The Painlevé equations and q-Askey scheme

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This relation relies on the τ-function

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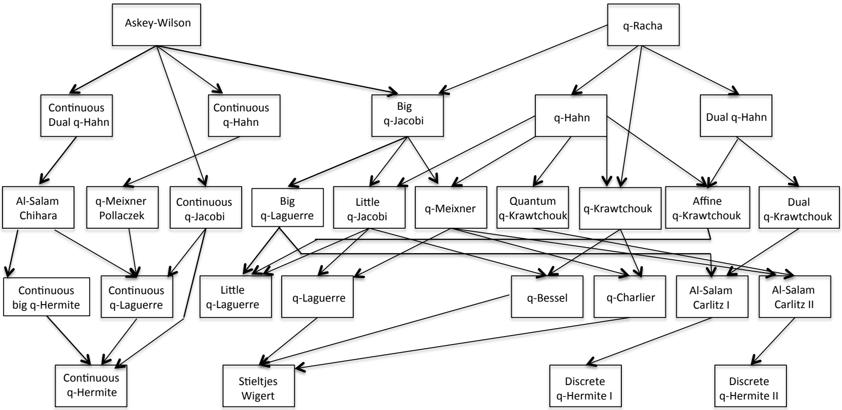
- These quantum algebras admit representations on the space of Laurent polynomials.
- In this representation an operator acts as multiplication, the other as q-difference operator
- The eigenvaules of the q-difference operator are basic hypergeometric polynomials (q-Askey scheme).

M.M. arXiv:1307.6140

q-Askey scheme

Koekoek, Lesky, Swarttouw 2010



Painlevé equations and q-Askey polynomials

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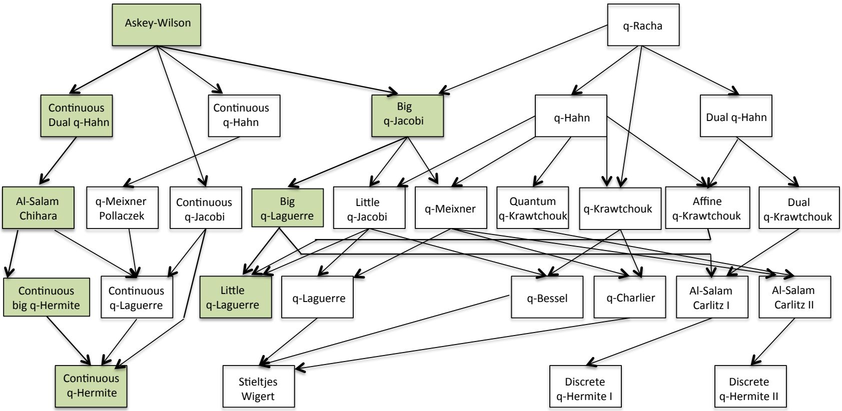

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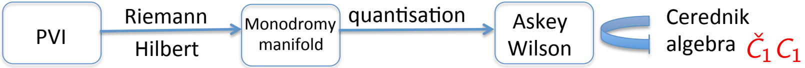

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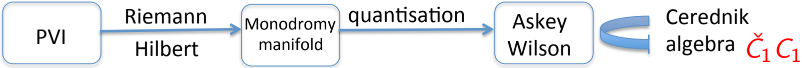

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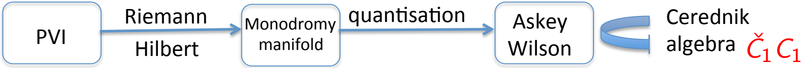

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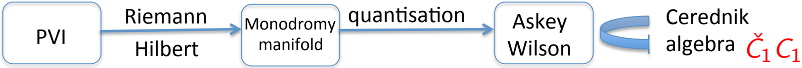

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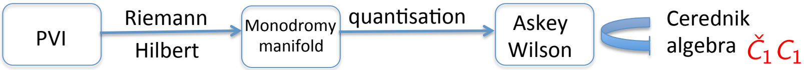

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- Confluence Askey Wilson algebra,
- Confluence Cherednik algebra $\Rightarrow$ seven new algebras. M.M. arXiv:1307.6140

Cherednik algebra of type $\check{C}_1 C_1$ Cherednik '92, Sahi '99

Algebra generated by $V_0, V_1, \check{V}_0, \check{V}_1$:

$$\begin{aligned}
(V_0 - k_0)(V_0 + k_0^{-1}) &= 0\\
(V_1 - k_1)(V_1 + k_1^{-1}) &= 0\\
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\check{V}_1 V_1 V_0 \check{V}_0 &= q^{-1/2},
\end{aligned}$$

k_0, k_1, u_0, u_1 scalars.

