Elliptic Asymptotics

of Discrete Painlevé Equations

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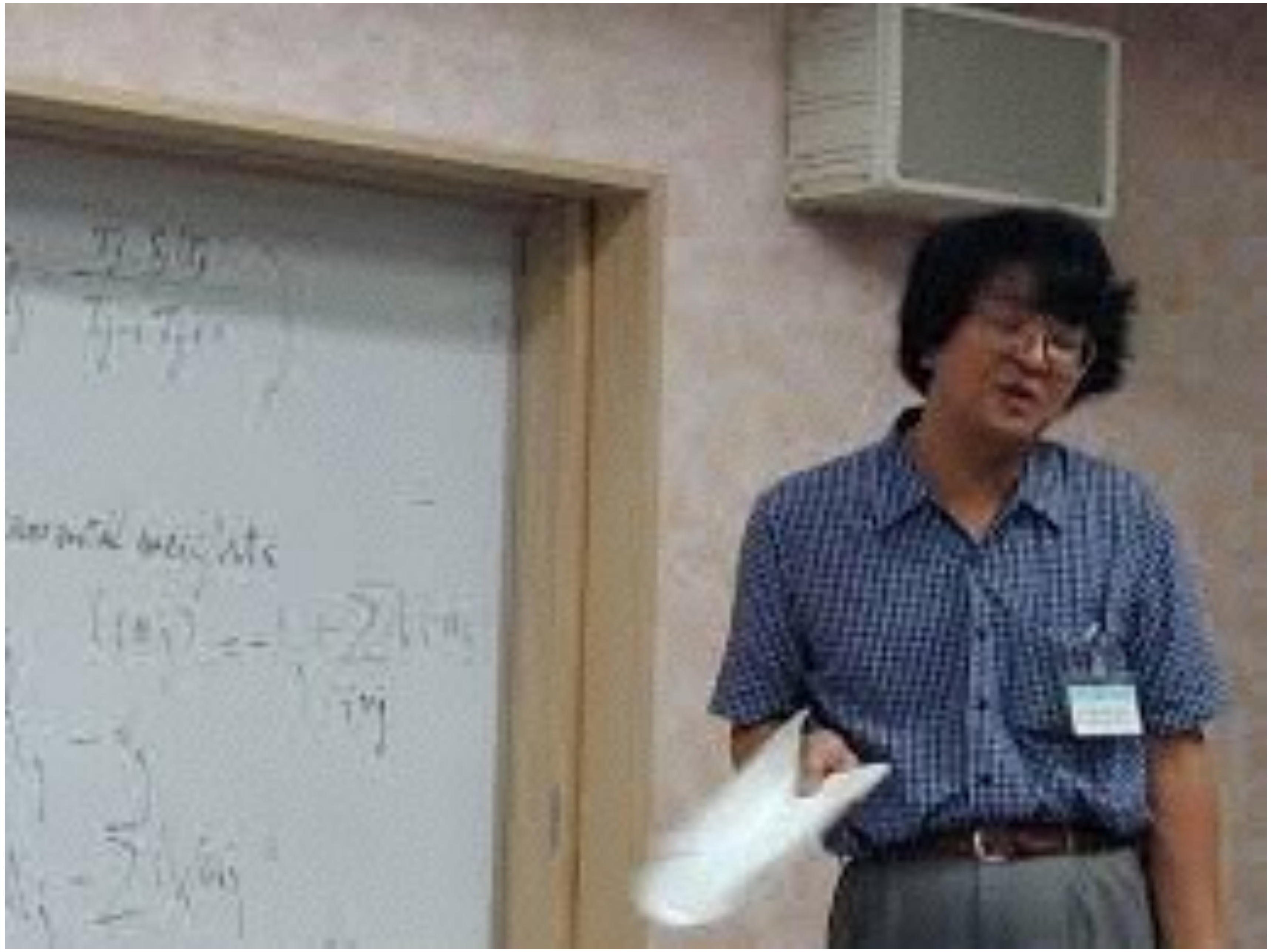
Supported by the Australian Research Council

THE UNIVERSITY OF SYDNEY



Happy Birthday, Noumi san!

