Families of bilinear transformations for basic hypergeometric series and their multivariate generalizations

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♠ Bilinear summation formula for orthogonal polynomials (to name few)

Motivated by the theory of moments and etc.

- A. Cayley, Orr (19th century)
- H. Bateman, W.N. Bailey, Burchnal-Chaundy, G.N. Watson (Beginning of the 20th century)
- E.D. Rainville, L. Carlitz

Many classical results can be found in R. Askey's lecture notes "Orthogonal polynomials and special functions (OPSF)" with fundamental properties of orthogonal polynomials.

More recently

- M. Rahman and his collaborators (the theory of orthogonal polynomials)
- T.H. Koornwinder, E. Koelink, J.V. Stokman and his collaborators (representation theoretic)

(among others)

In this talk, I will propose yet another simple approach towards the construction of bilinear transformations by using multiple hypergeometric transformations.

♡ Main theme of my talk

The results obtained here seem to be more general than ever before. Namely, I present a class of bilinear transformation formulas which include the following as a special case:

$$\begin{aligned}
&\sum_{K\in\mathbb{N}} q^K \frac{(b/s, c_1/s, c_2/s, q^{-N}, g, tq)_K}{(1/s, q, q/d, tq/e, tq/f_1, tq/f_2)_K} \\
&\times\ {}_8W_7\left[q^{-K}s; b, c_1, c_2, q^{-K}d, q^{-K}; q; \frac{s^2q^2}{bc_1c_2d}\right] \\
&\times\ {}_8W_7\left[q^K t; e, f_1, f_2, q^K g, q^{K-N}; q; \frac{t^2q^{N+2}}{ef_1f_2g}\right] \\
&=\ g^N \frac{(tq, tq/eg, tq/f_1g, tq/f_2g)_N}{(tq/e, tq/f_1, tq/f_2, tq/g)_N} \\
&\times\ {}_6\phi_5\left[\begin{matrix} g, sq/c_1d, sq/c_2d, sq/bd, q^{-N}g/t, q^{-N} \\ q/d, q^{-N}f_1g/t, q^{-N}f_2g/t, sq/d, q^{-N}eg/t \end{matrix}; q; q\right]
\end{aligned}$$

provided $s^2t^2q^{2+N} = bc_1c_2def_1f_2g$.

Our construction is very close to one of most famous and elementary proof of Sears transformation formula for terminating balanced ${}_4\phi_3$ series.

★ Notations of basic hypergeometric series

Throughout of this talk, we assume that $0 < |q| < 1$.
We denote the basic hypergeometric series ${}_{r+1}\phi_r$ as

$$ {}_{r+1}\phi_r \left[\begin{matrix} a_0, a_1, \dots a_r \\ c_1, \dots c_r \end{matrix} ; q; u \right] = \sum_{n \in \mathbb{N}} \frac{(a_0)_n (a_1)_n \dots (a_r)_n}{(c_1)_n \dots (c_r)_n (q)_n} u^n. $$

and

$$ (a)_\infty := \prod_{n \in \mathbb{N}} (1 - aq^n), \quad (a)_k := \frac{(a)_\infty}{(aq^k)_\infty} \quad \text{for } k \in \mathbb{C} $$

is a q-shifted factorial.
And we frequently use

$$ (a_1, a_2, \cdots, a_n)_N := (a_1)_N (a_2)_N \cdots (a_n)_N $$

★ Very well-poised basic hypergeometric series

The basic hypergeometric series ${}_{n+1}\phi_n$ is "well-poised" if $a_0 q = a_1 c_1 = \cdots = a_n c_n$. It is called very well-poised if it is well-poised and if $a_1 = q\sqrt{a_0}$ and $a_2 = -q\sqrt{a_0}$. Namely, the very well-poised ${}_{n+1}\phi_n$ is expressed as the following form:

$$
\begin{aligned}
&{}_{n+1}\phi_n \left[\begin{matrix} a_0, & q\sqrt{a_0}, & -q\sqrt{a_0}, & a_3, & \ldots, & a_n \\ & \sqrt{a_0}, & -\sqrt{a_0}, & a_0 q/a_3, & \ldots, & a_0 q/a_n \end{matrix} ; q, u \right] \\
&= \sum_{k \in \mathbb{N}} \frac{1 - a_o q^{2k}}{1 - a_0} \frac{(a_0)_k (a_3)_k \cdots (a_n)_k}{(q)_k (a_0 q/a_3)_k \cdots (a_0 q/a_n)_k} u^k \\
&\qquad := {}_{n+1}W_n \left[a_0; a_3, \ldots, a_n; q; u \right]
\end{aligned}
$$