Spherical sub-algebra algebra:

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$e\mathcal{H}e = \langle X_1, X_2, X_3 \rangle$

$$q^{-1/2}X_1X_2 - q^{1/2}X_2X_1 = (q^{-1}-q)X_3 - (q^{-1/2}-q^{1/2})\omega_3 e$$

$$q^{-1/2}X_2X_3 - q^{1/2}X_3X_2 = (q^{-1}-q)X_1 - (q^{-1/2}-q^{1/2})\omega_1 e$$

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Casimir:

$$q^{\frac{1}{2}}X_2X_1X_3 - qX_2^2 - q^{-1}X_1^2 - qX_3^2 + q^{\frac{1}{2}}\omega_2X_2 + q^{-\frac{1}{2}}\omega_1X_1 + q^{\frac{1}{2}}\omega_3X_3 = \omega_4 e.$$

Zhedanov '91, Oblomkov '04

Recap:

- Cherednik algebra of type $\check{C}_1 C_1$:

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Represented on Laurent polynomials Non-symmetric Askey-Wilson (Sahi, Noumi-Stokman)

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Represented on symmetric Laurent polynomials (Askey-Wilson)

Sixth Painlevé equation

$$
\begin{aligned}
y_{tt} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)y_t^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)y_t+\\
&+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right].
\end{aligned}
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Singular points at $0, 1, \infty$.
Parameters $\alpha, \beta, \gamma, \delta$.

All Painlevé equations are **isomonodromic deformation equations** (Jimbo-Miwa 1980)

$$\frac{\mathrm{d}B}{\mathrm{d}\lambda} - \frac{\mathrm{d}A}{\mathrm{d}t} = [A, B]$$

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$A = A(\lambda; t, y, y_t),\ B = B(\lambda; t, y, y_t) \in \mathfrak{sl}_2$.
This means that **the monodromy data** of the linear system

$$\frac{\mathrm{d}Y}{\mathrm{d}\lambda} = A(\lambda; t, y, y_t)Y$$

are **locally constant along solutions of the Painlevé equation**.

PVI as isomonodromic deformations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} Y = \sum_{k=1}^{3} \frac{A_k}{\lambda - a_k} Y, \qquad \lambda \in \mathbb{C} \backslash \{a_1, a_2, a_3\}$$

$A_1, A_2, A_3 \in \mathfrak{sl}(2, \mathbb{C}), \qquad \sum_{k=1}^{3} A_k = -A_\infty$, diagonal.

- Fundamental matrix: $Y_\infty(\lambda) = \left(1 + O(\frac{1}{\lambda})\right) \lambda^{-A_\infty}$
- Monodromy matrices: $\gamma_j(Y_\infty) = Y_\infty M_j$

$\mathrm{eigenv}(M_j) = \exp(\pm \frac{p_j}{2}), \quad i = 1, 2, 3, \infty,$

$M_\infty M_1 M_2 M_3 = 1.$

Jimbo, Miwa '81

Cherednik algebra as quantisation of the monodromy group

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\begin{aligned}
(M_3 - e^{\frac{p_3}{2}})(M_3 - e^{-\frac{p_3}{2}}) &= 0, \\
(M_2 - e^{\frac{p_2}{2}})(M_2 - e^{-\frac{p_2}{2}}) &= 0, \\
(M_1 - e^{\frac{p_1}{2}})(M_1 - e^{-\frac{p_1}{2}}) &= 0, \\
(M_\infty - e^{\frac{p_\infty}{2}})(M_\infty - e^{-\frac{p_\infty}{2}}) &= 0 \\
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There is a natural quantisation which works.

$$
M_1 \to i\check{V}_1, \quad M_2 \to iV_1, \quad M_3 \to iV_0, \quad M_\infty \to i\check{V}_0.
$$

$$
u_1 = -ie^{-\frac{p_1}{2}}, \quad k_0 = -ie^{-\frac{p_3}{2}}, \quad k_1 = -ie^{-\frac{p_2}{2}}, \quad u_0 = -ie^{-\frac{p_\infty}{2}}.
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Riemann Hilbert correspondence: $\forall (M_1, M_2, M_3)/SL_2(\mathbb{C})$ there exists a unique local solution to PVI modulo Okamoto transformations.