~1998


EXIT

EXIT

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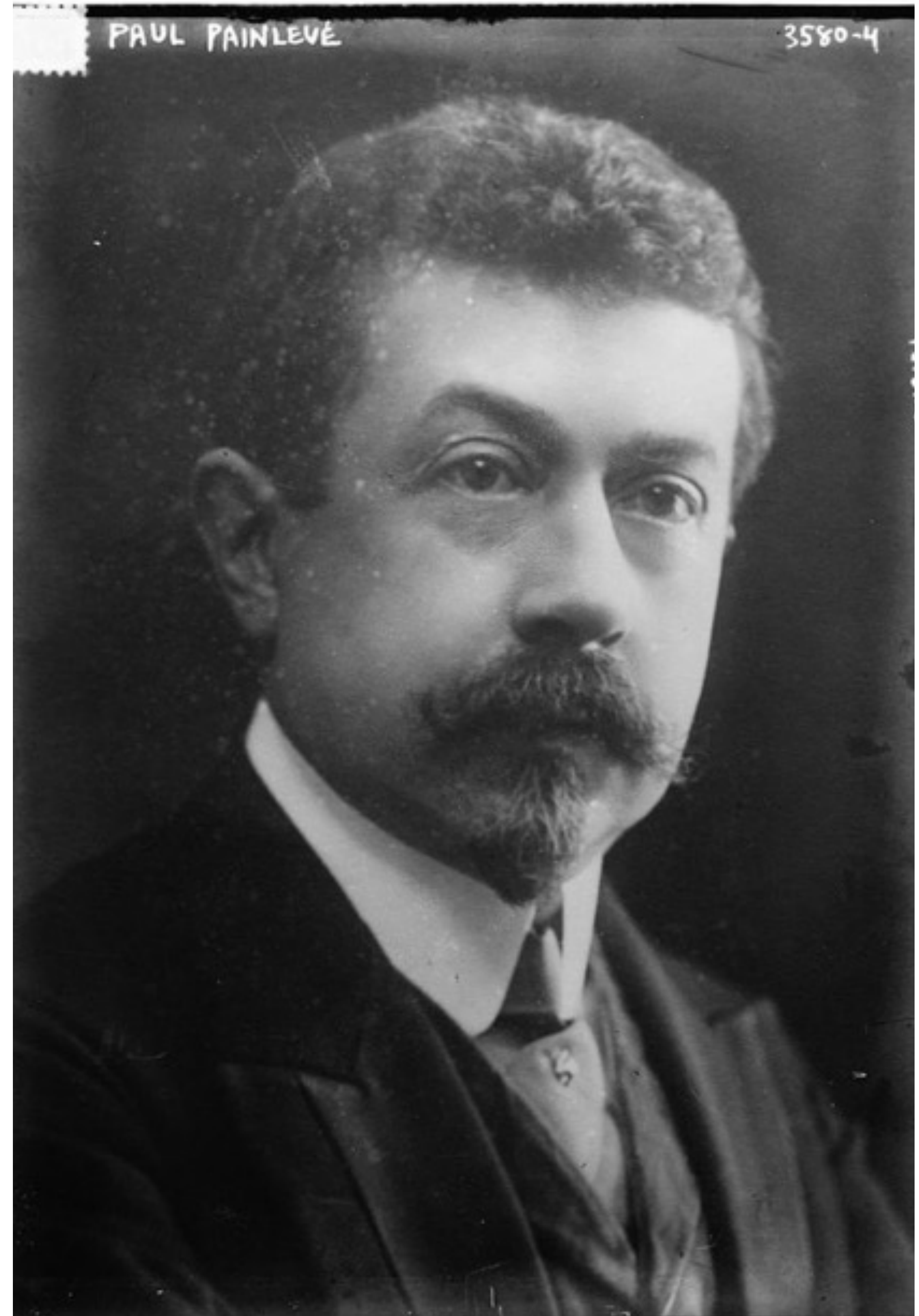
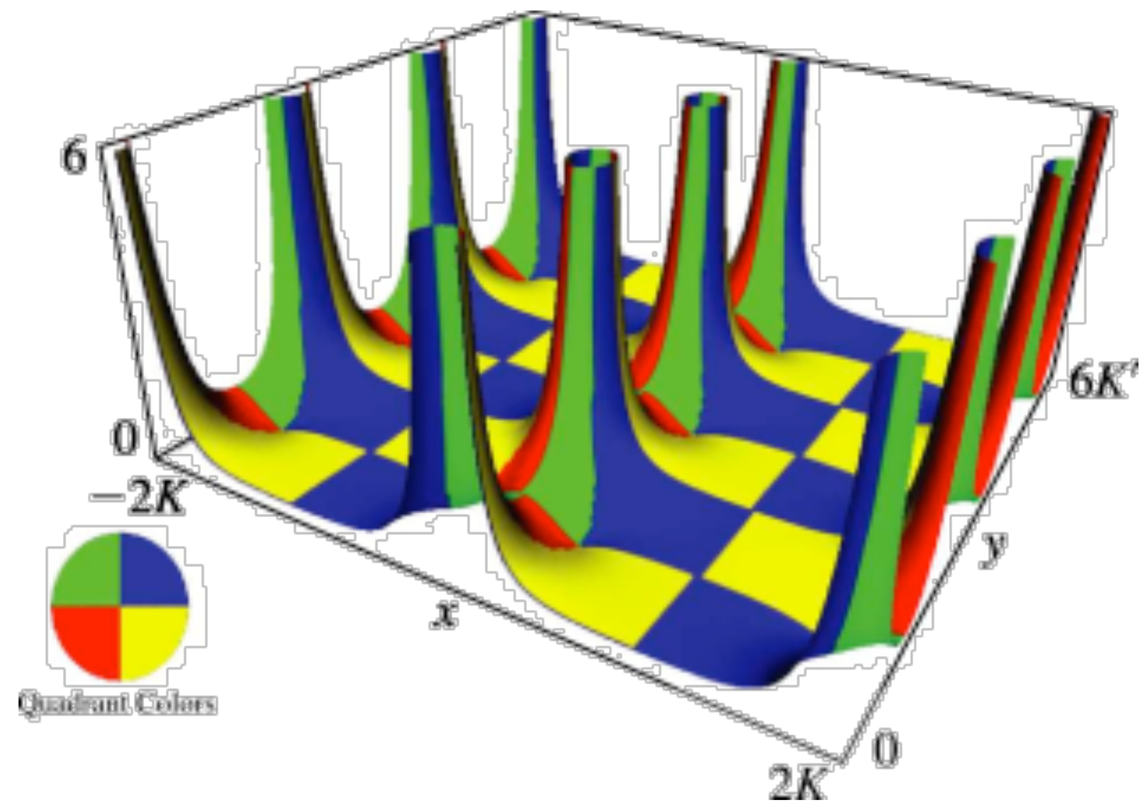
Paul Painlevé