♡ A proof of Sears transformation

Sears transformation formula for terminating balanced ${}_4\phi_3$ series

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, b, c, q^{-N} \\ d, e, f \end{matrix}; q; q\right] \\
=\ & a^N \frac{(e/a)_N (f/a)_N}{(e)_N (f)_N} {}_4\phi_3\left[\begin{matrix} a, d/b, d/c, q^{-N} \\ d, aq^{-N}/e, aq^{-N}/f \end{matrix}; q; q\right]
\end{aligned}
$$

provided the $(1-)$ balancing condition

$$abcq^{1-N} = def.$$

First, consider the following product of two ${}_2\phi_1$ series:

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q; u\right] \times \frac{(deu/f)_\infty}{(u)_\infty} {}_2\phi_1\left[\begin{matrix} f/e, f/d \\ f \end{matrix}; q; \frac{deu}{f}\right]. $$

Now assume that $ab/c = de/f$. By the 3rd Heine transformation (basic analogue of Euler transformation formula for Gauss' hypergeometric series ${}_2F_1$):

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q; u\right] = \frac{(abu/c)_\infty}{(u)_\infty} \quad {}_2\phi_1\left[\begin{matrix} c/b, c/a \\ c \end{matrix}; q; \frac{abu}{c}\right], $$

The following equality holds:

$$ \begin{aligned} &{}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q; u\right] \times {}_2\phi_1\left[\begin{matrix} f/e, f/d \\ f \end{matrix}; q; \frac{deu}{f}\right] \\ = {}&{}_2\phi_1\left[\begin{matrix} c/b, c/a \\ c \end{matrix}; q; \frac{abu}{c}\right] \times {}_2\phi_1\left[\begin{matrix} d, e \\ f \end{matrix}; q; u\right] \end{aligned} $$

Taking the coefficient of u^N in the equation above and relabeling the parameters gives Sears transformation.

★ Definition of A_n basic hypergeometric series

$\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$: multi-index
$|\beta| = \sum_{i=1}^{n} \beta_i$: length of β

In this talk, a multiple series $\sum_{\beta \in \mathbb{N}^n} S(\beta)$ is called A_n basic hypergeometric series if:

- the series has a form
$$\sum_{\beta \in \mathbb{N}^n} \frac{\Delta(xq^\beta)}{\Delta(x)} u_1^{\beta_1} \cdots u_n^{\beta_n} \times (\text{basic hypergeometric stuff})$$
where
$$\Delta(x) = \prod_{1 \le i < j \le n} (x_i - x_j)$$
and
$$\Delta(xq^\beta) = \prod_{1 \le i < j \le n} (x_i q^{\beta_i} - x_j q^{\beta_j})$$
are Vandermonde determinants of $x = (x_1, \ldots, x_n)$ and $xq^\beta = (x_1 q^{\beta_1}, \ldots, x_n q^{\beta_n})$ respectively.
- symmetric w.r.t. the subscript
- $n = 1 \qquad \Rightarrow$ basic hypergeometric series

★ Multiple basic hypergeometric series

♡ Origin (ordinary case)

W.Holman, L.Biedenharn, J.Louck Representation of $SU(n+1)$:

Clebsch-Gordan coefficients of $SU(2) \Rightarrow$ terminating ${}_3F_2$ series
(Hahn polynomials)

Racah-Wigner coefficients of $SU(2) \Rightarrow$ terminating balanced ${}_4F_3$ series (Racah polynomials)

♠ Derivation

- **S.Milne** A certain algebraic invariants and q-difference equation
- **S.Milne, G.Lily, G.Bhatnager, C.Krattenthaller, M.Schlosser** Multidimensional matrix inversion (Multidimensional Bailey lattice)
- **Y.K, M.Noumi-Y.K** Cauchy kernel and Macdonald's q-difference operators

♠ Application

- **C.Krattenthaller, I.Gessel** Cylindric partition enumeration
- **S.Milne** Analytic number theory —-sum of squares
- **S.Milne, V.Leininger** New infinite families of η function identities
- **J.F. van Diejen, M.Noumi-Y.K** Macdonald polynomials

In this talk, we wants to claim:

"Multivariate hypergeometric transformation is useful for (even in the case of) one-variable hypergeometric series case."

