big[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})+$$

$$\forall i,j=1,\ldots,N. \qquad +\zeta(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\big]$$

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$$\sigma(\tau)\Phi_\kappa(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\,\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_l-\alpha)\,\Phi_\kappa(\xi_l-\xi_j-\gamma)$$

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$$\left. +\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\right]$$

$$= \sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})\Phi_\kappa(\widetilde{\xi}_l-\xi_j-\alpha),$$

which subsequently yields:

$$\sigma(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\Big[\Phi_\kappa(\tau)\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Phi_{-\tau}(\xi_l-\xi_j-\gamma)+$$

$$+\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)\big(\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)+$$

$$+\zeta(\tau)+\zeta(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\big)\Big] =$$

$$= \sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})\left[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})+\right.$$

$$\forall i,j=1,\ldots,N. \qquad \left. +\,\zeta(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\right]$$

Thus, we end up with the form:

$$\begin{aligned}
&\sigma(\tau)\sum_{l=1}^{N} H_{il}\,S_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\\
&\times\Big[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)+\zeta(\widetilde{\xi}_i-\xi_l+\tau-\alpha)-\zeta(\xi_j-\xi_l+\gamma)\Big]\\
=\quad&\sum_{l=1}^{N}\widetilde{S}_{il}\,H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})\big[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})+\\
&(\forall i,j=1,\ldots,N).\qquad\qquad +\zeta(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\big]
\end{aligned}$$

Thus, by setting $\widetilde{\gamma}=\gamma-\tau$, the equations can be separated into a part depending on the spectral parameter κ, and the remainder independent of κ. This leads to the relations:

- $\sum_{l=1}^{N}\widetilde{S}_{il}H_{lj}=\sum_{l=1}^{N}H_{il}S_{lj}\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\,\sigma(\tau)$,

- $\sum_{l=1}^{N}\widetilde{S}_{il}H_{lj}\,\sigma(-\tau)\,\Phi_{-\tau}(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})\,\Phi_{-\tau}(\widetilde{\xi}_l-\xi_j-\alpha)=\sum_{l=1}^{N}H_{il}S_{lj}\,\Phi_{-\tau}(\xi_l-\xi_j-\gamma)$,

for all $i,j=1,\ldots,N$.

Thus, we end up with the form:

$$
\begin{aligned}
& \sigma(\tau) \sum_{l=1}^{N} H_{il}\, S_{lj}\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau) \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha) \\
& \times \Big[\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau) + \zeta(\widetilde{\xi}_i - \xi_l + \tau - \alpha) - \zeta(\xi_j - \xi_l + \gamma) \Big] \\
= \quad & \sum_{l=1}^{N} \widetilde{S}_{il}\, H_{lj}\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) \big[\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) + \\
& (\forall i, j = 1, \ldots, N). \qquad\qquad + \zeta(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma}) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) \big]
\end{aligned}
$$

Thus, by setting $\widetilde{\gamma} = \gamma - \tau$, the equations can be separated into a part depending on the spectral parameter κ, and the remainder independent of κ. This leads to the relations:

- $\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj} = \sum_{l=1}^{N} H_{il} S_{lj} \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \sigma(\tau)$,
- $\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj}\, \sigma(-\tau)\, \Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})\, \Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha) = \sum_{l=1}^{N} H_{il} S_{lj}\, \Phi_{-\tau}(\xi_l - \xi_j - \gamma)$,

for all $i, j = 1, \ldots, N$.

Thus, we end up with the form:

$$\begin{aligned}
&\sigma(\tau)\sum_{l=1}^{N} H_{il}\, S_{lj}\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha) \\
&\times\Big[\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau) + \zeta(\widetilde{\xi}_i - \xi_l + \tau - \alpha) - \zeta(\xi_j - \xi_l + \gamma)\Big] \\
= \quad &\sum_{l=1}^{N} \widetilde{S}_{il}\, H_{lj}\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma})\big[\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) + \\
&(\forall i, j = 1, \ldots, N). \qquad\qquad + \zeta(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma}) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha)\big]
\end{aligned}$$

Thus, by setting $\widetilde{\gamma} = \gamma - \tau$, the equations can be separated into a part depending on the spectral parameter κ, and the remainder independent of κ. This leads to the relations:

- $\displaystyle\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj} = \sum_{l=1}^{N} H_{il} S_{lj} \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \sigma(\tau)$,

- $\displaystyle\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj}\, \sigma(-\tau)\, \Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})\, \Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha) = \sum_{l=1}^{N} H_{il} S_{lj}\, \Phi_{-\tau}(\xi_l - \xi_j - \gamma),$

for all $i, j = 1, \ldots, N$.