1863 – 1933



Search for new functions

- To generalise elliptic functions: needs global definition of solutions.
- Painlevé property: single-valued around all movable singularities => ODEs defining new functions.



The Painlevé Equations

$$
\begin{aligned}
P_{\mathrm{I}}:\ y'' &= 6y^2 + x \\
P_{\mathrm{II}}:\ y'' &= 2y^3 + xy + \alpha \\
P_{\mathrm{III}}:\ y'' &= \frac{y'^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y} \\
P_{\mathrm{IV}}:\ y'' &= \frac{y'^2}{2y} + \frac{3y^3}{2} + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y} \\
P_{\mathrm{V}}:\ y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2 y}(\alpha y^2 + \beta) \\
&\quad + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1} \\
P_{\mathrm{VI}}:\ y'' &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' \\
&\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right)
\end{aligned}
$$

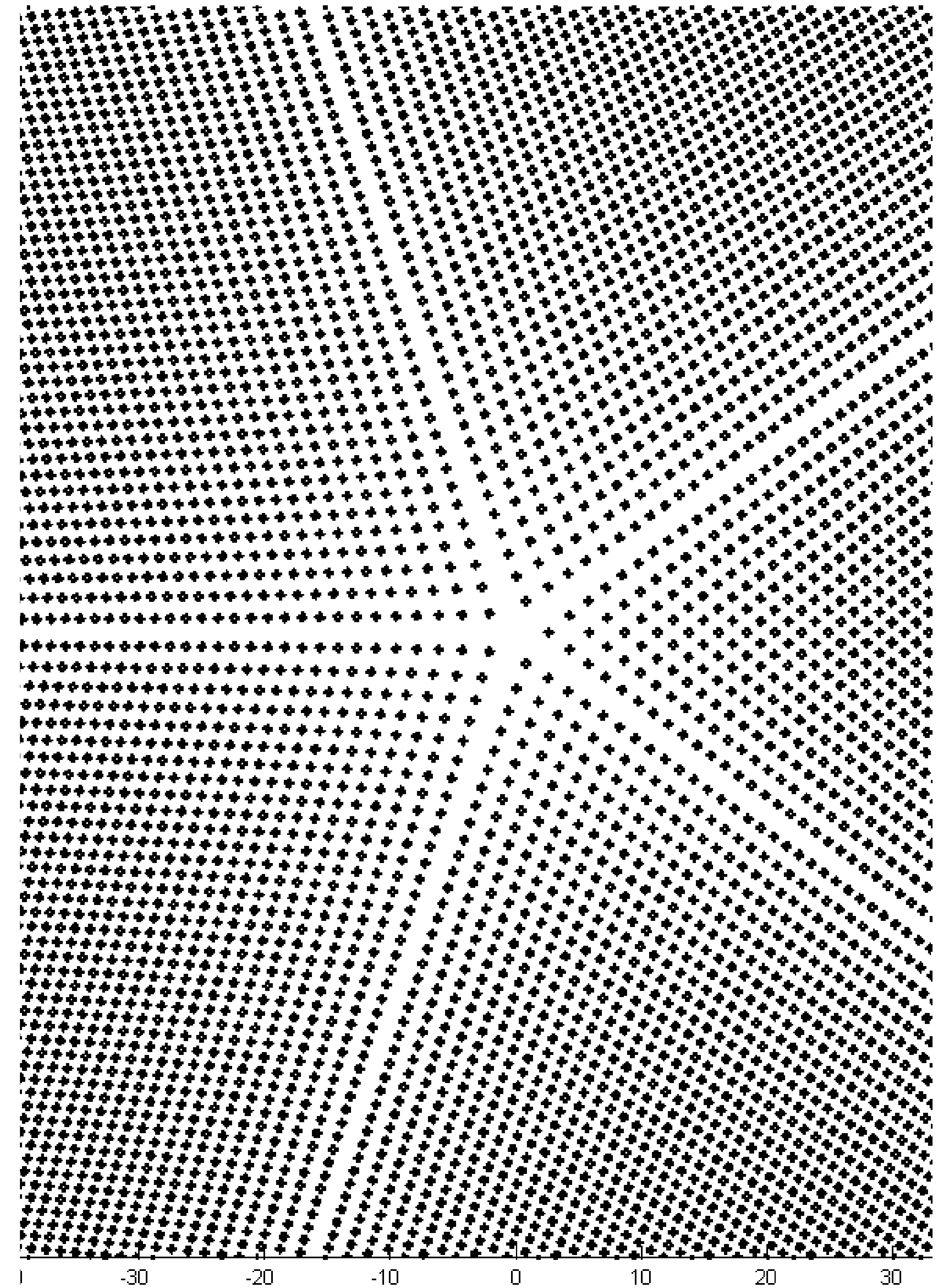
Asymptotic behaviours

- Studied since *Boutroux, 1913*
- Scaled elliptic-function behaviours within sectors as $|x| \to 1 \;\; (\mathrm{P_{VI}})$

