Monodromy representations associated with the generalized hypergeometric function ${}_{n+1}F_n$

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三町勝久

Katsuhisa Mimachi

(Osaka)

Joint work with Noumi-san ($+\alpha$)

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The generalized HGF

$$ {}_{n+1}F_n\left(\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_{n+1} \\ \beta_1, \ldots\ldots, \beta_n \end{array}; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_{n+1})_k}{(\beta_1)_k \ \cdots\cdots \ (\beta_n)_k \, k!} \, z^k, \quad |z| < 1, $$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

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The differential equation ${}_{n+1}E_n$:

$$ \left\{ \theta_z \Big\{ \prod_{1 \le i \le n} (\theta_z + \beta_i - 1) \Big\} - z \Big\{ \prod_{1 \le i \le n+1} (\theta_z + \alpha_i) \Big\} \right\} F = 0, $$

where $\theta_z = zd/dz$.

Characteristic exponents of ${}_{n+1}E_n$:

$$
\begin{array}{ll}
0,\ 1-\beta_1,\ 1-\beta_2,\ \ldots,\ 1-\beta_n & \text{at} \quad z=0, \\
0,\ 1,\ \ldots,\ n-1,\ \displaystyle\sum_{i=1}^{n}\beta_i-\sum_{i=1}^{n+1}\alpha_i & \text{at} \quad z=1, \\
\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{n+1} & \text{at} \quad z=\infty.
\end{array}
$$

Monodromy representation

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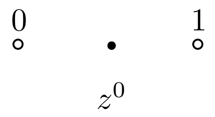

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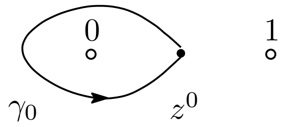

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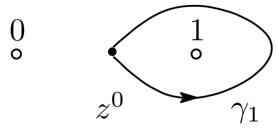

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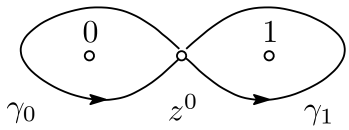

Integral representations of the solutions

$$\int_D u_D(t) dt_1 \cdots dt_m$$

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For $u(t) = \prod_i f_i(t)^{\mu_i}$, $u_D(t) = \prod_i (\epsilon_i f_i(t))^{\mu_i}$, where $\epsilon_i = \pm$ is determined so that $\epsilon_i f_i(t) > 0$ on D.

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$$u(t)=\prod_{i=1}^{n} t_i^{\lambda_i}\prod_{i=1}^{n+1}(t_{i-1}-t_i)^{\lambda_{i-1,i}}$$

with

$$\lambda_i=\alpha_{i+1}-\beta_i\quad \lambda_{i-1,i}=\beta_i-\alpha_i-1,\quad t_0=1,\quad t_{n+1}=z,\quad \beta_{n+1}=1.$$

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$$u(t) = \prod_{i=1}^{n} t_i^{\lambda_i} \prod_{i=1}^{n+1} (t_{i-1} - t_i)^{\lambda_{i-1,i}}$$

with

$$\lambda_i = \alpha_{i+1} - \beta_i \quad \lambda_{i-1,i} = \beta_i - \alpha_i - 1, \quad t_0 = 1, \quad t_{n+1} = z, \quad \beta_{n+1} = 1$$

on

$$T_z = \mathbb{C}^n \setminus \cup_{i=1}^{n} \{ t_i = 0 \} \cup \cup_{i=1}^{n+1} \{ t_{i-1} - t_i = 0 \}.$$

$$D_{j+1}^{(0)} = \begin{pmatrix} 0 < t_{j+1} < \cdots < t_n < z \\ 1 < t_1 < \cdots < t_j < \infty \end{pmatrix}, \quad j = 0, 1, \ldots, n.$$

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we have

$$\begin{aligned} I_{j+1}(z) &= \int_{D_{j+1}^{(0)}} u_{D_{j+1}^{(0)}}(t)\, dt_1 \cdots dt_n \\ &= \prod_{\substack{1 \le s \le n+1 \\ s \ne j+1}} B(\alpha_s - \beta_i + 1, \beta_s - \alpha_s) \times z^{1-\beta_{j+1}}(1 + O(z)). \end{aligned}$$

$$\gamma_0(D_{j+1}^{(0)}) = e(-2\beta_{n-j})D_{j+1}^{(0)}, \quad j = 0, 1, \ldots, n.$$

$$\gamma_0(D^{(0)}_{j+1}) = e(-2\beta_{n-j})D^{(0)}_{j+1}, \quad j = 0, 1, \ldots, n,$$

where $e(A) = \exp(\pi\sqrt{-1}A)$.

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What is the action of γ_1?

γ_0 acts diagonally

$$\gamma_0(D^{(0)}_{j+1}) = e(-2\beta_{n-j})D^{(0)}_{j+1}, \quad j = 0, 1, \ldots, n.$$

What is the action of γ_1?

$$\gamma_1 \begin{pmatrix} 0 < t_{j+1} < \cdots < t_n < z \\ 1 < t_1 < \cdots < t_j < \infty \end{pmatrix} = ?$$

In what follows we assume that

$$\beta_i - \beta_j \notin \mathbb{Z}, \quad 1 \leq i \leq j \leq n+1, \quad \text{and} \quad \beta_{n+1} = 1.$$

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and moreover,

$$\alpha_i - \beta_j \notin \mathbb{Z}, \quad 1 \leq i, j \leq n+1.$$

$$D_{j+1}^{(0)} = \begin{pmatrix} 0 < t_{j+1} < \cdots < t_n < z \\ 1 < t_1 < \cdots < t_j < \infty \end{pmatrix}, \quad j = 0, 1, \ldots, n,$$

$$D_{j+1}^{(0)} = \left(\begin{array}{c} 0 < t_{j+1} < \cdots < t_n < z \\ 1 < t_1 < \cdots < t_j < \infty \end{array} \right), \quad j = 0, 1, \ldots, n,$$

Case $n = 1$:

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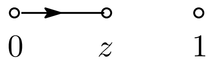
Case $n = 1$:

$$\gamma_1^*(D_1^{(0)}) = \gamma_1^* \left\{ \quad \right\}.$$



$$D^{(0)}_{j+1} = \begin{pmatrix} 0 < t_{j+1} < \cdots < t_n < z \\ 1 < t_1 < \cdots < t_j < \infty \end{pmatrix}, \quad j = 0, 1, \dots, n,$$

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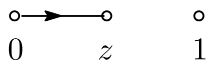

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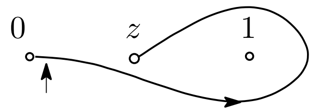

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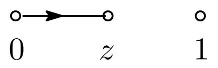
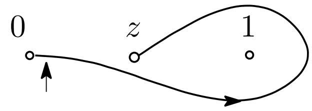
$$\gamma_1^*(D_1^{(0)}) = \gamma_1^* \Big\{$$ 

$$\Big\} = \Big\{$$ 

$$\Big\}.$$

Here and in what follows, the vertical arrow in each picture indicates the point at which the argument of each factor of the integrand is fixed to be zero, while we omit to write such an arrow when the chain in the picture is an interval of $T_{\mathbb{R}}$.

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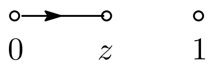
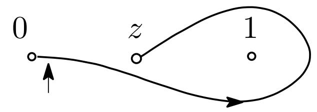
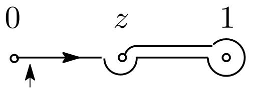
$\gamma_1^*(D_1^{(0)}) = \gamma_1^* \Big\{$ 

$\Big\} = \Big\{$ 

$\Big\}$

$= \Big\{$ 

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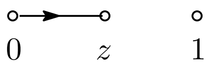

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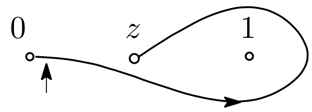

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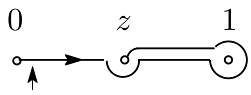

$$\Bigg\} = D_1^{(0)} + e(\lambda_{12})(1 - e(2\lambda_{01}))(z < t_1 < 1).$$

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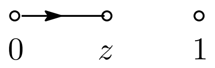

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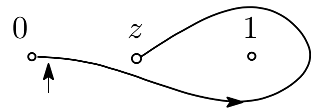

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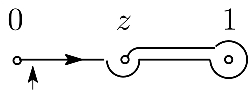

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Lemma 1.

$$\gamma_1^*(D_1^{(0)}) = D_1^{(0)}$$

$$+ (-)^{n-1}(1-e(2\lambda_{01}))e(\lambda_{12}+\lambda_{23}+\cdots+\lambda_{n,n+1})(z<t_n<\cdots<t_1<1).$$

$$D^{(0)}_{j+1} = \begin{pmatrix} 0 < t_{j+1} < \cdots < t_n < z \\ 1 < t_1 < \cdots < t_j < \infty \end{pmatrix}, \quad j = 0, 1, \ldots, n,$$

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Proof.

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Proof. Induction implies the assertion.