Case $N = 1$

This is the simplest case which can be explicitly solved. In this case all quantities $H_{i,j}$, $S_{i,j}$ are scalars, leading to the system of equations:

$$\widetilde{S}_{11} H_{11} = H_{11} S_{11} \Phi_\tau(\widetilde{\xi} - \xi - \alpha)\, \sigma(\tau) \ ,$$
$$\widetilde{S}_{11} H_{11}\, \sigma(-\tau)\, \Phi_{-\tau}(\widetilde{\xi} - \xi - \alpha) \Phi_{-\tau}(\widetilde{\xi} - \widetilde{\xi} - \widetilde{\gamma}) = H_{11} S_{11}\, \Phi_{-\tau}(\xi - \xi - \gamma).$$

Eliminating $\widetilde{S}_{11}$, S_{11} and H_{11} (simply by dividing the relations over each other) and using the definition of $\Phi_{\pm\tau}(x)$, as well as $\widetilde{\gamma} = \gamma - \tau$, we obtain:

$$\frac{\sigma(\gamma + \tau)\sigma(\gamma - \tau)}{\sigma^2(\gamma)} = \frac{\sigma(\widetilde{\xi} - \xi - \alpha + \tau)\sigma(\widetilde{\xi} - \xi - \alpha - \tau)}{\sigma^2(\widetilde{\xi} - \xi - \alpha)} \ .$$

Rearranging by using the addition formula

$$\frac{\sigma(x + y)\, \sigma(x - y)}{\sigma^2(x)\, \sigma^2(y)} = \wp(y) - \wp(x) \ ,$$

we find that

$$\wp(\widetilde{\xi} - \xi - \alpha) \ = \ \wp(\gamma) \ ,$$

which gives a first order difference equation for $\xi_1 =: \xi(n)$, namely

$$\widetilde{\xi} - \xi - \alpha = \pm\gamma (\mathrm{mod} \ \ \mathrm{period} \ \ \mathrm{lattice}) \ .$$

Integrating the latter, using $\gamma = \gamma_0 - n\tau$ we get

$$\xi(n) = \xi(0) + (\alpha \pm \gamma_0) n \pm \tfrac{1}{2} n(n-1)\tau \ .$$

This indicates that in the simplest case the scheme gives rise to functions obeying the rational version of the equations that are elliptic functions with arguments depending *quadratically* on the discrete independent variable n.

Higher N values

The general system was given as:

- $$\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj} = \sum_{l=1}^{N} H_{il} S_{lj} \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \sigma(\tau)\ ,$$

- $$\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj}\, \sigma(-\tau)\, \Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})\, \Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha) = \sum_{l=1}^{N} H_{il} S_{lj}\, \Phi_{-\tau}(\xi_l - \xi_j - \gamma),$$

for all $i, j = 1, \ldots, N$. As in the autonomous case we want to eliminate the variables H_{ij}, S_{ij} and obtain a closed form system of equations for the dependent variables $\xi_i =: \xi_i(n)$. To write the system more concisely we introduce matrices:

$$A^{\pm}_{ij} := \sigma(\pm\tau)\Phi_{\pm\tau}(\widetilde{\xi}_i - \xi_j - \alpha)\ , \quad \Gamma^{\pm}_{ij} := \sigma(\pm\tau)\Phi_{\pm\tau}(\xi_i - \xi_j - \gamma)\ ,$$

and the operation of *"glueing"* matrices: for any two matrices $\mathbf{A} = (A_{i,j})$, $\mathbf{B} = (B_{i,j})$ we introduce the *glued matrix* $[\mathbf{AB}]$, given by:

$$([\mathbf{AB}])_{i,j} := A_{i,j} B_{i,j}\ ,$$

In terms of this notation the above system takes the simple matrix form:

- $\widetilde{\mathbf{S}} \cdot \mathbf{H} = [\mathbf{A}^+\mathbf{H}] \cdot \mathbf{S}$,
- $[\widetilde{\boldsymbol{\Gamma}}^- \ \widetilde{\mathbf{S}}] \cdot [\mathbf{A}^-\mathbf{H}] = \mathbf{H} \cdot [\boldsymbol{\Gamma}^-\mathbf{S}]\ .$

We want the matrix $\mathbf{H}$ to be of rank 1 (as in the autonomous case) it follows from the first eq. that $[\mathbf{A}^+\mathbf{H}]$ is of rank 1, since $[\mathbf{A}^-\mathbf{H}]$ is generically not of rank 1, and the second eq. then implies that $[\boldsymbol{\Gamma}^-\mathbf{S}]$ is of rank 1 (and not $\mathbf{S}$ itself!), implying:

$$\det(\mathbf{A}^+) = \det(\Phi_\tau(\widetilde{\xi}_i - \xi_j - \alpha)_{i,j=1,\ldots,N} = 0 \quad \Rightarrow \quad \tau + \widetilde{\Xi} - \Xi - N\alpha = 0\ .$$

for $\Xi := \sum_{i=1}^{N} \xi_i$. (This follows from Frobenius' elliptic Cauchy determinant)