$$|x| \to 0 \;\; (\mathrm{P_{III}}, \mathrm{P_{V}}, \mathrm{P_{VI}})$$

$$|x| \to \infty \;\; (\mathrm{P_{I}}, \ldots, \mathrm{P_{VI}})$$

u(0) = 0,   u'(0) = 0

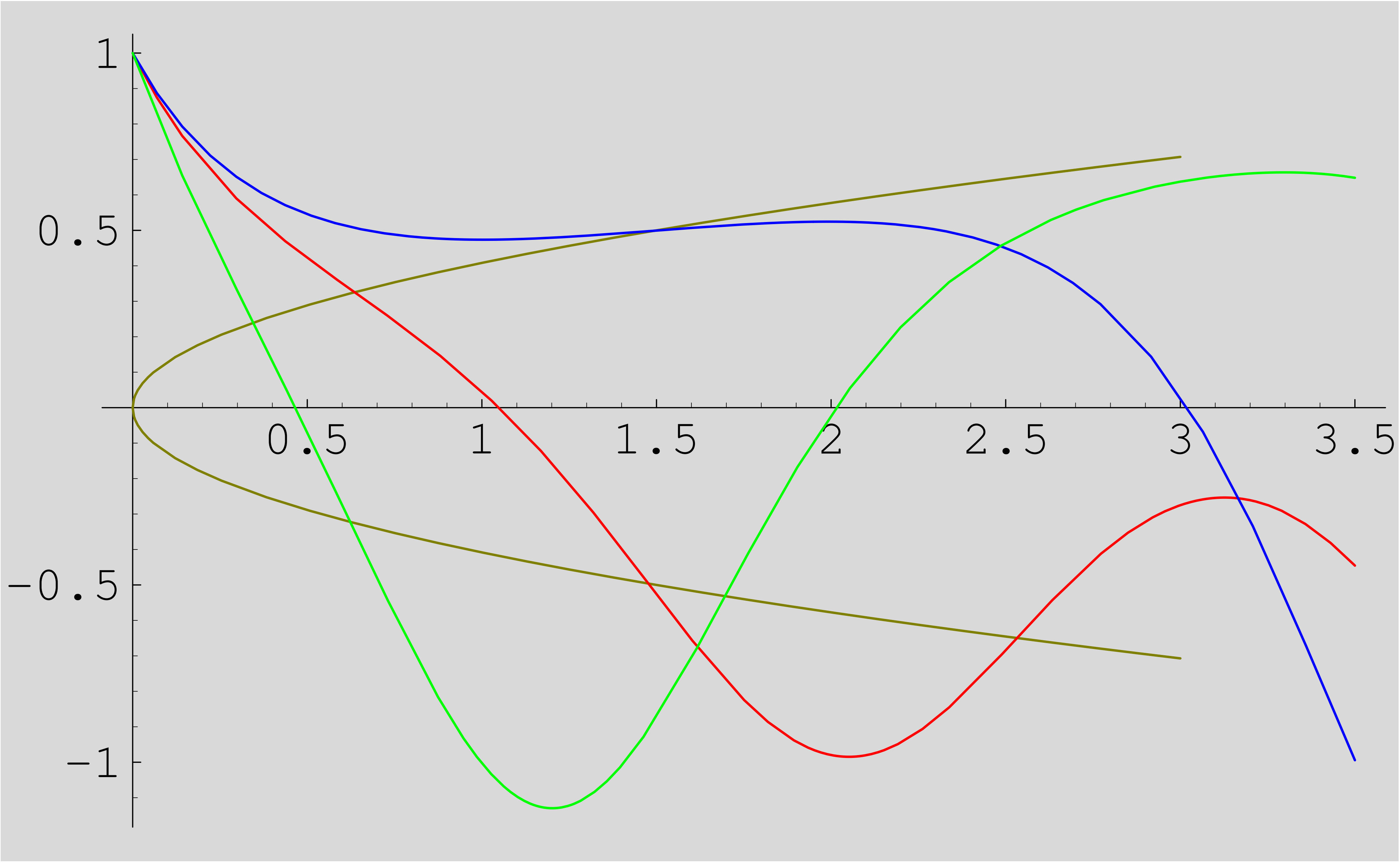