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Proof. Induction implies the assertion. (1) The case $n = 1$ is proved above. (2) Suppose the case $n - 1$.

$$D_{j+1}^{(0)} = \begin{pmatrix} 0 < t_{j+1} < \cdots < t_n < z \\ 1 < t_1 < \cdots < t_j < \infty \end{pmatrix}, \quad j = 0, 1, \ldots, n,$$

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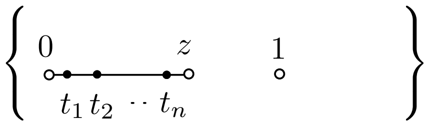
$$=$$



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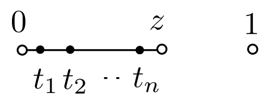
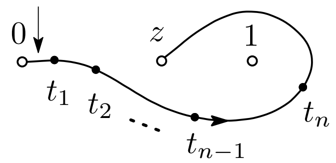
$= \Bigg\{$  $\Bigg\}$

$= \Bigg\{$  $\Bigg\}$

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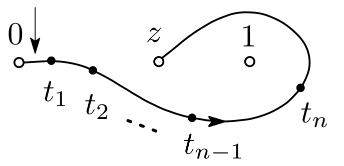
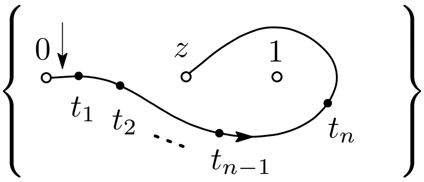
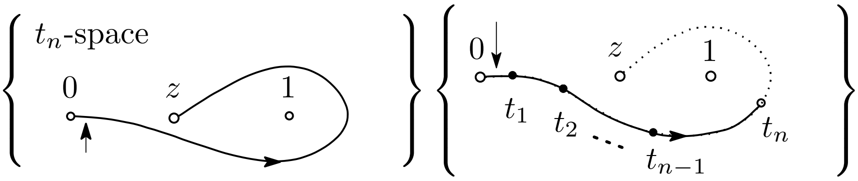
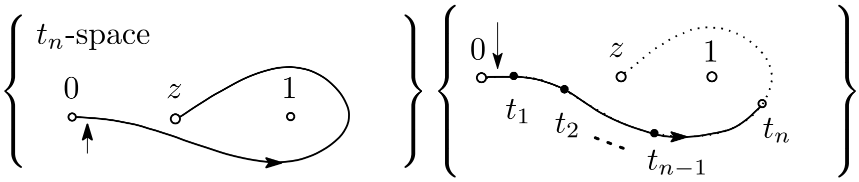
$$= \left\{ \quad \right\}$$



$$\gamma_1^*(D_1^{(0)}) = \gamma_1^*(0 < t_1 < \cdots < t_n < z)$$

$=$ 

$=$ 

$=$ 
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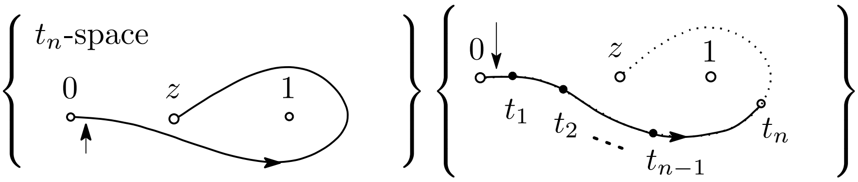
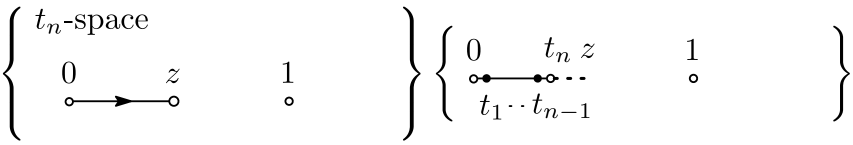
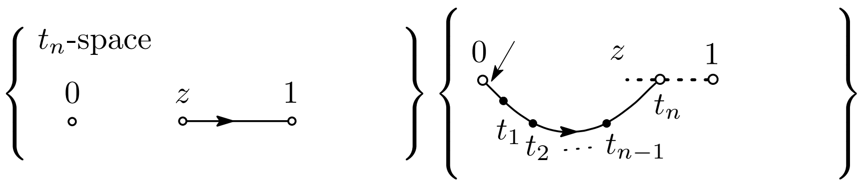
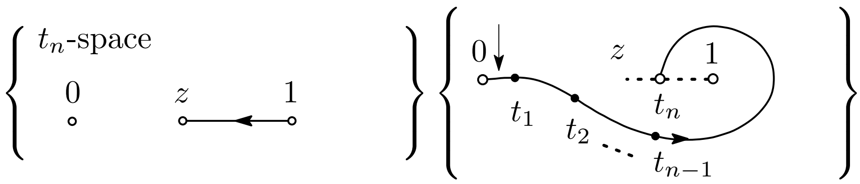
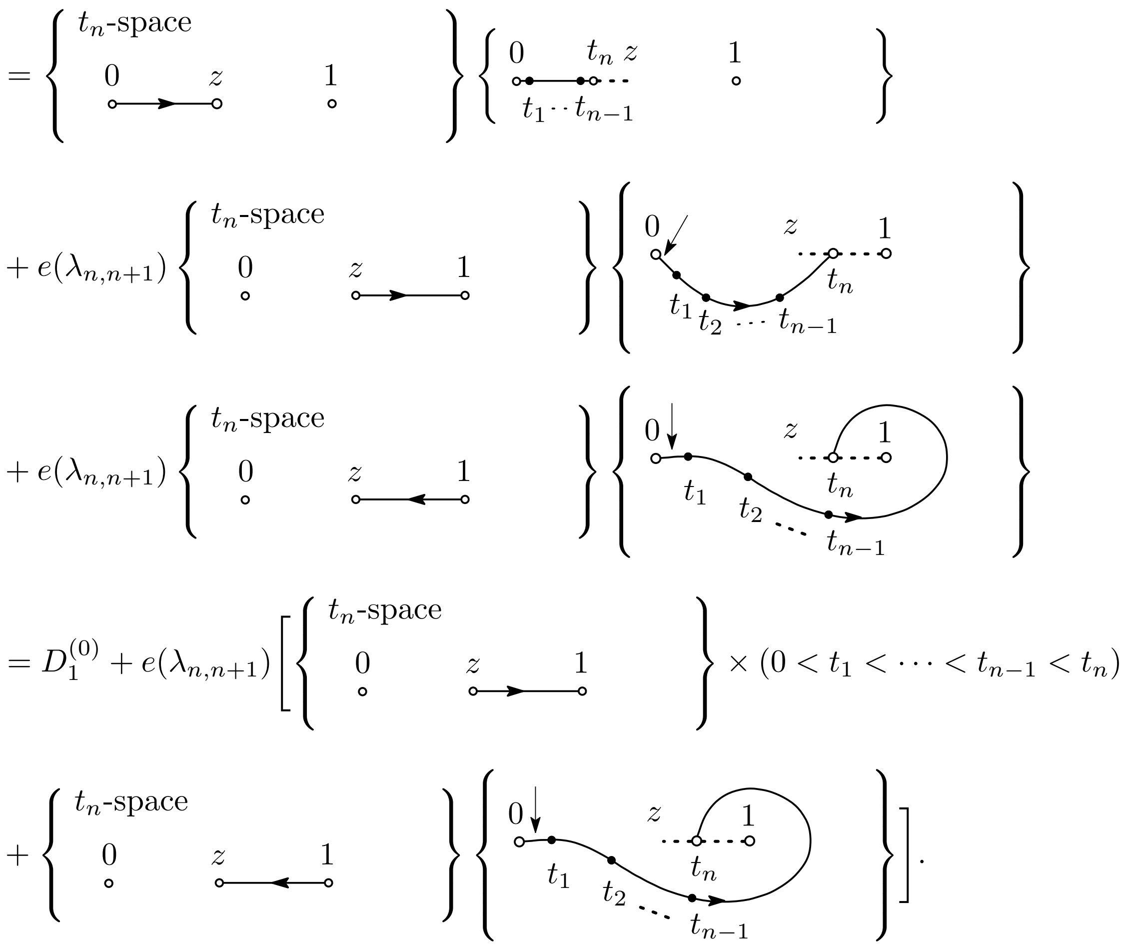
$=$ 

$=$ 

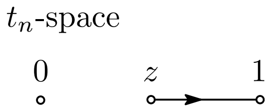
$+\, e(\lambda_{n,n+1})$ 

$+\, e(\lambda_{n,n+1})$ 



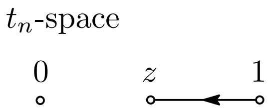
$= \{ t_n$-space $0 \quad z \quad 1 \} \{ 0 \quad t_n \; z \quad 1 \quad t_1 \cdots t_{n-1} \}$
$+ e(\lambda_{n,n+1}) \{ t_n$-space $0 \quad z \quad 1 \} \{ 0 \quad z \quad 1 \quad t_n \quad t_1 \; t_2 \cdots t_{n-1} \}$
$+ e(\lambda_{n,n+1}) \{ t_n$-space $0 \quad z \quad 1 \} \{ 0 \quad z \quad 1 \quad t_n \quad t_1 \quad t_2 \quad t_{n-1} \}$
$= D_1^{(0)} + e(\lambda_{n,n+1}) \Big[\{ t_n$-space $0 \quad z \quad 1 \} \times (0 < t_1 < \cdots < t_{n-1} < t_n)$
$+ \{ t_n$-space $0 \quad z \quad 1 \} \{ 0 \quad z \quad 1 \quad t_n \quad t_1 \quad t_2 \quad t_{n-1} \} \Big].$

$$= D_1^{(0)} + e(\lambda_{n,n+1})\Bigg[\Bigg\{$$

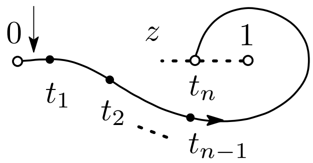


$$\Bigg\} \times (0 < t_1 < \cdots < t_{n-1} < t_n)$$

$$+ \Bigg\{$$

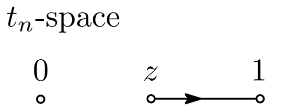


$$\Bigg\}\Bigg\{$$



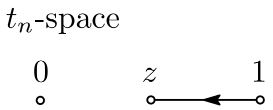
$$\Bigg\}\Bigg].$$

$$= D_1^{(0)} + e(\lambda_{n,n+1})\Bigg[\Bigg\{$$

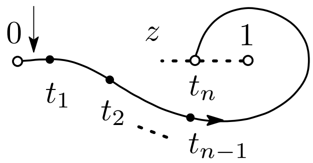


$$\Bigg\} \times (0 < t_1 < \cdots < t_{n-1} < t_n)$$

$$+ \Bigg\{$$



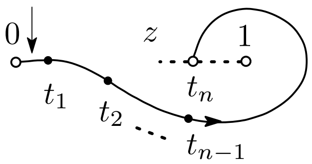
$$\Bigg\}\Bigg\{$$



$$\Bigg\}\Bigg].$$

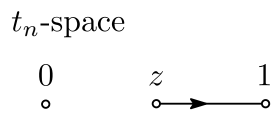
The assumption implies

$$\Bigg\{$$

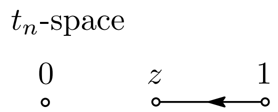


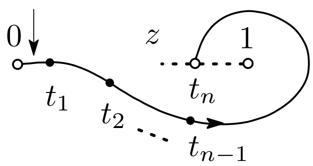
$$\Bigg\} = (0 < t_1 < \cdots < t_{n-1} < t_n)$$

$$+ (-1)^{n-2}(1 - e(2\lambda_{01}))e(\lambda_{12} + \cdots + \lambda_{n-1,n})(t_n < t_{n-1} < \cdots < t_1 < 1)$$