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The relation $M_\infty M_3 M_2 M_1 = 1$ gives:

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This cubic is the moduli space of monodromy representations:

$$\rho : \pi_1(\overline{\mathbb{C}} \setminus \{0, t, 1, \infty\}) \to SL_2(\mathbb{C}).$$

Jimbo '81, Iwasaki '03

Monodromy manifold and spherical sub–algebra

Natural Poisson bracket on

$$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 - \omega_4 = 0,$$

defines a natural Poisson bracket:

$$\{x_1, x_2\} = x_1x_2 + 2x_3 + \omega_3, \qquad \text{and cyclic.}$$

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....cyclic...

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- Complexify the flat coordinates to describe the character variety.
- Quantise flat coordinates.

Poincaré uniformsation

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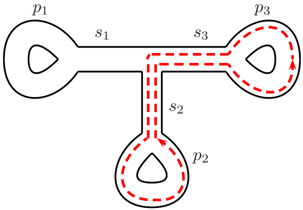
where Δ is a *Fuchsian group*, i.e. a discrete sub-group of $\mathbb{P}SL_2(\mathbb{R})$. Decompose each hyperbolic element in Right, Left and Edge matrices Fock, Thurston

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, X_y := \begin{pmatrix} 0 & -e^{\frac{y}{2}} \\ e^{-\frac{y}{2}} & 0 \end{pmatrix}.$$



Teichmüller space of a Riemann sphere with 4 singularities

Interpret each x_i as a geodesic length:

$$x_1 = e^{s_2+s_3} + e^{-s_2-s_3} + e^{-s_2+s_3} + (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{s_3} + (e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}})e^{-s_2}$$
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Complexify $s_1, s_2, s_3 \Rightarrow$ flat coordinates on the Character variety.

Quantisation:

$s_i \to$ quantum operator $s_i^\hbar$ with commutation relation

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$\Rightarrow$ Weyl ordering:

$$\exp\left(s_i^\hbar\right)\exp\left(s_j^\hbar\right) = \exp\left(s_i^\hbar + s_i^\hbar + \frac{1}{2}[s_i^\hbar, s_j^\hbar]\right),$$

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Quantum algebra. Zhedanov algebra $q = e^{-i\pi\hbar}$:

$$\begin{aligned} q^{-1/2}x_1^\hbar x_2^\hbar - q^{1/2}x_2^\hbar x_1^\hbar &= (q^{-1}-q)x_3^\hbar + (q^{-1/2}-q^{1/2})\omega_3 \\ q^{-1/2}x_2^\hbar x_3^\hbar - q^{1/2}x_3^\hbar x_2^\hbar &= (q^{-1}-q)x_1^\hbar + (q^{-1/2}-q^{1/2})\omega_1 \\ q^{-1/2}x_3^\hbar x_1^\hbar - q^{1/2}x_1^\hbar x_3^\hbar &= (q^{-1}-q)x_2^\hbar + (q^{-1/2}-q^{1/2})\omega_2 \end{aligned}$$

(L. Chekhov and M.M. J.Phys A 2010).

Quantise the monodromy matrices: each monodromy matrix corresponds to a half–geodesic on our Riemann surface.

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Quantise them in the same way: we obtain the Cherednik algebra of type $\check{C}_1 C_1$ *(M.M. arXiv:1307.6140)*

$$M_1 \to i\check{V}_1, \quad M_2 \to iV_1, \quad M_3 \to iV_0, \quad M_\infty \to i\check{V}_0.$$

$$u_1 = -ie^{-\frac{p_1}{2}}, \quad k_0 = -ie^{-\frac{p_3}{2}}, \quad k_1 = -ie^{-\frac{p_2}{2}}, \quad u_0 = -ie^{-S_1-S_2-S_3}.$$

Embedding of the Cherednik algebra of type $\check{C}_1 C_1$ into $Mat(2,\mathbb{T}_q)$

$$V_0=\begin{pmatrix} k_0-k_0^{-1}-\ ie^{-s_3} & -i\,e^{-s_3}\\ k_0^{-1}-k_0+i\,e^{-s_3}+i\,e^{s_3} & i\,e^{-s_3}\end{pmatrix},$$

$$V_1=\begin{pmatrix} k_1-k_1^{-1}-i\,e^{s_2} & k_1-k_1^{-1}-i\,e^{-s_2}-i\,e^{s_2}\\ i\,e^{s_2} & i\,e^{s_2}\end{pmatrix},$$

$$\check{V}_1=\begin{pmatrix} 0 & -\ ie^{s_1}\\ i\,e^{-s_1} & u_1-u_1^{-1}\end{pmatrix},\qquad \check{V}_0=\begin{pmatrix} u_0 & 0\\ s & -\frac{1}{u_0}\end{pmatrix},$$