(Especially, hypergeometric transformations with different dimensions in our previous work.)

♡ **Euler transformation formula for basic hypergeometric series of type A**

Theorem 1. (Y.K. 2004 Adv. Math.) Suppose that none of denominators vanish. Then we have the Euler transformation formula for basic hypergeometric series of type A with different dimension:

$$\sum_{\gamma\in\mathbb{N}^n} u^{|\gamma|}\frac{\Delta(xq^{\gamma})}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}$$
$$\prod_{1\le i\le n,1\le k\le m}\frac{(b_kx_iy_k/x_ny_m)_{\gamma_i}}{(cx_iy_k/x_ny_m)_{\gamma_i}}$$
$$= \frac{(a_1\cdots a_nb_1\cdots b_mu/c^m)_\infty}{(u)_\infty}$$
$$\sum_{\delta\in\mathbb{N}^m}(a_1\cdots a_nb_1\cdots b_mu/c^m)^{|\delta|}\frac{\Delta(yq^{\delta})}{\Delta(y)}$$
$$\prod_{1\le k,l\le m}\frac{((c/b_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}\prod_{1\le i\le n,1\le k\le m}\frac{((c/a_i)x_iy_k/x_ny_m)_{\delta_k}}{(cx_iy_k/x_ny_m)_{\delta_k}}$$

for $a_1^{-1},\ldots,a_n^{-1},b_1/c,\ldots,b_m/c\in\mathbb{C}$.

♡ Sears transformation formula for basic hypergeometric series of type A

Theorem 2. **(Y.K. 2004 Adv. Math.)**

$$\sum_{\gamma\in\mathbb{N}^n} q^{|\gamma|}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}$$
$$\frac{(q^-N,a)_{|\gamma|}}{(e,f)_{|\gamma|}}\prod_{1\le i\le n,1\le k\le m}\frac{(c_kx_iy_k/x_ny_m)_{\gamma_i}}{(dx_iy_k/x_ny_m)_{\gamma_i}}$$
$$= a^N\frac{(e/a,f/a)_N}{(e,f)_N}\sum_{\delta\in\mathbb{N}^m} q^{|\delta|}\frac{\Delta(yq^\delta)}{\Delta(y)}\prod_{1\le k,l\le m}\frac{((d/c_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}$$
$$\frac{(q^{-N},a)_{|\delta|}}{(q^{1-N}a/e,q^{1-N}a/f)_{|\delta|}}\prod_{1\le i\le n,1\le k\le m}\frac{((d/b_i)x_iy_k/x_ny_m)_{\delta_k}}{(dx_iy_k/x_ny_m)_{\delta_k}}$$

provided $aBCq^{1-N} = d^mef$.

However it was obtained from the case when we consider the product:

$$\sum_{\gamma\in\mathbb{N}^n} u^{|\gamma|}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}$$
$$\prod_{1\le i\le n,1\le k\le m}\frac{(b_kx_iy_k/x_ny_m)_{\gamma_i}}{(cx_iy_k/x_ny_m)_{\gamma_i}}$$
$$\times\ {}_2\phi_1\left[\begin{matrix}f/e, f/d\\ f\end{matrix};q;\frac{deu}{f}\right]$$

Question

What arises when we consider the product of multiple series in general??