$= D_1^{(0)} + e(\lambda_{n,n+1}) \Bigg[\Bigg\{$ 

$\Bigg\} \times (0 < t_1 < \cdots < t_{n-1} < t_n)$

$+ \Bigg\{$ 

$\Bigg\} \Bigg\{$ 

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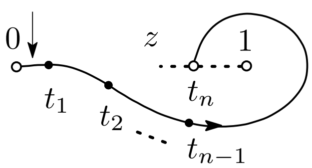
$\Bigg\{$ 

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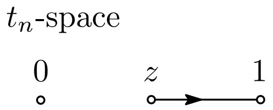
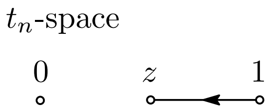
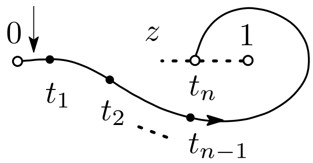
$$= D_1^{(0)} + e(\lambda_{n,n+1})\Bigg[\Bigg\{$$ 

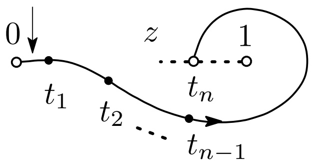
$$\Bigg\} \times (0 < t_1 < \cdots < t_{n-1} < t_n)$$

$$+ \Bigg\{$$ 

$$\Bigg\}\Bigg\{$$ 

$$\Bigg\}\Bigg].$$

The assumption implies

$$\Bigg\{$$ 

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□

Lemma 2.

$$\gamma_1(D^{(0)}_{j+1}) = D^{(0)}_{j+1}$$
$$+ (-)^{n-1} e(\sum_{\substack{1\leq s\leq n+1\\ s\neq j+1}} \lambda_{s-1,s})(1 - e(2\lambda_{j,j+1}))\,(z < t_n < \cdots < t_1 < 1).$$

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Proof.

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Proof. Induction on j implies the assertion.

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$$\gamma_1^*(D^{(0)}_{j+1}) = \gamma_1^* \begin{pmatrix} 0 < t_{j+1} < \cdots < t_n < z \\ 1 < t_1 < \cdots < t_j < \infty \end{pmatrix}$$

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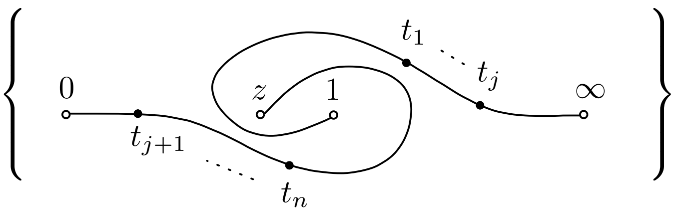
$=$



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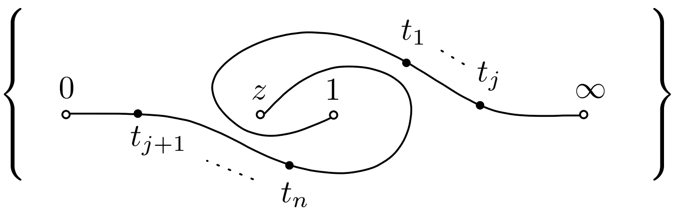

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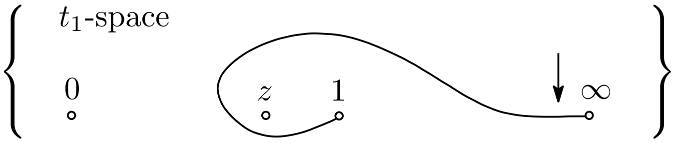

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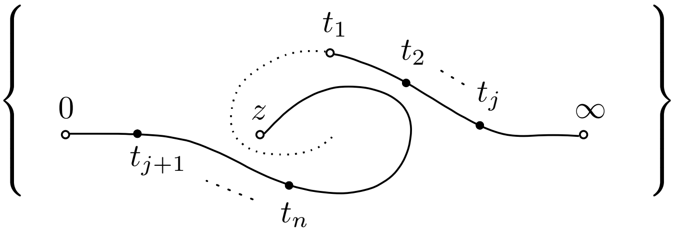

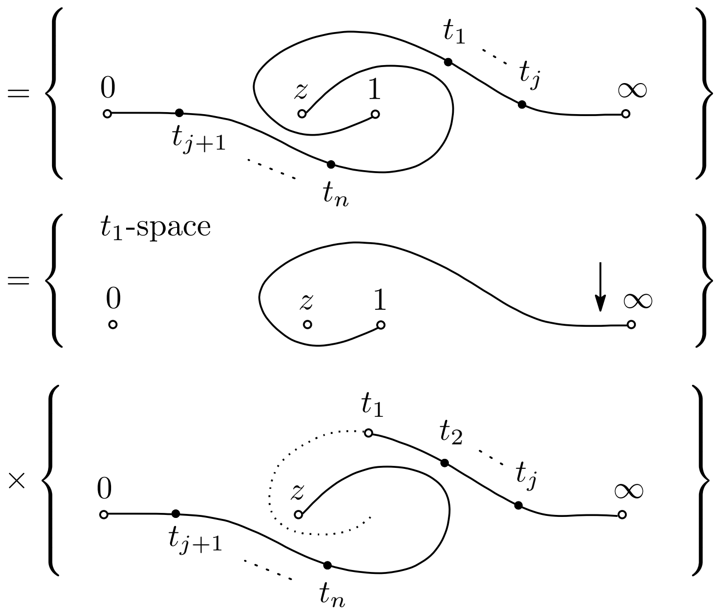
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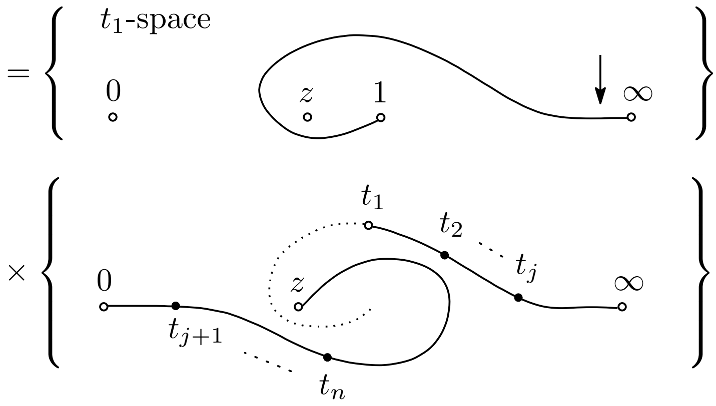
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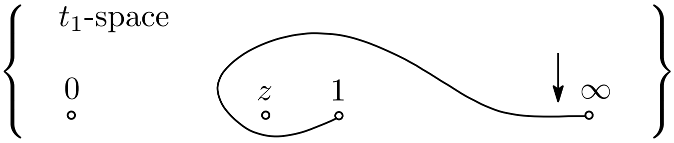
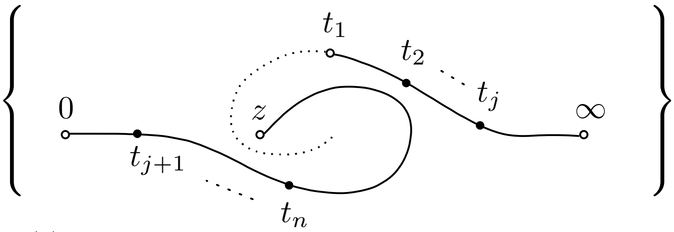
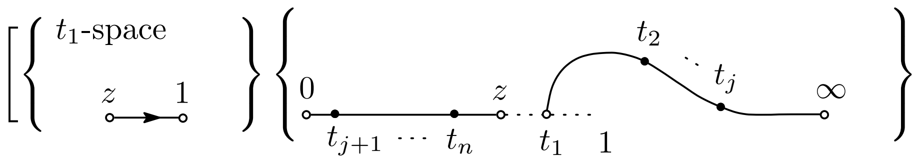
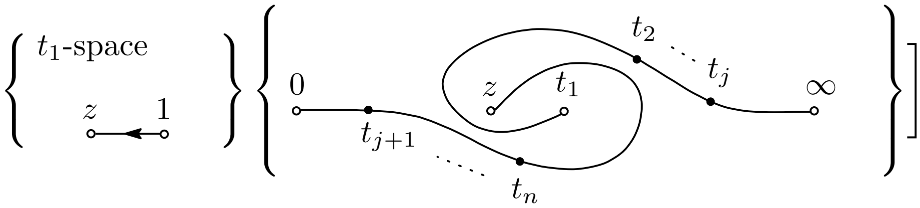
$=$ 