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The general system was given as:

- $$\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj} = \sum_{l=1}^{N} H_{il} S_{lj} \Phi_{\tau}(\widetilde{\xi}_i - \xi_l - \alpha)\, \sigma(\tau) \ ,$$

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for all $i, j = 1, \dots, N$. As in the autonomous case we want to eliminate the variables H_{ij}, S_{ij} and obtain a closed form system of equations for the dependent variables $\xi_i =: \xi_i(n)$. To write the system more concisely we introduce matrices:

$$A^{\pm}_{ij} := \sigma(\pm\tau)\Phi_{\pm\tau}(\widetilde{\xi}_i - \xi_j - \alpha) \ , \quad \Gamma^{\pm}_{ij} := \sigma(\pm\tau)\Phi_{\pm\tau}(\xi_i - \xi_j - \gamma) \ ,$$

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We want the matrix $\mathbf{H}$ to be of rank 1 (as in the autonomous case) it follows from the first eq. that $[\mathbf{A}^{+}\mathbf{H}]$ is of rank 1, since $[\mathbf{A}^{-}\mathbf{H}]$ is generically not of rank 1, and the second eq. then implies that $[\mathbf{\Gamma}^{-}\mathbf{S}]$ is of rank 1 (and not $\mathbf{S}$ itself!), implying:

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The general system was given as:

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for all $i, j = 1, \dots, N$. As in the autonomous case we want to eliminate the variables H_{ij}, S_{ij} and obtain a closed form system of equations for the dependent variables $\xi_i =: \xi_i(n)$. To write the system more concisely we introduce matrices:

$$A^{\pm}_{ij} := \sigma(\pm\tau)\Phi_{\pm\tau}(\widetilde{\xi}_i - \xi_j - \alpha)\ , \quad \Gamma^{\pm}_{ij} := \sigma(\pm\tau)\Phi_{\pm\tau}(\xi_i - \xi_j - \gamma)\ ,$$

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- $[\widetilde{\mathbf{\Gamma}}^-\ \widetilde{\mathbf{S}}] \cdot [\mathbf{A}^-\mathbf{H}] = \mathbf{H} \cdot [\mathbf{\Gamma}^-\mathbf{S}]\ .$

We want the matrix $\mathbf{H}$ to be of rank 1 (as in the autonomous case) it follows from the first eq. that $[\mathbf{A}^+\mathbf{H}]$ is of rank 1, since $[\mathbf{A}^-\mathbf{H}]$ is generically not of rank 1, and the second eq. then implies that $[\mathbf{\Gamma}^-\mathbf{S}]$ is of rank 1 (and not $\mathbf{S}$ itself!), implying:

$$\det(\mathbf{A}^+) = \det(\Phi_\tau(\widetilde{\xi}_i - \xi_j - \alpha)_{i,j=1,\dots,N} = 0 \quad \Rightarrow \quad \tau + \widetilde{\Xi} - \Xi - N\alpha = 0\ .$$

for $\Xi := \sum_{i=1}^{N} \xi_i$. (This follows from Frobenius' elliptic Cauchy determinant.)

Higher N values

The general system was given as:

- $$\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj} = \sum_{l=1}^{N} H_{il} S_{lj} \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \sigma(\tau)\ ,$$

- $$\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj}\, \sigma(-\tau)\, \Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})\, \Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha) = \sum_{l=1}^{N} H_{il} S_{lj}\, \Phi_{-\tau}(\xi_l - \xi_j - \gamma),$$

for all $i, j = 1, \ldots, N$. As in the autonomous case we want to eliminate the variables H_{ij}, S_{ij} and obtain a closed form system of equations for the dependent variables $\xi_i =: \xi_i(n)$. To write the system more concisely we introduce matrices:

$$A^{\pm}_{ij} := \sigma(\pm\tau)\Phi_{\pm\tau}(\widetilde{\xi}_i - \xi_j - \alpha)\ , \quad \Gamma^{\pm}_{ij} := \sigma(\pm\tau)\Phi_{\pm\tau}(\xi_i - \xi_j - \gamma)\ ,$$

and the operation of *"glueing"* matrices: for any two matrices $\mathbf{A} = (A_{i,j})$, $\mathbf{B} = (B_{i,j})$ we introduce the *glued matrix* $[\mathbf{AB}]$, given by:

$$([\mathbf{AB}])_{i,j} := A_{i,j} B_{i,j}\ ,$$

In terms of this notation the above system takes the simple matrix form:

- $\widetilde{\mathbf{S}} \cdot \mathbf{H} = [\mathbf{A}^+ \mathbf{H}] \cdot \mathbf{S}$,
- $[\widetilde{\mathbf{\Gamma}}^- \ \widetilde{\mathbf{S}}] \cdot [\mathbf{A}^- \mathbf{H}] = \mathbf{H} \cdot [\mathbf{\Gamma}^- \mathbf{S}]$.