★ Definition of the multiple very well-poised BHS $W^{n,m}$

To simplify expressions of formulas, we introduce a notation of multiple very well-poised basic hypergeometric series as follows:

$$W^{n,m}\left(\begin{matrix} \{a_i\}_n \\ \{x_i\}_n \end{matrix} \,\middle|\, s; \{u_k\}_m; \{v_k\}_m; z \right)$$

$$= \sum_{\gamma \in \mathbb{N}^n} z^{|\mu|} \prod_{1 \le i < j \le n} \frac{\Delta(xq^{\gamma})}{\Delta(x)} \prod_{1 \le i \le n} \frac{1 - q^{|\gamma| + \gamma_i} s x_i / x_n}{1 - s x_i / x_n}$$

$$\prod_{1 \le j \le n} \frac{(s x_j / x_n)_{|\gamma|}}{((sq/a_j) x_j / x_n)_{|\gamma|}} \left(\prod_{1 \le i \le n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \right)$$

$$\prod_{1 \le k \le m} \frac{(v_k)_{|\gamma|}}{(sq/u_k)_{|\gamma|}} \left(\prod_{1 \le i \le n} \frac{(u_k x_i / x_n)_{\gamma_i}}{((sq/v_k) x_i / x_n)_{\gamma_i}} \right),$$

where $\{u_i\}_n$ means $u_1, \ldots, u_n$ according to this order.

This series contains "well-poised" combinations of factors

$$\frac{(u x_1 / x_n)_{\gamma_1} \cdots (u x_{n-1} / x_n)_{\gamma_{n-1}} \cdot (u)_{\gamma_n}}{(sq/u)_{|\gamma|}},$$

$$\frac{(v)_{|\gamma|}}{((sq/v) x_1 / x_n)_{\gamma_1} \cdots ((sq/v) x_{n-1} / x_n)_{\gamma_{n-1}} \cdot (sq/v)_{\gamma_n}}$$

Note that in the case when $n = 1$, $W^{1,m}$ reduces ${}_{2m+4}W_{2m+3}$ series.

♡ Key Lemma

Homogeneous part of multiple hypergeometric series

$\simeq$

(multiple) very well-poised hypergeometric series

♠ (Example) q-multiple binomial theorem and Rogers' terminating multiple very-well-poised ${}_6\phi_5$ summation

An A_n q-binomial theorem **S.C. Milne (Adv. Math (1985))**

$$\frac{(b_1 \cdots b_{n+1} u)_\infty}{(u)_\infty} = \sum_{\beta \in \mathbb{N}^{n+1}} u^{|\beta|} \frac{\Delta(xq^\beta)}{\Delta(x)} \prod_{1 \le i,j \le n+1} \frac{(b_j x_i/x_j)_{\beta_i}}{(qx_i/x_j)_{\beta_i}}.$$

In the case when $n = 1$, it reduces to q-binomial theorem:

$$\frac{(bu)_\infty}{(u)_\infty} = \sum_{k \in \mathbb{N}} u^k \frac{(b)_k}{(q)_k}.$$

After changing $n \to n+1$, take the coefficients of u^N in the both side of A_n q-binomial theorem. The coefficient which appear from the right hand side can be expressed in terms of $W^{n,1}$ series. Then by changing the parameter appropriately, we obtain A_n Rogers' terminating ${}_6W_5$ summation formula due to Milne,

$$\frac{(aq/b_1\cdots b_n c)_N}{(aq/c)_N}\prod_{1\le i\le n}\frac{(aqx_i/x_n)_N}{((aq/b_i)x_i/x_n)_N}$$
$$= W^{n,1}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a; c; q^{-N}; \frac{aq^{1+N}}{b_1\cdots b_n c}\right).$$

The case when $n=1$, this formula reduces to Rogers' terminating ${}_6W_5$ summation formula

$${}_6W_5\left[a; b, c, q^{-N}; q; \frac{aq^{1+N}}{bc}\right] = \frac{(aq/bc)_N}{(aq/c)_N}\frac{(aq)_N}{(aq/b)_N}.$$

Set the homogeneous part in multiple basic hypergeometric series as Φ_N:

$$\Phi_N^{n,m}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_m\\ \{c y_k\}_m\end{matrix}\right)$$
$$:= \sum_{\gamma\in\mathbb{N}^n, |\gamma|=N} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} \frac{(a_j x_i/x_j)_{\gamma_i}}{(q x_i/x_j)_{\gamma_i}}$$
$$\times \prod_{1\le i\le n, 1\le k\le m} \frac{(b_k x_i y_k/x_n y_m)_{\gamma_i}}{(c x_i y_k/x_n y_m)_{\gamma_i}}.$$