$\times$ 

$= D^{(0)}_{j+1} + e(\lambda_{01})$

$\times$ 

$+$ 

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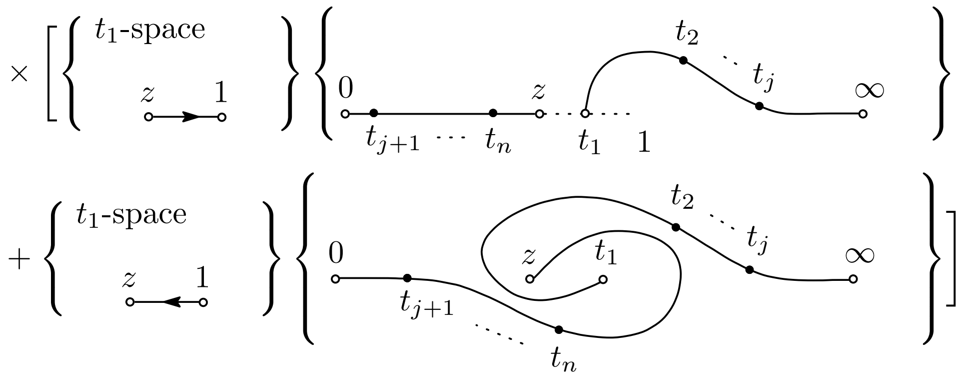

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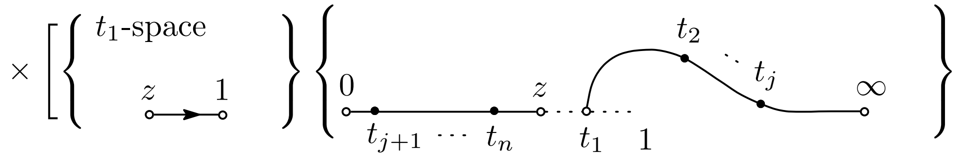

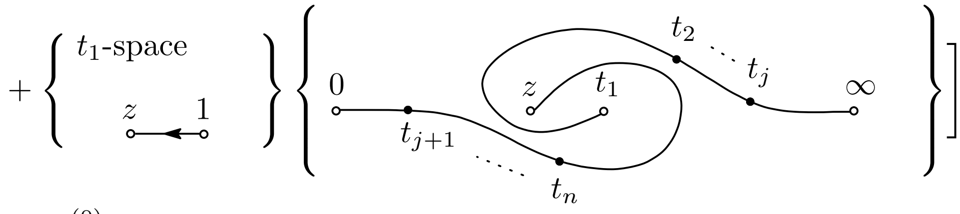

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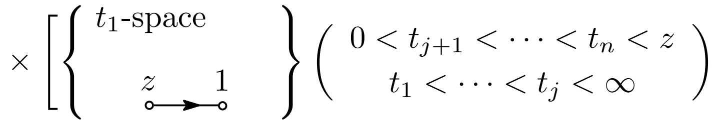

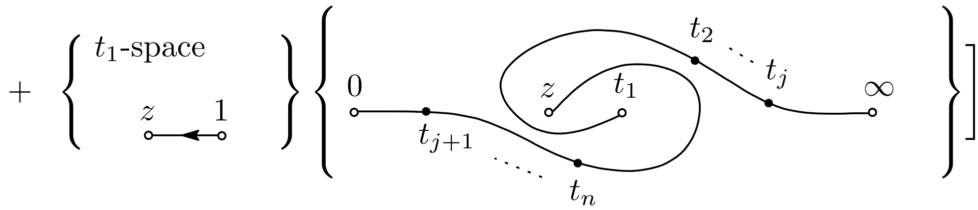

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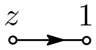
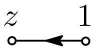
$\times \Big[\Big\{ t_1\text{-space}$



$\Big\} \left(\begin{matrix} 0 < t_{j+1} < \cdots < t_n < z \\ t_1 < \cdots < t_j < \infty \end{matrix} \right)$

$+ \Big\{ t_1\text{-space}$



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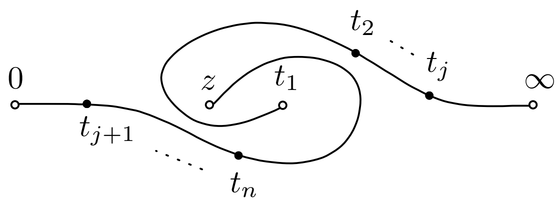

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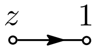
$\times \Big[\Big\{$ t_1-space



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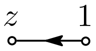
$+ \Big\{$ t_1-space



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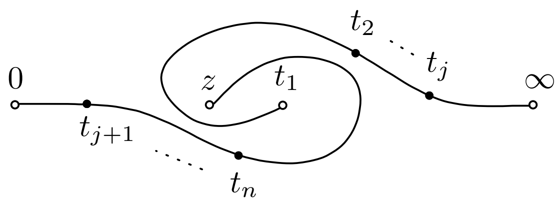

$\Big\} \Big]$

The assumption implies

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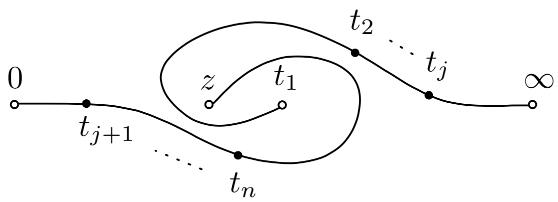

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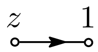

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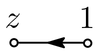
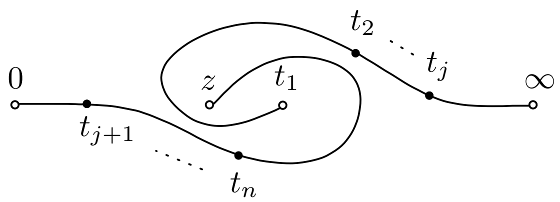


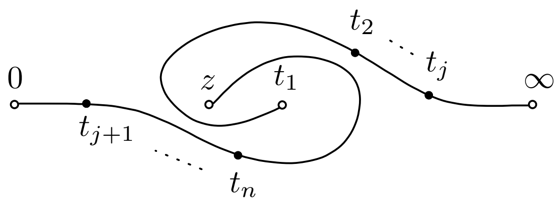
$\Bigg\} \Bigg\{$



$\Bigg\} \Bigg]$

The assumption implies

$\Bigg\{$



$$\Bigg\} = \begin{pmatrix} 0 < t_{j+1} < \cdots < t_n < z \\ t_1 < \cdots < t_j < \infty \end{pmatrix}$$

$$+ (-)^{n-2} e\Big(\sum_{\substack{2 \le s \le n+1 \\ s \ne j+1}} \lambda_{s-1,s}\Big)(1 - e(2\lambda_{j,j+1}))\,(z < t_n < \cdots < t_2 < t_1).$$

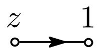
Hence we have

$$\gamma_1(D^{(0)}_{j+1}) = D^{(0)}_{j+1}$$

$$+ (-)^{n-1} e\left(\sum_{\substack{1 \le s \le n+1 \\ s \ne j+1}} \lambda_{s-1,s}\right)(1 - e(2\lambda_{j,j+1}))\,(z < t_n < \cdots < t_1 < 1).$$

$= D^{(0)}_{j+1} + e(\lambda_{01})$

$\times \Bigg[\Bigg\{$ t_1-space



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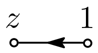

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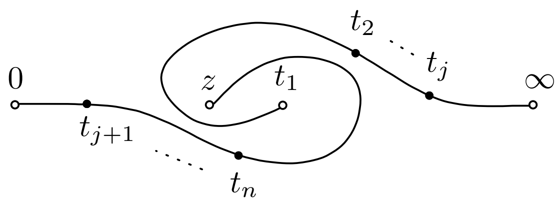

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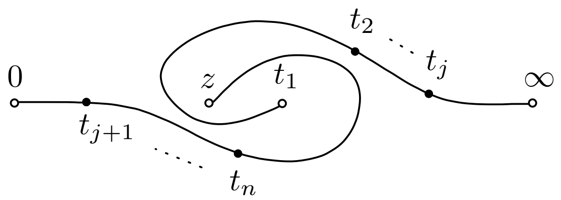

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&= D^{(0)}_{j+1} \\
&+ 2\sqrt{-1} e\left(\sum_{s=1}^{n} \beta_s - \sum_{s=1}^{n+1} \alpha_s\right) s(\alpha_{j+1} - \beta_{j+1})\,(z < t_n < \cdots < t_1 < 1),
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where $s(A) = \sin(\pi A)$.

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$$
(z<t_n<\cdots<t_1<1) = \sum_{i=1}^{n+1} c_i D^{(0)}_i \quad ?
$$

Twisted homology group $H_n^{\mathrm{lf}}(T,\mathcal{L})$ and $H_n(T,\mathcal{L})$

$H_n^{\mathrm{lf}}(T,\mathcal{L})$ or $H_n(T,\mathcal{L})$, where $\mathcal{L}$ is determined by

$$u(t) = \prod_{i=1}^{n} t_i^{\alpha_{i+1}-\beta_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\beta_i-\alpha_i-1}, \quad (\beta_{n+1}=1, t_0=1, t_{n+1}=z)$$

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$$T=\mathbb{C}^n\setminus\cup_{i=1}^{n}\{\,t_i=0\,\}\cup\cup_{i=1}^{n+1}\{\,t_i-t_{i-1}=0\,\}.$$

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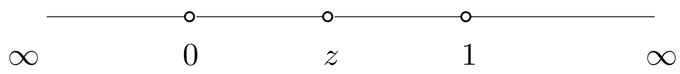
At first, fix z to be $0<z<1$.