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$$V_1 = \begin{pmatrix} k_1 - k_1^{-1} - i\,e^{s_2} & k_1 - k_1^{-1} - i\,e^{-s_2} - i\,e^{s_2} \\ i\,e^{s_2} & i\,e^{s_2} \end{pmatrix},$$

$$\check{V}_1 = \begin{pmatrix} 0 & -ie^{s_1} \\ i\,e^{-s_1} & u_1 - u_1^{-1} \end{pmatrix}, \qquad \check{V}_0 = \begin{pmatrix} u_0 & 0 \\ s & -\frac{1}{u_0} \end{pmatrix},$$

$$e^{S_2}e^{S_1} = qe^{S_1}e^{S_2}, \quad e^{S_3}e^{S_2} = qe^{S_2}e^{S_3}, \quad e^{S_1}e^{S_3} = qe^{S_3}e^{S_1}.$$

Monodromy manifolds for the Painlevé equations

PVI $$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

PV $$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

PIV $$x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_2 x_3 + 1 = \omega_4$$

$PIII$ $$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 = \omega_1 - 1$$

PII $$x_1x_2x_3 + x_1 + x_2 + x_3 = \omega_4$$

PI $$x_1x_2x_3 + x_1 + x_2 + 1 = 0$$

Saito and van der Put

The confluence from PVI to PV is realised by

$$s_3 \to s_3 - \log[\epsilon], \qquad p_3 \to p_3 - 2\log[\epsilon], \qquad \epsilon \to 0$$

$$x_1 = -e^{s_2+s_3} - e^{-s_2+s_3} - (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{s_3} - e^{\frac{p_3}{2}} e^{-s_2}$$
$$x_2 = -e^{s_3+s_1} - e^{\frac{p_3}{2}} e^{s_1},$$
$$x_3 = -e^{s_1+s_2} - e^{-s_1-s_2} - e^{-s_1+s_2} - (e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}})e^{s_2} - (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{-s_1}$$

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$$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

Quantum PV algebra:

$$\begin{aligned}
q^{-1/2}x_1^\hbar x_2^\hbar - q^{1/2}x_2^\hbar x_1^\hbar &= (q^{-1/2} - q^{1/2})\omega_3 \\
q^{-1/2}x_2^\hbar x_3^\hbar - q^{1/2}x_3^\hbar x_2^\hbar &= (q^{-1} - q)x_1^\hbar + (q^{-1/2} - q^{1/2})\omega_1 \\
q^{-1/2}x_3^\hbar x_1^\hbar - q^{1/2}x_1^\hbar x_3^\hbar &= (q^{-1} - q)x_2^\hbar + (q^{-1/2} - q^{1/2})\omega_2
\end{aligned}$$

M.M. and V.Rubtsov arXiv:1212.6723

$$V_0 = \begin{pmatrix} -1 & 0 \\ 1 + i\, e^{S_3} & 0 \end{pmatrix}, \qquad V_0^2 + V_0 = 0$$

$$V_1 = \begin{pmatrix} k_1 - k_1^{-1} - i\, e^{S_2} & k_1 - k_1^{-1} - i\, e^{-S_2} - i\, e^{S_2} \\ i\, e^{S_2} & i\, e^{S_2} \end{pmatrix},$$

$$\check{V}_1 = \begin{pmatrix} 0 & -\, i e^{S_1} \\ i\, e^{-S_1} & u_1 - u_1^{-1} \end{pmatrix},$$

$$\check{V}_0 = \begin{pmatrix} 0 & 0 \\ s & -\frac{1}{u_0} \end{pmatrix}, \qquad \check{V}_0{}^2 + u_0^{-1}\check{V}_0 = 0$$

$$q^{1/2}\,\check{V}_1\,V_1\,V_0 = \check{V}_0 + u_0^{-1},$$

$$q^{1/2}\,\check{V}_0\,\check{V}_1\,V_1 = V_0 + 1.$$

$\mathcal{H}_V$:

$$\begin{aligned} V_0^2 + V_0 &= 0, \\ (V_1 - k_1)(V_1 + k_1^{-1}) &= 0, \\ \check{V}_0{}^2 + u_0^{-1}\check{V}_0 &= 0, \\ (\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) &= 0, \\ q^{1/2}\check{V}_1 V_1 V_0 &= \check{V}_0 + u_0^{-1}, \\ q^{1/2}\check{V}_0 \check{V}_1 V_1 &= V_0 + 1. \end{aligned}$$