Then we have

Lemma

$$\Phi_N^{1,m}\left(\begin{matrix}a\\ 1\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_m\\ \{c y_k\}_m\end{matrix}\right) = \left(\text{the coeff. of } u^N \text{ in } {}_{m+1}\phi_m \text{ series}\right)$$
$$= \frac{(a)_N}{(q)_N} \prod_{1\le k\le m} \frac{(b_k y_k)_N}{(c y_k)_N}$$

and

$$\Phi_N^{n+1,m}\left(\begin{matrix}\{a_i\}_{n+1}\\ \{x_i\}_{n+1}\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_m\\ \{c y_k\}_m\end{matrix}\right)$$
$$= \frac{(a_{n+1})_N}{(q)_N} \prod_{1\le i\le n} \frac{(a_i x_{n+1}/x_i)_N}{(x_{n+1}/x_i)_N} \prod_{1\le k\le m} \frac{(b_k x_{n+1} y_k)_N}{(c x_{n+1} y_k)_N}$$
$$W^{n,m+1}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\middle| q^{-N}/x_{n+1}; \{b_k y_k\}_m, a_{n+1}/x_{n+1};\right.$$
$$\left.\{(q^{1-N}/x_{n+1}) c^{-1} y_k^{-1}\}_m, q^{-N};\ \frac{c^m q}{a_1\cdots a_n a_{n+1} B}\right)$$

♡ The master formula and its reversing version

Theorem 3

$$\sum_{K\in\mathbb{N}} \Phi_K^{n_1,m_1}\left(\begin{matrix}\{a_i\}_{n_1}\\ \{x_i\}_{n_1}\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_{m_1}\\ \{c y_k\}_{m_1}\end{matrix}\right) \tag{1}$$

$$\times\ \Phi_{N-K}^{n_2,m_2}\left(\begin{matrix}\{f/e_p\}_{n_2}\\ \{z_p\}_{n_2}\end{matrix}\middle|\begin{matrix}\{(f/d_s)w_s\}_{m_2}\\ \{f w_s\}_{m_2}\end{matrix}\right)\left(\frac{f^{n_2}}{DE}\right)^{N-K}$$

$$= \sum_{L\in\mathbb{N}} \Phi_L^{m_1,b_1}\left(\begin{matrix}\{c/b_k\}_{m_1}\\ \{y_k\}_{m_1}\end{matrix}\middle|\begin{matrix}\{(c/a_i)x_i\}_{n_1}\\ \{c x_i\}_{n_1}\end{matrix}\right)$$

$$\times\ \Phi_{N-L}^{n_2,m_2}\left(\begin{matrix}\{d_s\}_{m_2}\\ \{w_s\}_{m_2}\end{matrix}\middle|\begin{matrix}\{e_p z_p\}_{n_2}\\ \{f z_p\}_{n_2}\end{matrix}\right)\left(\frac{c^{m_1}}{AB}\right)^{L}$$

when the "balancing" condition $AB/c^{m_1} = DE/f^{n_2}$ holds.

And

$$\sum_{K\in\mathbb{N}} \Phi_K^{n_1,m_1}\left(\begin{matrix}\{a_i\}_{n_1}\\ \{x_i\}_{n_1}\end{matrix}\middle|\begin{matrix}\{b_k y_k\}_{m_1}\\ \{c y_k\}_{m_1}\end{matrix}\right) \tag{2}$$

$$\times\ \Phi_{N-K}^{n_2,m_2}\left(\begin{matrix}\{f/e_p\}_{n_2}\\ \{z_p\}_{n_2}\end{matrix}\middle|\begin{matrix}\{(f/d_s)w_s\}_{m_2}\\ \{f w_s\}_{m_2}\end{matrix}\right)\left(\frac{f^{n_2}}{DE}\right)^{N-K}$$