$n=1$



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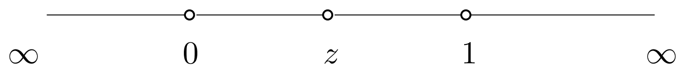

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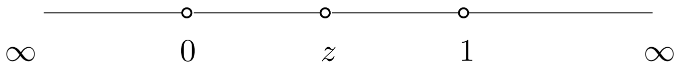

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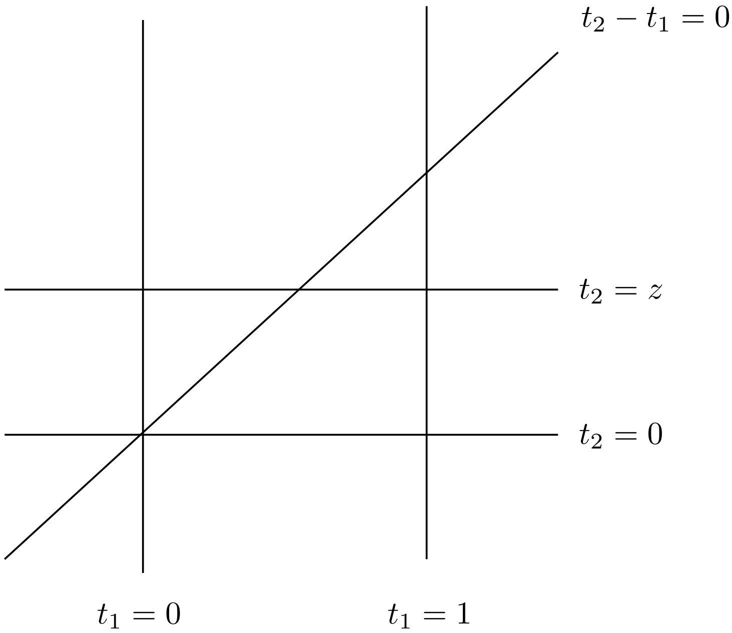

Basis of $H_n^{\mathrm{lf}}(T, \mathcal{L})$:

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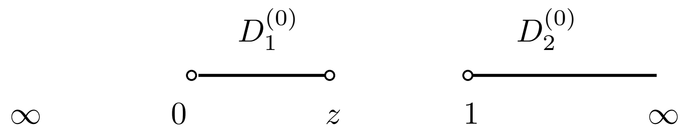

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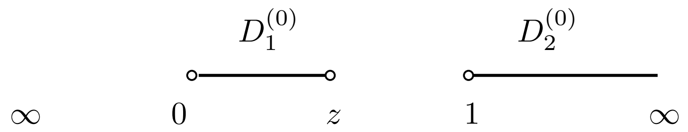

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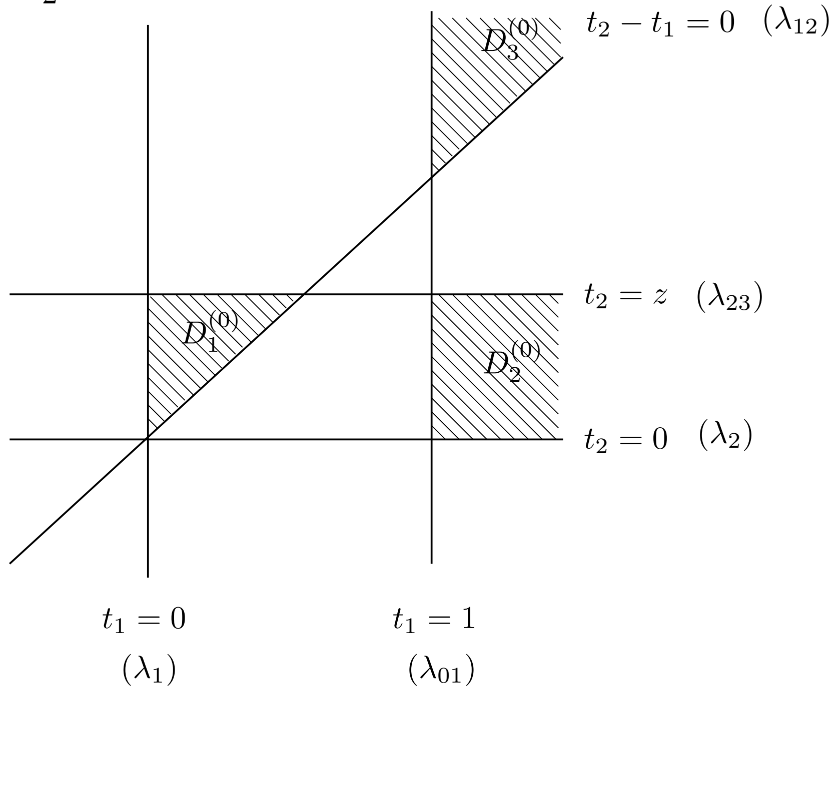

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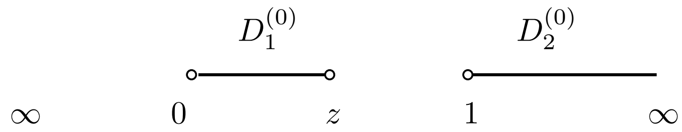

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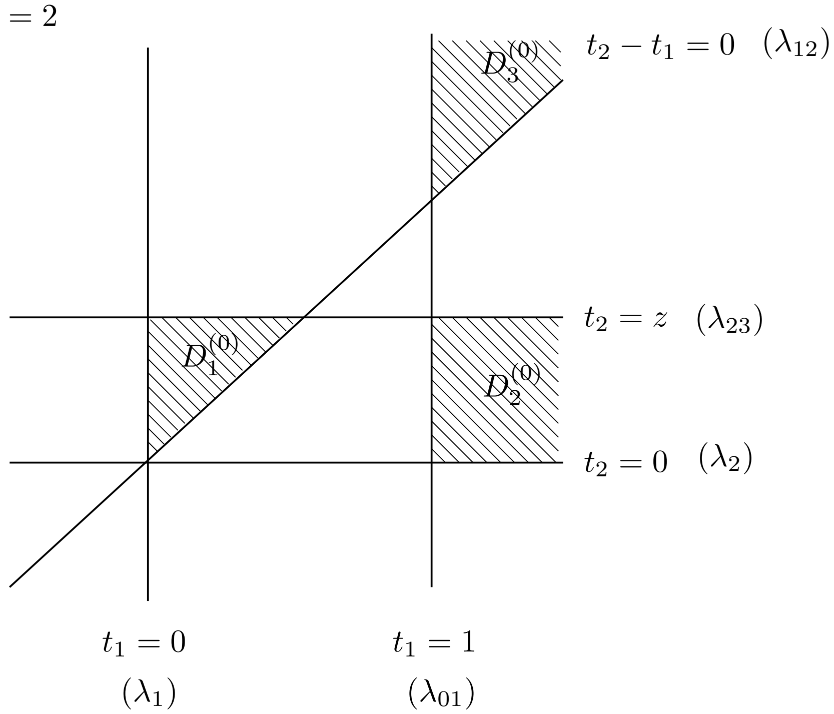

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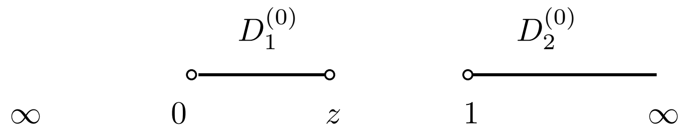

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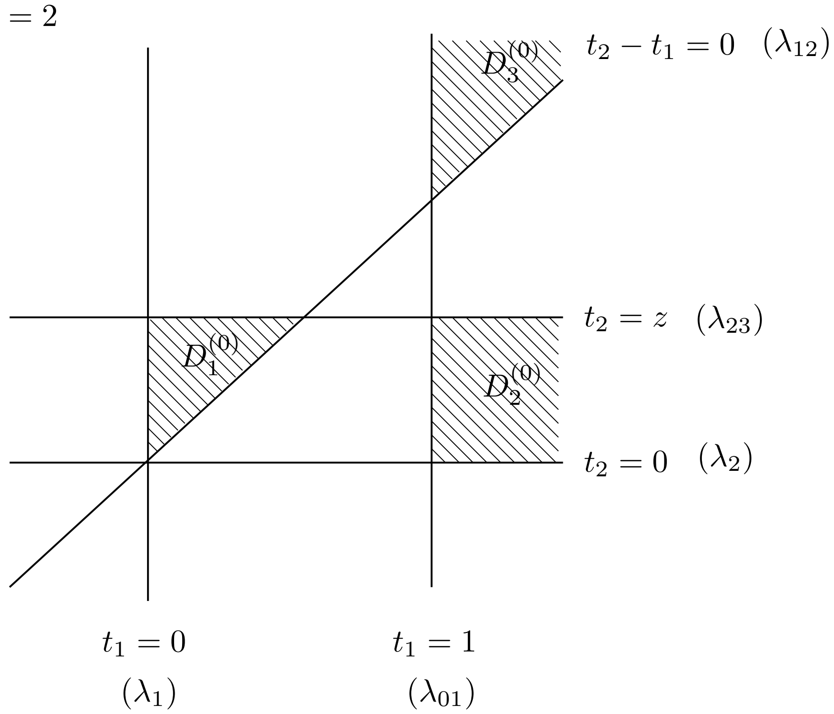

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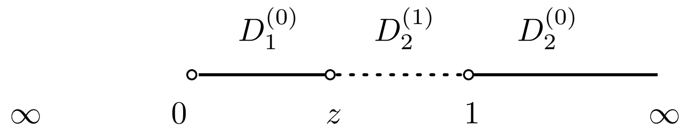

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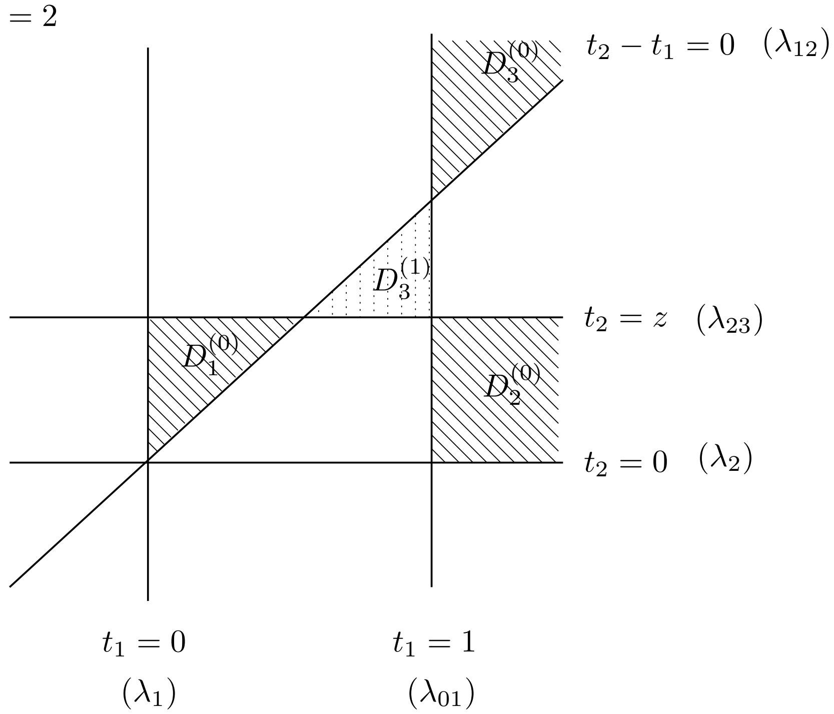