We want the matrix $\mathbf{H}$ to be of rank 1 (as in the autonomous case) it follows from the first eq. that $[\mathbf{A}^+ \mathbf{H}]$ is of rank 1, since $[\mathbf{A}^- \mathbf{H}]$ is generically not of rank 1, and the second eq. then implies that $[\mathbf{\Gamma}^- \mathbf{S}]$ is of rank 1 (and not $\mathbf{S}$ itself!), implying:

$$\det(\mathbf{A}^+) = \det(\Phi_\tau(\widetilde{\xi}_i - \xi_j - \alpha)_{i,j=1,\ldots,N} = 0 \quad \Rightarrow \quad \tau + \widetilde{\Xi} - \Xi - N\alpha = 0\ .$$

for $\Xi := \sum_{j=1}^{N} \xi_j$. (This follows from Frobenius' elliptic Cauchy determinant).

Revised scheme

From the implied condition that the matrix $[\boldsymbol{\Gamma}\mathbf{S}]$ must be of rank 1, it is convenient to revise the scheme absorb the matrix $\boldsymbol{\Gamma} = (\Gamma_{i,j})$ in the coefficient, leading to a revised Lax scheme. This yields an alternative Lax pair of the form:

$$\widetilde{\chi}_\kappa = \mathscr{L}_\kappa\, \chi_\kappa \ , \quad \chi_{\kappa+\tau} = \mathscr{T}_{\kappa+\tau}\, \chi_\kappa \ ,$$

with revised Lax matrices containing rank 1 matrix coefficients $H_{i,j} = h_i^+ h_j^-$, $K_{i,j} = k_i^+ k_j^-$ (assumed independent of κ):

$$\begin{aligned} (\mathscr{L}_\kappa)_{i,j} &= h_i^+\, \sigma(\kappa)\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha)\ h_j^- \ , \\ (\mathscr{T}_\kappa)_{i,j} &= s_i^+\, \sigma(\kappa)\, \Phi_\kappa(\xi_i - \xi_j - \gamma)\ s_j^- \ . \qquad (i,j = 1,\dots,N) \end{aligned}$$

For this system the calculation proceeds in a similar way as before, and using the addition formulae the compatibility yields the following system of equations:

- $h_i^+ \Big(\sum_{l=1}^N h_l^- s_l^+ \Big)\ s_j^- = \sigma(-\tau)\ \widetilde{s}_i^+ \sum_{l=1}^N \widetilde{s}_l^-\, h_l^+ \Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha)\ h_j^- \ ,$

- $h_i^+ \sum_{l=1}^N h_l^- s_l^+\, \sigma(\tau)\, \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\ \Phi_\tau(\xi_l - \xi_j - \gamma) s_j^- = \widetilde{s}_i^+ \sum_{l=1}^N \widetilde{s}_l^-\, h_l^+\, \Phi_\tau(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma}) h_j^- ,$

(for all $i,j = 1,\dots,N$), which as before can be cast in the matrix form:

- $\widetilde{S} \cdot [A^- H] = H \cdot S \ ,$
- $[A^+ H] \cdot [\Gamma^+ S] = [\widetilde{\Gamma}^+ \widetilde{S}] \cdot H \ ,$

with now the rank 1 matrices $H = h^+(h^-)^T$, $S = s^+(s^-)^T$ and the matrix Γ^+ (instead of Γ^-). Again, we need to impose that $\det(A^+) = 0$, implying:

$$\widetilde{\Xi} - \Xi = N\alpha - \tau \quad \Rightarrow \quad \Xi(n) = \Xi(0) + (N\alpha - \tau) n \ .$$

Revised scheme

From the implied condition that the matrix **[ΓS]** must be of rank 1, it is convenient to revise the scheme absorb the matrix $\boldsymbol{\Gamma} = (\Gamma_{i,j})$ in the coefficient, leading to a revised Lax scheme. This yields an alternative Lax pair of the form:

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$$(\mathscr{L}_\kappa)_{i,j} = h_i^+\, \sigma(\kappa)\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha)\, h_j^- \ ,$$

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- $h_i^+ \sum_{l=1}^N h_l^- s_l^+\, \sigma(\tau)\, \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_\tau(\xi_l - \xi_j - \gamma) s_j^- = \widetilde{s}_i^+ \sum_{l=1}^N \widetilde{s}_l^- h_l^+\, \Phi_\tau(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma}) h_j^-$,