$$= \sum_{L\in\mathbb{N}} \Phi_{N-L}^{m_1,b_1}\left(\begin{matrix}\{c/b_k\}_{m_1}\\ \{y_k\}_{m_1}\end{matrix}\middle|\begin{matrix}\{(c/a_i)x_i\}_{n_1}\\ \{c x_i\}_{n_1}\end{matrix}\right)$$

$$\times\ \Phi_{L}^{n_2,m_2}\left(\begin{matrix}\{d_s\}_{m_2}\\ \{w_s\}_{m_2}\end{matrix}\middle|\begin{matrix}\{e_p z_p\}_{n_2}\\ \{f z_p\}_{n_2}\end{matrix}\right)\left(\frac{c^{m_1}}{AB}\right)^{N-L}$$

when $AB/c^{m_1} = DE/f^{m_2}$ holds.

♡ Example of bilinear transformation formula for multivariate hypergeometric series

(a) $n_1, n_2 \geq 2, m_1 = m_2 = 1$ case of (1) (after an appropriate replacement of parameters):

$$\begin{aligned}
&\sum_{K\in\mathbb{N}} q^K \frac{(c_1/s, c_2/s)_K}{(q, q/d)_K} \prod_{1\leq i\leq n_1} \frac{((b_i/s)x_i^{-1})_K}{((1/s)x_i^{-1})_K} \\
&\times\ \frac{(q^{-N}, g)_K}{(tq/f_1, tq/f_2)_K} \prod_{1\leq p\leq n_2} \frac{((tq)z_p)_K}{((tq/e_p)z_p)_K} \\
&\times\ W^{n_1,2}\Big(\begin{matrix} \{b_i\}_{n_1} \\ \{x_i\}_{n_1} \end{matrix} \Big| q^{-K}s; c_1, c_2; q^{-K}d, q^{-K}; \frac{s^2q^2}{Bc_1c_2d}\Big) \\
&\times\ W^{n_2,2}\Big(\begin{matrix} \{e_p\}_{n_2} \\ \{z_p\}_{n_2} \end{matrix} \Big| q^{K}t; f_1, f_2; q^{K}g, q^{K-N}; \frac{t^2q^{N+2}}{Ef_1f_2g}\Big) \\
&=\ g^N \frac{(tq/f_1g, tq/f_2g)_N}{(tq/f_1, tq/f_2)_N} \prod_{1\leq p\leq n_2} \frac{((tq)z_p, (tq/e_pg)z_p)_N}{((tq/g)z_p, (tq/e_p)z_p)_N} \\
&\times\ {}_{n_1+n_2+4}\phi_{n_1+n_2+3} \left[\begin{matrix} g, sq/c_1d, sq/c_2d, \{(sq/b_id)x_i\}_{n_1}, \\ q/d, q^{-N}f_1g/t, q^{-N}f_2g/t, \{(sq/d)x_i\}_{n_1}, \end{matrix} \right. \\
&\qquad\qquad \left. \begin{matrix} \{(q^{-N}g/t)z_p\}_{n_2}, q^{-n} \\ \{(q^{-N}e_pg/t)z_p\}_{n_2} \end{matrix} ; q; q \right]
\end{aligned}$$

provided $s^2t^2q^{2+N} = Bc_1c_2dEf_1f_2g$.

In the case when $n_1 = n_2 = 1$ and $x_1 = z_1 = 1$, the formula above reduces to:

$$
\begin{aligned}
&\sum_{K\in\mathbb{N}} q^K \frac{(b/s, c_1/s, c_2/s, q^{-N}, g, tq)_K}{(1/s, q, q/d, tq/e, tq/f_1, tq/f_2)_K} \\
&\times\ {}_8W_7\left[q^{-K}s; b, c_1, c_2, q^{-K}d, q^{-K}; q; \frac{s^2q^2}{bc_1c_2d}\right] \\
&\times\ {}_8W_7\left[q^K t; e, f_1, f_2, q^K g, q^{K-N}; q; \frac{t^2q^{N+2}}{ef_1f_2g}\right] \\
&=\ g^N \frac{(tq, tq/eg, tq/f_1g, tq/f_2g)_N}{(tq/e, tq/f_1, tq/f_2, tq/g)_N} \\
&\times\ {}_6\phi_5\left[\begin{matrix} g, sq/c_1d, sq/c_2d, sq/bd, q^{-N}g/t, q^{-N} \\ q/d, q^{-N}f_1g/t, q^{-N}f_2g/t, sq/d, q^{-N}eg/t \end{matrix}; q; q\right]
\end{aligned}
$$

provided $s^2t^2q^{2+N} = bc_1c_2def_1f_2g$, which we showed first of the present talk.