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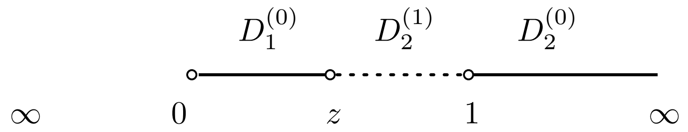

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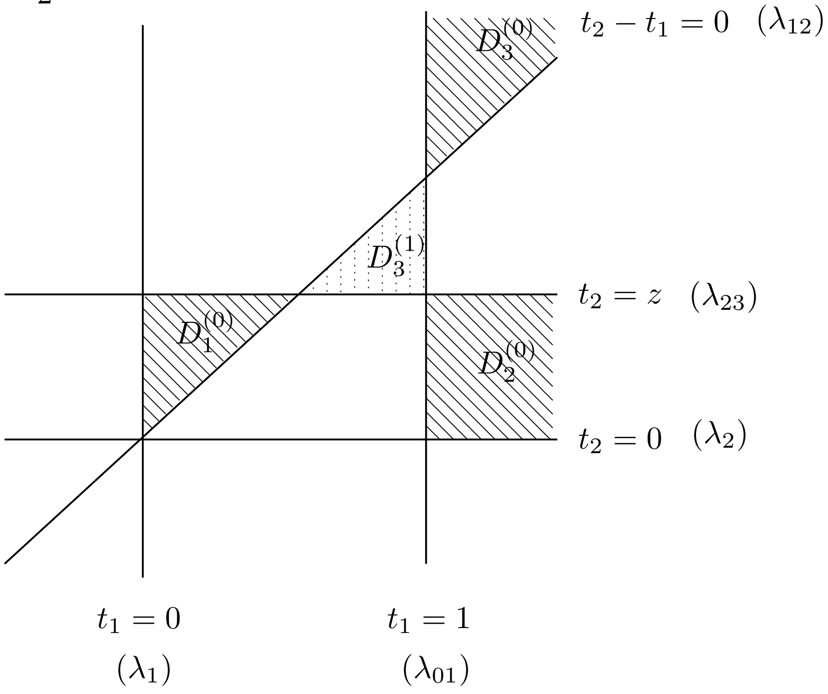


$\Longrightarrow \exists\, c_j$ such that

$$D_{n+1}^{(1)} = \sum_{1 \le j \le n+1} c_j\, D_j^{(0)}$$

Intersection form (Intersection numbers)

The map

$$\mathrm{reg}\ :\ H_m^{\mathrm{lf}}(T,\mathcal{L})\ \longrightarrow\ H_m(T,\mathcal{L})$$

is defined as an inverse of the natural map

$$\iota\ :\ H_m(T,\mathcal{L})\ \longrightarrow\ H_m^{\mathrm{lf}}(T,\mathcal{L}).$$

To define the intersection numbers for $C, C' \in H_m^{\mathrm{lf}}(T,\mathcal{L})$, we first regularize one of them, secondly compute the intersection number of the consequent cycles and finally sum up them. Actually, the *intersection form*

$$\langle\quad,\quad\rangle : H_n^{\mathrm{lf}}(T,\mathcal{L}) \times H_n^{\mathrm{lf}}(T,\mathcal{L}) \longrightarrow \mathbb{C}$$

is the Hermitian form defined by

$$(C, C') \longmapsto \langle C, C'\rangle = \sum_{\rho,\,\sigma} a_\rho\, \overline{a'_\sigma} \sum_{t\in\rho\cap\sigma} I_t(\rho,\sigma) v_\rho(t)\overline{v'_\sigma(t)}/|u|^2,$$

for C, $C' \in H_m^{\mathrm{lf}}(T,\mathcal{L})$, if

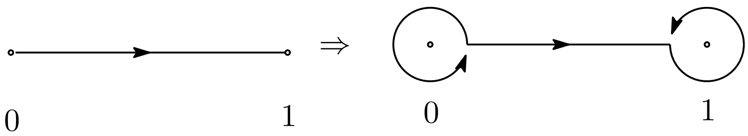
$$\mathrm{reg}\ C = \sum_\rho a_\rho\, \rho \otimes v_\rho, \quad C' = \sum_\sigma a'_\sigma\, \sigma \otimes v'_\sigma,$$

where $a_\rho, a'_\sigma \in \mathbb{C}$, ρ, σ:n-simplex, v_ρ, v'_σ: a section of $\mathcal{L}$ on ρ, σ, $^-$: the complex conjugation, $I_t(\rho,\sigma)$: the topological intersection number of ρ and σ at t.

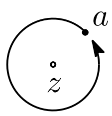
The value $\langle C, C'\rangle$ is called the *intersection number* of C and C' and written also by $C \bullet C'$

Example of regularization. $T = \mathbb{C}\backslash\{0,1\}$, $u(t) = t^\alpha(1-t)^\beta$.

$$\overrightarrow{(0,1)} \quad \Rightarrow \quad \mathrm{reg}\,\overrightarrow{(0,1)} = \left\{ \frac{1}{d_\alpha} S(\epsilon\,;0) + \overrightarrow{[\epsilon, 1-\epsilon]} - \frac{1}{d_\beta} S(1-\epsilon\,;1) \right\}$$



Here $d_a = e(2a) - 1, \quad e(a) = \exp(\pi\sqrt{-1}a)$. The symbol $S(a\,;z)$ stands for the positively oriented circle centered at the point z with starting and ending point a, ϵ is a small positive number and the argument of each factor of $u(t)$ on the oriented circle $S(\epsilon\,;0)$ or $S(1-\epsilon\,;1)$ is defined so that $\arg t$ takes value from 0 to 2π on $S(\epsilon\,;0)$, and $\arg(1-t)$ from 0 to 2π on $S(1-\epsilon\,;1)$.



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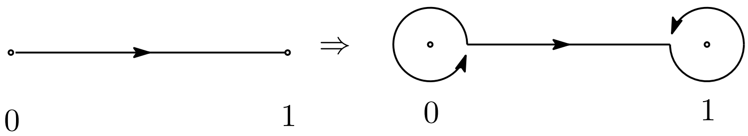

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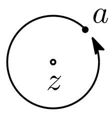

Examples of intersection numbers.

$$\overrightarrow{(0,1)} \bullet \overrightarrow{(0,1)} = -\frac{1}{d_\alpha} - 1 + \frac{-1}{d_\beta}$$
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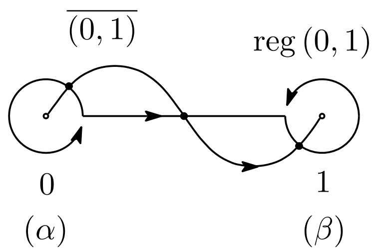
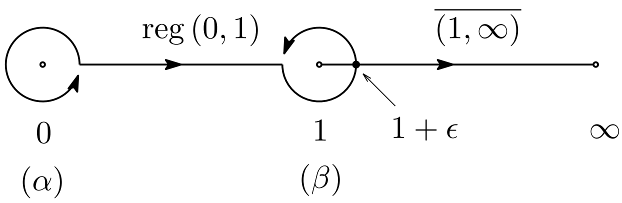
where $s(a) = \sin(\pi a)$



$$\overrightarrow{(0,1)} \bullet \overrightarrow{(1,\infty)} = \frac{e(\beta)}{e(2\beta) - 1}$$



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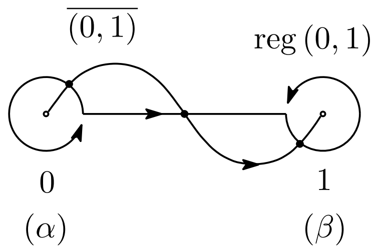

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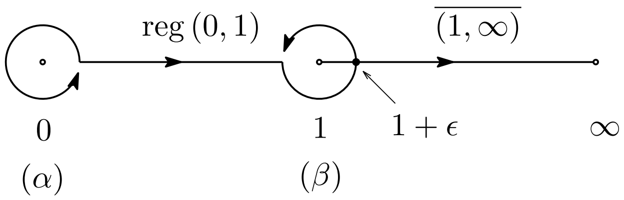

Connection coefficients

$$D_{n+1}^{(1)} = \sum_{1 \leq j \leq n+1} c_j \, D_j^{(0)}.$$

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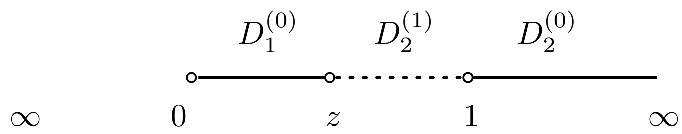

$$\Longrightarrow \quad \begin{array}{l} D_1^{(0)} \bullet D_2^{(0)} = 0 \\ D_2^{(0)} \bullet D_1^{(0)} = 0 \end{array}$$

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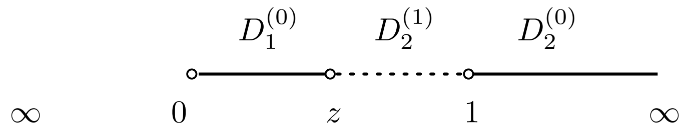