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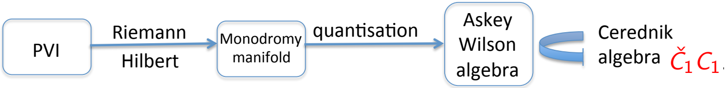
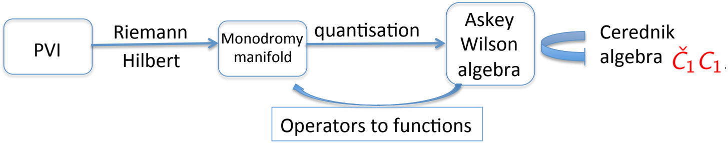
- Represented on the space of Laurent polynomials.
- Non-symmetric continuous dual q–Hahn polynomials. M.M. SIGMA 2014
- Spherical sub-algebra represented on the space of symmetric Laurent polynomials.

PIV:	PII:	PI:
$V_0^2+V_0=0,$	$V_0^2+V_0=0,$	$V_0^2=0,$
$(V_1-k_1)(V_1+k_1^{-1})=0,$	$V_1^2+V_1=0,$	$V_1^2+V_1=0,$
$\check{V}_1{}^2+u_1^{-1}\check{V}_1=0,$	$\check{V}_1{}^2+u_1^{-1}\check{V}_1=0,$	$\check{V}_1{}^2+\check{V}_1=0,$
$\check{V}_0{}^2+\check{V}_0=0,$	$\check{V}_0{}^2+\check{V}_0=0,$	$\check{V}_0{}^2+\check{V}_0=0,$
$\check{V}_1V_1V_0=\frac{\check{V}_0+1}{\sqrt{q}},$	$\check{V}_1V_1V_0=\frac{\check{V}_0+1}{\sqrt{q}},$	$\check{V}_1V_1V_0=\frac{\check{V}_0+1}{\sqrt{q}},$
$\check{V}_0\check{V}_1=0,$	$\check{V}_0\check{V}_1=0,$	$\check{V}_0\check{V}_1=0,$
$V_0\check{V}_0=0,$	$V_0\check{V}_0=0,$	$V_0\check{V}_0=0,$

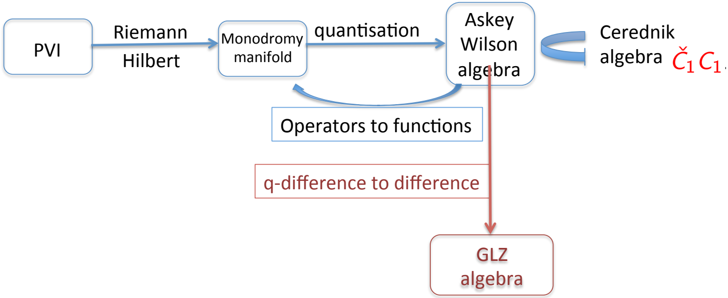
Outlook


PVI
Riemann
Hilbert
Monodromy
manifold
quantisation
Askey
Wilson
algebra
Cerednik
algebra

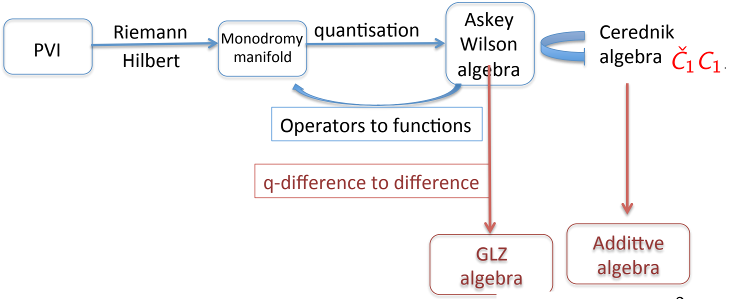
Outlook


PVI
Riemann
Hilbert
Monodromy
manifold
quantisation
Askey
Wilson
algebra
Cerednik
algebra $\check{C}_1 C_1$
Operators to functions

Outlook


PVI
Riemann
Hilbert
Monodromy
manifold
quantisation
Askey
Wilson
algebra
Cerednik
algebra
Operators to functions
q-difference to difference
GLZ
algebra