♡ Another example of bilinear transformation formulas for (multivariate) hypergeometric series

(b) $n_1 = n_2 = m_1 = m_2 = 2$ case of (1) :

If the "balancing" condition

$$t^3 q^2 \sigma^3 q^{N+2} \;=\; bcd_1 d_2 ef \beta\gamma\delta_1\delta_2\epsilon\phi$$

holds, then we have:

$$\sum_{K\in\mathbb{N}} \frac{(b/t, c/t, d_1/t, d_2/t)_K}{(1/t, q/e, q/f, q)_K} \times \frac{(\sigma q, \epsilon, \phi, q^{-N})_K}{(\sigma q/\beta, \sigma q/\gamma, \sigma q/\delta_1, \sigma q/\delta_2)_K} q^K$$
$$\times\ {}_{10}W_9\left[tq^{-K}; b, c, d_1, d_2, eq^{-K}, fq^{-K}, q^{-K}; q; \frac{t^3q^3}{bcd_1d_2ef}\right]$$
$$\times\ {}_{10}W_9\left[\sigma q^{K}; \beta, \gamma, \delta_1, \delta_2, \epsilon q^{K}, \phi q^{K}, q^{K-N}; q; \frac{\sigma^3 q^{N+3}}{\beta\gamma\delta_1\delta_2\epsilon\phi}\right]$$

$$
\begin{aligned}
= \ & \phi^N \frac{(\sigma q/\delta_1\phi, \sigma q/\delta_2\phi)_N}{(\sigma q/\delta_1, \sigma q/\delta_2)_N} \cdot \frac{(\epsilon, \sigma q/\gamma\phi)_N}{(\epsilon/\phi, (\sigma q/\gamma))_N} \\
& \qquad\qquad\qquad\qquad \times \frac{(\sigma q, \sigma q/\beta\phi)_N}{(\sigma q/\phi, \sigma q/\beta)_N} \\
\times \ & \sum_{L\in\mathbb{N}} \frac{(tq/cf, tq/bf, tq/d_1 f, tq/d_2 f)_L}{(e/f, tq/f, q/f, q)_L} \\
& \qquad \times \frac{(q^{1-N}\epsilon/\phi, q^{-N}\phi/\sigma, \phi, q^{-N})_L}{(q^{-N}\gamma\phi/\sigma, q^{-N}\beta\phi/\sigma, q^{-N}\delta_1\phi/\sigma, q^{-N}\delta_2\phi/\sigma)_L} q^L \\
\times \ & {}_{10}W_9 \left[q^{-L} f/e; tq/ce, tq/be, tq/d_1 e, tq/d_2 e, \right. \\
& \qquad\qquad \left. f q^{-L}/t, f q^{-L}, q^{-L}; q; \frac{bcd_1 d_2 e f q^{-1}}{t^3} \right] \\
\times \ & {}_{10}W_9 \left[q^{L-N}\phi/\epsilon; \sigma q/\gamma\epsilon, \sigma q/\beta\epsilon, \sigma q/\delta_1\epsilon, \sigma q/\delta_2\epsilon, \right. \\
& \qquad\qquad \left. q^{-N+L}\phi/\sigma, q^L\phi, q^{L-N}; q; \frac{\beta\gamma\delta_1\delta_2\epsilon\phi q^{-N-1}}{\sigma^3} \right],
\end{aligned}
$$

which can be considered as one of the most general formulas of our class of bilinear transformations in one dimensional setting.

Note Each of ${}_{10}W_9$ series is **NOT** necessary to be balanced. But the product of the argument of ${}_{10}W_9$ series is q^2 by the "balancing" condition above. So, if one of the ${}_{10}W_9$ series is balanced, all of those become to be balanced.