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Case $N = 2$

To resolve this case, the first identity allows us to identify $h^+ = \rho \widetilde{s}^+$ (for some scalar function ρ), and consequently:

$$s_j^- = \frac{-\sigma(\tau)}{(h^- \cdot s^+)} \sum_{l=1}^{2} \widetilde{s}_l^+ \widetilde{s}_l^- \; \Phi_{-\tau}(\widetilde{\xi}_l - \widetilde{\xi}_j - \alpha) \; h_j^- .$$

Expressing all the entries of the first and second relation in terms of $s_l^+ s_l^- =: S_l$, $s_l^+ h_l^- =: H_l$ we get:

$$\left(1 + \frac{H_2}{H_1}\right) S_1 = A_{11}^- \widetilde{S}_1 + A_{21}^- \widetilde{S}_2 \ , \quad \left(\frac{H_1}{H_2} + 1\right) S_2 = A_{12}^- \widetilde{S}_1 + A_{22}^- \widetilde{S}_2 \ ,$$

The entries of the other matrix relation yields the system:

$$\left(A_{11}^+ \Gamma_{11}^+ + A_{12}^+ \Gamma_{21}^+ \frac{H_2}{H_1}\right) S_1 = \left(A_{11}^+ \Gamma_{12}^+ \frac{H_1}{H_2} + A_{12}^+ \Gamma_{22}^+\right) S_2 = \widetilde{\Gamma}_{11}^+ \widetilde{S}_1 + \widetilde{\Gamma}_{12}^+ \widetilde{S}_2 \ ,$$
$$\left(A_{21}^+ \Gamma_{11}^+ + A_{22}^+ \Gamma_{21}^+ \frac{H_2}{H_1}\right) S_1 = \left(A_{21}^+ \Gamma_{12}^+ \frac{H_1}{H_2} + A_{22}^+ \Gamma_{22}^+\right) S_2 = \widetilde{\Gamma}_{21}^+ \widetilde{S}_1 + \widetilde{\Gamma}_{22}^+ \widetilde{S}_2 \ .$$

To analyse these further, taking into account that $\det(A^+) = 0$, we rewrite the relations in terms of the ratios $X = H_2/H_1$, $Y = S_2/S_1$ and $Z = \widetilde{S}_1/S_1$, leading to:

$$Y/X = (A_{12}^- + A_{22}^- \widetilde{Y})/(A_{11}^- + A_{21}^- \widetilde{Y}) \ , \quad Z = (1 + X)/(A_{11}^- + A_{21}^- \widetilde{Y}) \ ,$$
$$A_{11}^+ \Gamma_{11}^+ + A_{12}^+ \Gamma_{21}^+ X = (A_{11}^+ \Gamma_{12}^+ + A_{12}^+ \Gamma_{22}^+ X) Y/X = (\widetilde{\Gamma}_{11}^+ + \widetilde{\Gamma}_{12}^+ \widetilde{Y}) Z \ ,$$
$$A_{11}^+ / A_{21}^+ = A_{12}^+ / A_{22}^+ = (\widetilde{\Gamma}_{11}^+ + \widetilde{\Gamma}_{12}^+ \widetilde{Y})/(\widetilde{\Gamma}_{21}^+ + \widetilde{\Gamma}_{22}^+ \widetilde{Y}) \ .$$

Eliminating X, Y, Z this leads to a 1^{st} order difference equation for $\xi_j(n)$ $(j = 1, 2)$.

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$$\Big(A_{21}^+ \Gamma_{11}^+ + A_{22}^+ \Gamma_{21}^+ \frac{H_2}{H_1}\Big) S_1 = \Big(A_{21}^+ \Gamma_{12}^+ \frac{H_1}{H_2} + A_{22}^+ \Gamma_{22}^+\Big) S_2 = \widetilde{\Gamma}_{21}^+ \widetilde{S}_1 + \widetilde{\Gamma}_{22}^+ \widetilde{S}_2 \;.$$

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Eliminating X, Y, Z this leads to a 1^{st} order difference equation for $\xi_j(n)$ $(j = 1, 2)$.