Outlook



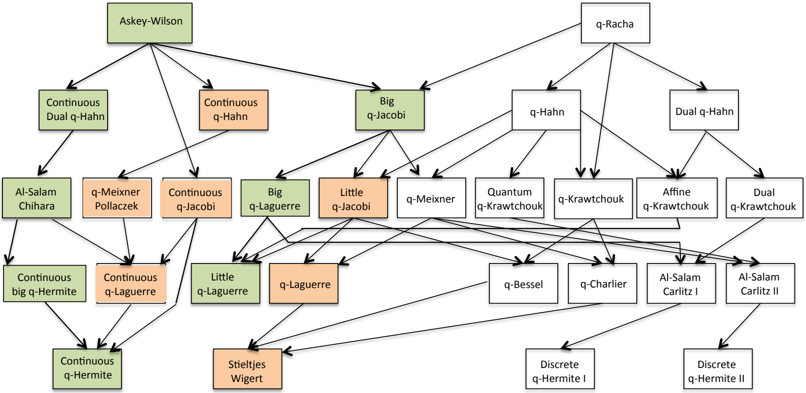
$$v_0^2 + \frac{(\delta+\gamma-1)^2}{4} = 0,$$

$$\check{v_0}^2 + \frac{(\delta-\gamma)^2}{4} = 0,$$

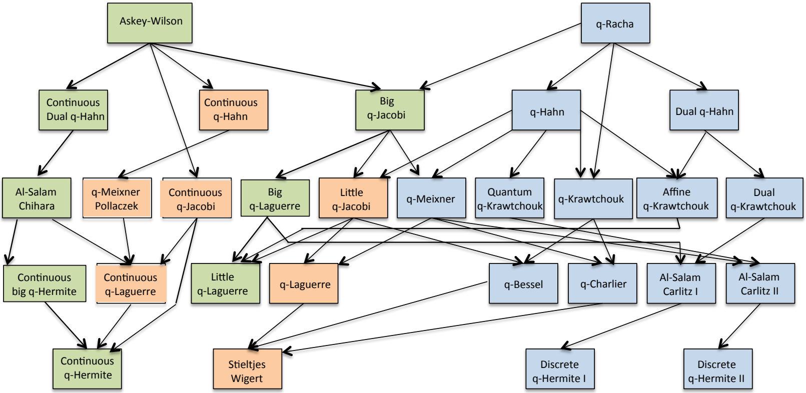
$$v_1^2 + \frac{(\alpha-\beta)^2}{4} = 0,$$

$$\check{v_1}^2 + \frac{(\alpha+\beta)^2}{4} = 0,$$

Outlook


Askey-Wilson
q-Racha
Continuous Dual q-Hahn
Continuous q-Hahn
Big q-Jacobi
q-Hahn
Dual q-Hahn
Al-Salam Chihara
q-Meixner Pollaczek
Continuous q-Jacobi
Big q-Laguerre
Little q-Jacobi
q-Meixner
Quantum q-Krawtchouk
q-Krawtchouk
Affine q-Krawtchouk
Dual q-Krawtchouk
Continuous big q-Hermite
Continuous q-Laguerre
Little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam Carlitz I
Al-Salam Carlitz II
Continuous q-Hermite
Stieltjes Wigert
Discrete q-Hermite I
Discrete q-Hermite II

Outlook


Askey-Wilson
q-Racha
Continuous
Dual q-Hahn
Continuous
q-Hahn
Big
q-Jacobi
q-Hahn
Dual q-Hahn
Al-Salam
Chihara
q-Meixner
Pollaczek
Continuous
q-Jacobi
Big
q-Laguerre
Little
q-Jacobi
q-Meixner
Quantum
q-Krawtchouk
q-Krawtchouk
Affine
q-Krawtchouk
Dual
q-Krawtchouk
Continuous
big q-Hermite
Continuous
q-Laguerre
Little
q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam
Carlitz I
Al-Salam
Carlitz II
Continuous
q-Hermite
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Discrete
q-Hermite I
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Outlook

- Additive discrete Painlevé equations

Outlook

- Additive discrete Painlevé equations$\rightarrow$ cover the whole q-Askey scheme.

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- q-difference and elliptic Painlevé equations may correspond to elliptic hypergeometric bi-orthogonal polynomials.
- Multivariable high order analogues of the Painlevé equations $\rightarrow$ *confluence scheme for Macdonald polynomials*