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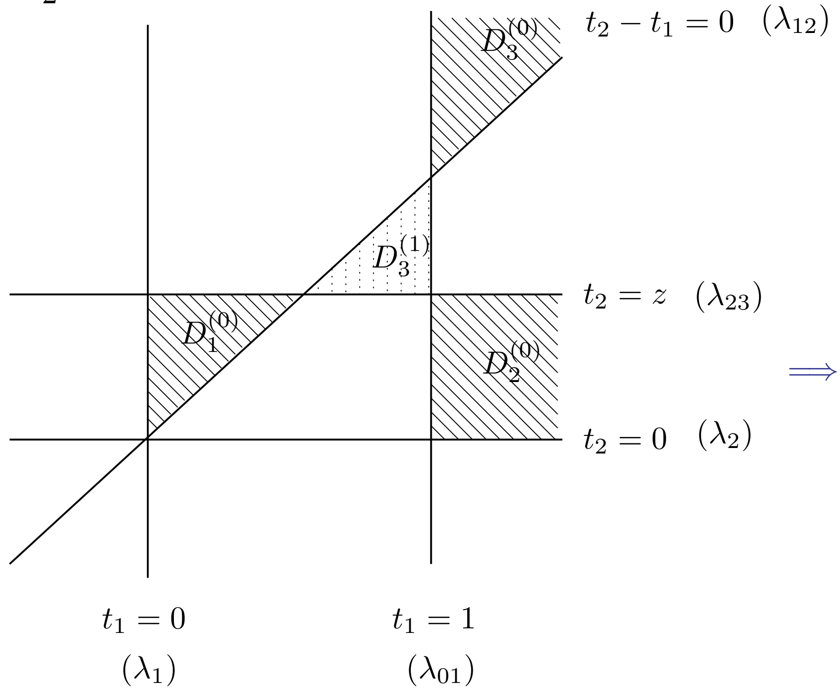

$$\Longrightarrow \quad \begin{array}{l} D_i^{(0)} \bullet D_j^{(0)} = 0 \\ (i \neq j) \end{array}$$

$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\frac{\sqrt{-1}}{2} \right)^n \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \frac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s) \sin(\alpha_s - \beta_j)},$$

$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\frac{\sqrt{-1}}{2} \right)^n \prod_{\substack{1 \leq s \leq n+1 \\ s \neq j}} \frac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s) \sin(\alpha_s - \beta_j)},$$

$$D_{n+1}^{(1)} \bullet D_j^{(0)} = \left(\frac{\sqrt{-1}}{2} \right)^n \prod_{\substack{1 \leq s \leq n+1 \\ s \neq j}} \frac{1}{\sin(\beta_s - \alpha_s)},$$

where $\beta_{n+1} = 1$.

$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\frac{\sqrt{-1}}{2} \right)^n \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \frac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s) \sin(\alpha_s - \beta_j)},$$

$$D_{n+1}^{(1)} \bullet D_j^{(0)} = \left(\frac{\sqrt{-1}}{2} \right)^n \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \frac{1}{\sin(\beta_s - \alpha_s)},$$

where $\beta_{n+1} = 1$.

$$\Longrightarrow \quad c_j = \frac{D_{n+1}^{(1)} \bullet D_j^{(0)}}{D_j^{(0)} \bullet D_j^{(0)}} = \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \frac{\sin(\alpha_s - \beta_j)}{\sin(\beta_s - \beta_j)}$$

Lemma 3. *For i, j such that $1 \leq i, j \leq n+1$, if $\alpha_i - \beta_j \notin \mathbb{Z}$ and $\beta_i - \beta_j \notin \mathbb{Z}$ $(i \neq j)$, then we have*

$$D_{n+1}^{(1)} = \sum_{j=1}^{n+1} \prod_{\substack{1 \leq s \leq n+1 \\ s \neq j}} \frac{s(\alpha_s - \beta_j)}{s(\beta_s - \beta_j)} \times D_j^{(0)}.$$

Lemma 3. *For i, j such that $1 \leq i, j \leq n+1$, if $\alpha_i - \beta_j \notin \mathbb{Z}$ and $\beta_i - \beta_j \notin \mathbb{Z}$ $(i \neq j)$, then we have*

$$D^{(1)}_{n+1} = \sum_{j=1}^{n+1} \prod_{\substack{1 \leq s \leq n+1 \\ s \neq j}} \frac{s(\alpha_s - \beta_j)}{s(\beta_s - \beta_j)} \times D^{(0)}_j.$$

Therefore, we get

$$\begin{aligned}
&\gamma_1(D^{(0)}_{j+1}) \\
&= D^{(0)}_{j+1} + 2\sqrt{-1}e\left(\sum_{s=1}^{n} \beta_s - \sum_{s=1}^{n+1} \alpha_s\right) s(\alpha_{j+1} - \beta_{j+1}) D^{(1)}_{n+1} \\
&= D^{(0)}_{j+1} + 2\sqrt{-1}e\left(\sum_{s=1}^{n} \beta_s - \sum_{s=1}^{n+1} \alpha_s\right) s(\alpha_{j+1} - \beta_{j+1}) \sum_{i=1}^{n+1} \prod_{\substack{1 \leq s \leq n+1 \\ s \neq i}} \frac{s(\alpha_s - \beta_i)}{s(\beta_s - \beta_i)} \times D^{(0)}_i.
\end{aligned}$$

Theorem. Suppose that

$$\Xi = {}^t(s(\beta_1-\alpha_1)^{-1}D_1, s(\beta_2-\alpha_2)^{-1}D_2, \ldots, s(\beta_{n+1}-\alpha_{n+1})^{-1}D_{n+1}).$$

Then

$$\gamma_i^*\Xi = \rho(\gamma_i)\Xi \quad (i=1,2),$$

where

$$\rho(\gamma_0) = \begin{bmatrix} e(-2\beta_n) & 0 & 0 & \cdots & 0 & 0 \\ 0 & e(-2\beta_{n-1}) & 0 & \cdots & 0 & 0 \\ 0 & 0 & e(-2\beta_{n-2}) & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & e(-2\beta_1) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

and

$$\rho(\gamma_1) = I - 2\sqrt{-1}\, e(\sum_{i=1}^{n}\beta_i - \sum_{i=1}^{n+1}\alpha_i) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Bigg[\frac{\sum_{1\le s\le n+1} s(\beta_1-\alpha_s)}{\sum_{\substack{1\le s\le n+1 \\ s\ne 1}} s(\beta_1-\beta_s)},$$

$$\ldots, \frac{\sum_{1\le s\le n+1} s(\beta_n-\alpha_s)}{\sum_{\substack{1\le s\le n+1 \\ s\ne n}} s(\beta_n-\beta_s)}, \frac{\sum_{1\le s\le n+1} s(\beta_{n+1}-\alpha_s)}{\sum_{\substack{1\le s\le n+1 \\ s\ne n+1}} s(\beta_{n+1}-\beta_s)} \Bigg].$$

Theorem. Suppose that

$$\Xi = {}^t(s(\beta_1-\alpha_1)^{-1}D_1, s(\beta_2-\alpha_2)^{-1}D_2, \ldots, s(\beta_{n+1}-\alpha_{n+1})^{-1}D_{n+1}).$$

Then

$$\gamma_i^*\Xi = \rho(\gamma_i)\Xi \quad (i=1,2),$$

where

$$\rho(\gamma_0) = \mathrm{diag}\big(\, e(-2\beta_n),\, e(-2\beta_{n-1}),\, \ldots,\, e(-2\beta_1),\, 1\,\big)$$

and

$$\rho(\gamma_1) = I - 2\sqrt{-1}\, e(\sum_{i=1}^{n}\beta_i - \sum_{i=1}^{n+1}\alpha_i) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \left[\frac{\sum_{1\le s\le n+1} s(\beta_1-\alpha_s)}{\sum_{\substack{1\le s\le n+1\\ s\neq 1}} s(\beta_1-\beta_s)}, \right.$$

$$\left. \ldots, \frac{\sum_{1\le s\le n+1} s(\beta_n-\alpha_s)}{\sum_{\substack{1\le s\le n+1\\ s\neq n}} s(\beta_n-\beta_s)}, \frac{\sum_{1\le s\le n+1} s(\beta_{n+1}-\alpha_s)}{\sum_{\substack{1\le s\le n+1\\ s\neq n+1}} s(\beta_{n+1}-\beta_s)} \right].$$

$(\beta_{n+1} = 1)$

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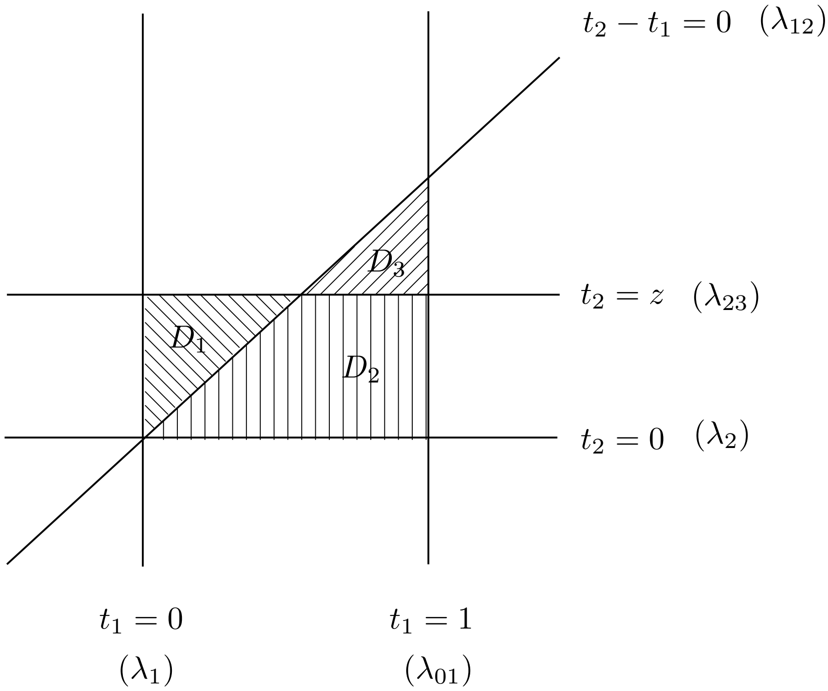