The result of this calculation is the following:

$$
\begin{aligned}
&\big[\sigma(\alpha+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_1)\,\sigma(-\gamma+\tau+\widetilde{\xi}_1-\widetilde{\xi}_2)\\
&\sigma(2\tau-\alpha-\gamma-\xi_1-\widetilde{\xi}_2)\,\big(\sigma(-\gamma+\xi_1-\xi_2)\sigma(\alpha-\gamma+\xi_2-\widetilde{\xi}_1)\\
&\sigma(-\alpha-\gamma+\tau-\xi_1+\widetilde{\xi}_1)\,\sigma(\alpha+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_2)-\sigma(-\gamma-\xi_1+\xi_2)\\
&\sigma(\alpha-\gamma+\xi_1-\widetilde{\xi}_1)\,\sigma(-\alpha-\gamma+\tau-\xi_2+\widetilde{\xi}_1)\,\sigma(\alpha+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha+\tau+\xi_2-\widetilde{\xi}_2)\big)\\
&+\big(\sigma(-\gamma+\xi_1-\xi_2)\,\sigma(\alpha+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_1)\,\sigma(\alpha-\gamma+\xi_2-\widetilde{\xi}_1)\\
&\sigma(-\alpha-\gamma+\tau-\xi_1+\widetilde{\xi}_1)+\sigma(-\gamma-\xi_1+\xi_2)\,\sigma(\alpha+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha-\gamma+\xi_1-\widetilde{\xi}_1)\\
&\sigma(\alpha+\tau+\xi_2-\widetilde{\xi}_1)\,\sigma(-\alpha-\gamma+\tau-\xi_2+\widetilde{\xi}_1)\big)\sigma(-\alpha-\gamma+2\tau-\xi_1+\widetilde{\xi}_1)\\
&\sigma(\alpha+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_2)\,\sigma(-\gamma+\tau-\widetilde{\xi}_1+\widetilde{\xi}_2)\big]\\
&+\frac{\sigma(-2\gamma+\tau)\,\sigma(-\gamma+\tau)\,\sigma(\xi_1-\xi_2)\,\sigma(\widetilde{\xi}_1-\xi_2-\alpha+\tau)}{\sigma(\gamma)\,\sigma(3\tau-2\gamma)\,\sigma(\widetilde{\xi}_1-\widetilde{\xi}_2)\,\sigma(\widetilde{\xi}_2-\xi_1-\alpha)}\big[\sigma(\alpha+\xi_1-\widetilde{\xi}_1)\\
&\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\tau-\gamma+\widetilde{\xi}_1-\widetilde{\xi}_2)\\
&\sigma(\widetilde{\xi}_2-\xi_1+2\tau-\alpha-\gamma)-\sigma(\alpha+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\widetilde{\xi}_1-\xi_1+2\tau-\gamma-\alpha)\\
&\sigma(\alpha+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\tau-\gamma-\widetilde{\xi}_1+\widetilde{\xi}_2)\big]\\
&\big[\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_1)\,\sigma(\alpha+\tau+\xi_2-\widetilde{\xi}_2)\,\sigma(\tau-\gamma+\widetilde{\xi}_1-\widetilde{\xi}_2)\\
&\sigma(\widetilde{\xi}_2-\xi_1+2\tau-\alpha-\gamma)-\sigma(\alpha+\tau+\xi_2-\widetilde{\xi}_1)\,\sigma(\widetilde{\xi}_1-\xi_1+2\tau-\gamma-\alpha)\\
&\sigma(\alpha+\xi_2-\widetilde{\xi}_2)\,\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\tau-\gamma-\widetilde{\xi}_1+\widetilde{\xi}_2)\big]=0
\end{aligned}
$$

which is subject to the condition $\xi_1+\xi_2=(2\alpha-\tau)n+\Xi(0)$. This a first order nonlinear ordinary elliptic difference equation.

Higher order scheme

In order to derive higher-order O∆Es we extend the isomonodromic problem to a higher order one as follows:

$$\chi_{\kappa+\tau} = \mathbf{T}'_\kappa \, \chi_\kappa \ ,$$

$$(\mathbf{T}'_\kappa)_{i,j} := \sigma^2(\kappa) \sum_{l=1}^{N} S_{i,j}^{(l)} \Phi_\kappa(\xi_i - \eta_l) \Phi_\kappa(\eta_l - \xi_j - \gamma) \ , \quad (i,j = 1, \dots, N) \ ,$$

where the η_l variables as well as the extended coefficients $S_{i,j}^{(l)}$ remain to be determined. We consider this difference equation on the torus in conjunction with he lattice Lax system (as before):

$$\widetilde{\chi}_\kappa = \mathbf{L}_\kappa \, \chi_\kappa \ , \quad (\mathbf{L}_\kappa)_{i,j} = H_{i,j} \, \sigma(\kappa) \, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha) \ ,$$

$$\widehat{\chi}_\kappa = \mathbf{M}_\kappa \, \chi_\kappa \ , \quad (\mathbf{M}_\kappa)_{i,j} = K_{i,j} \, \sigma(\kappa) \, \Phi_\kappa(\widehat{\xi}_i - \xi_j - \beta) \ ,$$

where as before we like to take the coefficient matrices **H** and **K** of rank 1 and independent of the spectral variable κ.