$t_2 - t_1 = 0 \quad (\lambda_{12})$
D_3
$t_2 = z \quad (\lambda_{23})$
D_1
D_2
$t_2 = 0 \quad (\lambda_2)$
$t_1 = 0$
(λ_1)
$t_1 = 1$
(λ_{01})

$$D_j = \{\ 0 < t_j < \cdots < t_n < z,\ 1 < t_1 < \cdots < t_{j-1} < \infty\ \}$$

Proposition 1. *For $k = 0, 1$ and $1 \leq j \leq n+1$, we have*

$$\gamma_k^*(D_j) = \sum_{i=1}^{n+1} D_i m_{ij}^{(k)},$$

where

$$m_{ij}^{(0)} = \begin{cases} 0 & \ldots\ 1 \leq j < i \leq n+1, \\ e(2\sum_{t=i}^{n} \mu_t) & \ldots\ 1 \leq i = j \leq n, \\ 1 & \ldots\ i = j = n+1, \\ (-1)^{i-j} e(\sum_{t=i}^{j-1} \mu_t + \sum_{t=i+1}^{j-1} \lambda_t + 2\sum_{t=j}^{n} \mu_t)\langle e(\lambda_i)\rangle & \ldots\ 1 \leq i < j \leq n+1, \end{cases}$$

and

$$m_{ij}^{(1)} = \begin{cases} 0 & \ldots\ 1 \leq j < i \leq n+1, \\ 1 & \ldots\ 1 \leq j = i \leq n, \\ e(2\sum_{t=0}^{n} \lambda_{t,t+1}) & \ldots\ j = i = n+1, \\ (-1)^{n-j+1} e(\sum_{t=0}^{n} \lambda_{t,t+1})\langle e(\sum_{t=0}^{j-1} \lambda_{t,t+1})\rangle & \ldots\ i = n+1, 1 \leq j \leq n, \end{cases}$$

with $\mu_t = \lambda_t + \lambda_{t,t+1}$.

Connection formulas

$$f_{n+1}^{(1)}(z) = \sum_{j=1}^{n+1} \prod_{s\neq i} \frac{\Gamma(1+\sum_{s=1}^{n}\beta_s - \sum_{s=1}^{n+1}\alpha_s)\prod_{\substack{1\leq s\leq n+1\\ s\neq i}}\Gamma(\beta_j-\beta_s)}{\prod_{1\leq s\leq n+1}\Gamma(\beta_j-\alpha_s)} \times f_j^{(0)}(z)\,,$$

where $f_i^{(0)}(z) = z^{1-\beta_i}(1+O(z)), f_{n+1}^{(1)}(z) = (1-z)^{\sum_{i=1}^{n}\beta_i - \sum_{i=1}^{n+1}\alpha_i}(1+O(1-z))$.

New title: Monodromy representations associated with the generalized hypergeometric function ${}_{n+1}F_n$

Abstract: We denote by ${}_{n+1}E_n$ the ordinary differential equation satisfied by the generalized hypergeometric function ${}_{n+1}F_n$. The equation ${}_{n+1}E_n$ is of rank $n+1$ with regular singular points $0, 1$ and ∞. Let Ξ be a fundamental set of solutions of ${}_{n+1}E_n$. Let γ_0, γ_1 be the generators of the fundamental group $\pi_1(\mathbb{C}\backslash\{0,1\})$, where γ_0 is the path encircling the point 0 with counterclockwise direction and γ_0 is the path encircling the point 1 with counterclockwise direction. The action of $\pi_1(\mathbb{C}\backslash\{0,1\})$ on the sheaf of the solution space of ${}_{n+1}E_n$ is the *monodromy representation on the solution space* of ${}_{n+1}E_n$. Our concern is such a representation with respect to Ξ. Indeed, choose a set Ξ on which γ_0 acts diagonally, and determine the matrix elements of the action of γ_1. The purpose of the present talk is to solve this problem, especially from the view point of integrals of multivalued functions, or the twisted homology theory. A connection relation obtained by using the intersection numbers of the twisted cycles play a crucial role on the way. If we have enough time, we would discuss another realization of the monodromy representation on the solution space of ${}_{n+1}E_n$.

On the other hand,

$$\int_{D_i^{(0)}} u_{D_i^{(0)}}(t)\, dt_1 \cdots dt_n = \prod_{\substack{1\le s\le n+1 \\ s\neq i}} B(\alpha_s - \beta_i + 1, \beta_s - \alpha_s) \times f_i^{(0)}(z),$$

$$\int_{D_{n+1}^{(1)}} u_{D_{n+1}^{(1)}}(t)\, dt_1 \cdots dt_n$$

$$= \prod_{1\le s\le n} B(\beta_1 + \cdots + \beta_s - \alpha_1 - \cdots - \alpha_s, \beta_{s+1} - \alpha_{s+1}) \times f_{n+1}^{(1)}(z).$$

For $u(t) = \prod_i f_i(t)^{\mu_i}$, $u_D(t) = \prod_i (\epsilon_i\, f_i(t))^{\mu_i}$, where $\epsilon_i = \pm$ is determined so that $\epsilon_i\, f_i(t) > 0$ on D.

Proposition 3.1. (1) *For a fixed i such that $1 \leq i \leq n+1$, if* $\mathrm{Re}\,(\alpha_s - \beta_i + 1) > 0$ *and* $\mathrm{Re}\,(\beta_s - \alpha_s) > 0$ *for $1 \leq s \leq n+1$ with $s \neq i$, and $|z| < 1$, we have*

$$\int_{D_i^{(0)}} u_{D_i^{(0)}}(t)\, dt_1 \cdots dt_n$$

$$= \int_{\substack{0<t_i<\cdots<t_n<z \\ 1<t_1<\cdots<t_{i-1}<\infty}} (t_{i-1} - t_i)^{\beta_i - \alpha_i - 1} \prod_{s=1}^{i-1} \left\{ t_s^{\alpha_{s+1} - \beta_s} (t_s - t_{s-1})^{\beta_s - \alpha_s - 1} \right\}$$

$$\times \prod_{s=i}^{n} \left\{ t_s^{\alpha_{s+1} - \beta_s} (t_{s+1} - t_s)^{\beta_{s+1} - \alpha_{s+1} - 1} \right\} dt_1 \cdots dt_n$$

$$= \prod_{\substack{1 \leq s \leq n+1 \\ s \neq i}} B(\alpha_s - \beta_i + 1, \beta_s - \alpha_s) f_i^{(0)}(z),$$

where $t_0 = 1, t_{n+1} = z, \beta_{n+1} = 1$, *and*

$$f_i^{(0)}(z) = z^{1-\beta_i}$$

$$\times\ {}_{n+1}F_n \left(\begin{matrix} \alpha_1 - \beta_i + 1,\ \alpha_2 - \beta_i + 1,\ \ldots\ \ldots\ \ldots\ \alpha_{n+1} - \beta_i + 1 \\ \beta_1 - \beta_i + 1,\ \ldots,\ \widehat{\beta_i - \beta_i + 1},\ \ldots,\ \beta_{n+1} - \beta_i + 1 \end{matrix} ; z \right).$$

(2) *If* $\mathrm{Re}\,(\beta_1 + \cdots + \beta_s - \alpha_1 - \cdots - \alpha_s) > 0$ *for* $1 \leq s \leq n$, $\mathrm{Re}\,(\beta_s - \alpha_s) > 0$ *for* $1 \leq s \leq n+1$, *and* $|1 - z| < 1$, *we have*

$$\int_{D_{n+1}^{(1)}} u_{D_{n+1}^{(1)}}(t)\, dt_1 \cdots dt_n$$

$$= \int_{z<t_n<\cdots<t_1<1} \prod_{s=1}^{n} t_s^{\alpha_{s+1} - \beta_s} \prod_{s=1}^{n+1} (t_{s-1} - t_s)^{\beta_s - \alpha_s - 1}\, d\,t_1 \cdots dt_n$$

$$= \prod_{s=1}^{n} B(\beta_1 + \cdots + \beta_s - \alpha_1 - \cdots - \alpha_s, \beta_{s+1} - \alpha_{s+1})\, f_{n+1}^{(1)}(z),$$

where $t_0 = 1, t_{n+1} = z, \beta_{n+1} = 1$, *and*

$$f_{n+1}^{(1)}(z) = (1-z)^{\beta_1+\cdots+\beta_n-\alpha_1-\cdots-\alpha_{n+1}}$$

$$\times \sum_{i_1,\ldots,i_n\geq 0} \prod_{s=1}^{n} \frac{(\beta_s-\alpha_{s+1})_{i_s}}{i_s!} \prod_{s=1}^{n} \frac{(\sum_{k=1}^{s}(\beta_k-\alpha_k))_{i_1+\cdots+i_s}}{(\sum_{k=1}^{s+1}(\beta_k-\alpha_k))_{i_1+\cdots+i_s}} (1-z)^{i_1+\cdots+i_n}.$$

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The *genericity condition* for the exponents λ_i and $\lambda_{i-1,i}$ is:

$$\lambda_{t_{i-1}=t_i},\ \lambda_{0=t_p=t_{p+1}=\cdots=t_q},\ \lambda_{t_p=t_{p+1}=\cdots=t_q=\infty} \notin \mathbb{Z},$$

for $1 \le i \le n+1$ and $1 \le p \le q \le n$, where

$$\lambda_{t_{i-1}=t_i} := \lambda_{i-1,i}\ ,$$

$$\lambda_{0=t_p=t_{p+1}=\cdots=t_q} := \sum_{s=p}^{q} \lambda_s + \sum_{s=p+1}^{q} \lambda_{s-1,s}\ ,$$

$$\lambda_{t_p=t_{p+1}=\cdots=t_q=\infty} := -\sum_{s=p}^{q} \lambda_s - \sum_{s=p}^{q+1} \lambda_{s-1,s}\ .$$

and

$$\lambda_{t_{i-1}=t_i} = \beta_i - \alpha_i - 1\,,$$

$$\lambda_{0=t_p=t_{p+1}=\cdots=t_q} = \alpha_{q+1} - \beta_p - (q-p)\,,$$

$$\lambda_{t_p=t_{p+1}=\cdots=t_q=\infty} = \alpha_p - \beta_{q+1} + 2 + q - p\,.$$