We can think of the scheme above as an elliptic de-autonomization of a higher-order periodic reduction on the lattice.

2-step periodic reduction:

$\chi \ \rightarrow \ \overline{\chi} \ \rightarrow \ \overline{\overline{\chi}} = \lambda\chi$

followed by de-autonomization:

$\lambda\chi \ \rightsquigarrow \ \chi_{\kappa+\tau}$

However, now we want to keep the midpoint unspecified associated with some value η for ξ.

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$$\widehat{\boldsymbol{\chi}}_\kappa = \mathbf{M}_\kappa \, \boldsymbol{\chi}_\kappa \ , \quad (\mathbf{M}_\kappa)_{i,j} = K_{i,j} \, \sigma(\kappa) \, \Phi_\kappa(\widehat{\xi}_i - \xi_j - \beta) \ ,$$

where as before we like to take the coefficient matrices **H** and **K** of rank 1 and independent of the spectral variable κ.

We can think of the scheme above as an elliptic de-autonomization of a higher-order periodic reduction on the lattice.

2-step periodic reduction:

$\chi \ \to \ \overline{\chi} \ \to \ \overline{\overline{\chi}} = \lambda\chi$

followed by de-autonomization:

$\lambda\chi \ \rightsquigarrow \ \chi_{\kappa+\tau}$

However, now we want to keep the midpoint unspecified associated with some value η for ξ.

Higher order scheme

In order to derive higher-order OΔEs we extend the isomonodromic problem to a higher order one as follows:

$$\boldsymbol{\chi}_{\kappa+\tau} = \mathbf{T}'_{\kappa}\,\boldsymbol{\chi}_{\kappa}\ ,$$

$$(\mathbf{T}'_{\kappa})_{i,j} := \sigma^2(\kappa)\sum_{l=1}^{N} S_{i,j}^{(l)}\Phi_{\kappa}(\xi_i-\eta_l)\Phi_{\kappa}(\eta_l-\xi_j-\gamma)\ , \quad (i,j=1,\dots,N)\ ,$$

where the η_l variables as well as the extended coefficients $S_{i,j}^{(l)}$ remain to be determined. We consider this difference equation on the torus in conjunction with he lattice Lax system (as before):

$$\widetilde{\boldsymbol{\chi}}_{\kappa} = \mathbf{L}_{\kappa}\,\boldsymbol{\chi}_{\kappa}\ , \quad (\mathbf{L}_{\kappa})_{i,j} = H_{i,j}\,\sigma(\kappa)\,\Phi_{\kappa}(\widetilde{\xi}_i-\xi_j-\alpha)\ ,$$
$$\widehat{\boldsymbol{\chi}}_{\kappa} = \mathbf{M}_{\kappa}\,\boldsymbol{\chi}_{\kappa}\ , \quad (\mathbf{M}_{\kappa})_{i,j} = K_{i,j}\,\sigma(\kappa)\,\Phi_{\kappa}(\widehat{\xi}_i-\xi_j-\beta)\ ,$$

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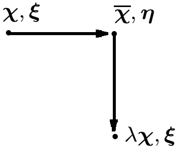

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Compatibility higher order scheme

This system leads to the system of compatibility conditions:

$$\widehat{\mathbf{L}}_\kappa \, \mathbf{M}_\kappa = \widetilde{\mathbf{M}}_\kappa \, \mathbf{L}_\kappa \ , \quad \mathbf{L}_{\kappa+\tau} \, \mathbf{T}'_\kappa = \widetilde{\mathbf{T}}'_\kappa \, \mathbf{L}_\kappa \ , \qquad \mathbf{M}_{\kappa+\tau} \, \mathbf{T}'_\kappa = \widehat{\mathbf{T}}'_\kappa \, \mathbf{M}_\kappa$$

.

To do this most effectively we need a new elliptic addition rule:

$$\frac{\Phi_\kappa(x)\,\Phi_\kappa(y)\,\Phi_\kappa(z)}{\Phi_\kappa(x+y+z)} = \tfrac{1}{2}\Big[\big(\zeta(\kappa)+\zeta(x)+\zeta(y)+\zeta(z)-\zeta(\kappa+x+y+z)\big)^2 + \wp(\kappa) - \big(\wp(x)+\wp(y)+\wp(z)+\wp(\kappa+x+y+z)\big)\Big] \ ,$$

$\forall\, \kappa, x, y, z$. The second consistency condition $\mathbf{L}_{\kappa+\tau} \, \mathbf{T}'_\kappa = \widetilde{\mathbf{T}}'_\kappa \, \mathbf{L}_\kappa$ works as follows:

$$\begin{aligned} & \sigma(\tau)\Phi_\kappa(\tau) \sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')}\, \Phi_{\kappa+\tau}(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_\kappa(\xi_l - \eta_{l'})\, \Phi_\kappa(\eta_{l'} - \xi_j - \gamma) \\ = \quad & \sigma(\tau) \sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')}\, \Phi_\kappa(\tau + \widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_\kappa(\xi_l - \eta_{l'})\, \Phi_\kappa(\eta_{l'} - \xi_j - \gamma) \\ = \quad & \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj} \Phi_\kappa(\widetilde{\xi}_i - \widetilde{\eta}_{l'})\, \Phi_\kappa(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma})\, \Phi_\kappa(\widetilde{\xi}_l - \xi_j - \alpha) \end{aligned}$$

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Thus, we find:

$$\sum_{l,l'=1}^{N} \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, H_{il}\, S_{lj}^{(l')}$$

$$\times\Big[\Big(\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau) + \zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \zeta(\xi_l - \eta_{l'}) + \zeta(\eta_{l'} - \xi_j - \gamma)\Big)^2$$

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This breaks down in the following constitutive relations:

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How to proceed?

The general scheme is of relations derived is rather complicated. The first and last relation can be written in the form:

- $\widetilde{\mathbf{S}} \cdot \mathbf{H} = [\mathbf{A}^+\mathbf{H}] \cdot \mathbf{S}$
- $[\widetilde{\mathbf{\Delta}}^- \widetilde{\mathbf{\Gamma}}^- \widetilde{\mathbf{S}}] \cdot [\mathbf{A}^-\mathbf{H}] = \mathbf{H} \cdot [\mathbf{\Delta}^- \mathbf{\Gamma}^- \mathbf{S}]$,

where we have used the same notation as before, with the matrix $\mathbf{S}$ as the matrix with entries $(\mathbf{S})_{i,j} = \sum_{l'} S_{ij}^{(l')}$, and where the "doubly glued" matrix $[\mathbf{\Delta}^- \mathbf{\Gamma}^- \mathbf{S}]$ is the matrix with entries:

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It follows, as before, that if $\mathbf{H}$ is of rank 1, then the latter matrix should be of rank 1 as well. This suggests that we again want to revise the scheme, and replace $\mathbf{T}_\kappa$ by $\mathbf{T}_{\kappa+\tau}$ in the monodromy problem.
Strategy is once again to eliminate $\mathbf{H}$, $\mathbf{S}$ and from the middle relation to solve the η_i.
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Discussion

- We gave a general framework for elliptic Lax pairs in 1+1 dimensions associated with elliptic (classical) 2D lattice equations containing two broad classes: LL (or spin non-zero), and KN (spin zero) type systems;
- There are two main reductions: one-or multi-step period reductions leading to the discrete-time Ruijsenaars model (and its "hierarchy") and non-autonomous reductions usually associated with scaling invariance;
- The isomonodromic problems obtained by de-autonomization of isospectral problems on the torus lead to systems non-autonomous elliptic ordinary difference equations and, we believe, will eventually yield elliptic discrete Painlevé equations and possibly higher-order analogues (elliptic Garnier systems?), but further analysis is needed;
- Recently isomonodromic deformation problems for Sakai's elliptic discrete Painlevé equation has been considered by several authors (Rains, Borodin, Yamada, Noumi) and a comparison with those works would be interesting;
- Our constructions involve mostly basic addition formulae for (Weierstrass) elliptic functions, while other constructions often theta-functions in multiplicative form;
- It would be interesting to further explore elliptic discrete integrable systems in higher dimensions, such as the elliptic lattice KP equation we recently constructed[9], which is essentially a system in 3+1 dimensions.

[9] P Jennings & FWN, *On an elliptic extension of the Kadomtsev-Petviashvili equation*, J.Phys. A:Math. Theor. **47** (2014) 055205.

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[9] P Jennings & FWN, *On an elliptic extension of the Kadomtsev-Petviashvili equation*, J.Phys. A:Math. Theor. **47** (2014) 055205.

Discussion

- We gave a general framework for elliptic Lax pairs in 1+1 dimensions associated with elliptic (classical) 2D lattice equations containing two broad classes: LL (or spin non-zero), and KN (spin zero) type systems;
- There are two main reductions: one-or multi-step period reductions leading to the discrete-time Ruijsenaars model (and its "hierarchy") and non-autonomous reductions usually associated with scaling invariance;
- The isomonodromic problems obtained by de-autonomization of isospectral problems on the torus lead to systems non-autonomous elliptic ordinary difference equations and, we believe, will eventually yield elliptic discrete Painlevé equations and possibly higher-order analogues (elliptic Garnier systems?), but further analysis is needed;
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Thank you for your attention!