Fornberg & Weideman 2009

$\mathrm{P_I}$

Problems still open…

Consider P_I $y'' = 6\,y^2 - x$ for y(x), x∈$\mathbb{R}$

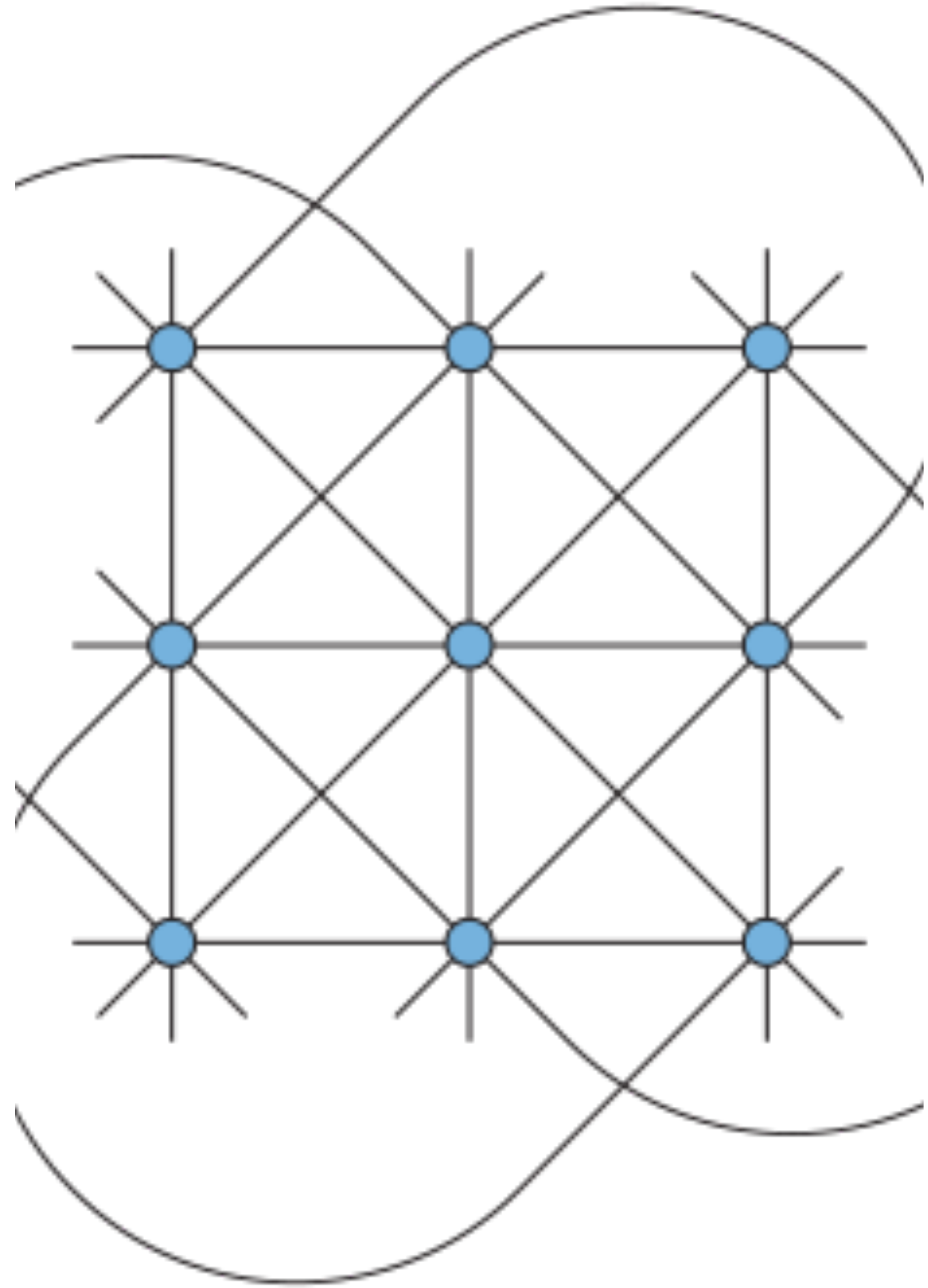


Kazuo Okamoto

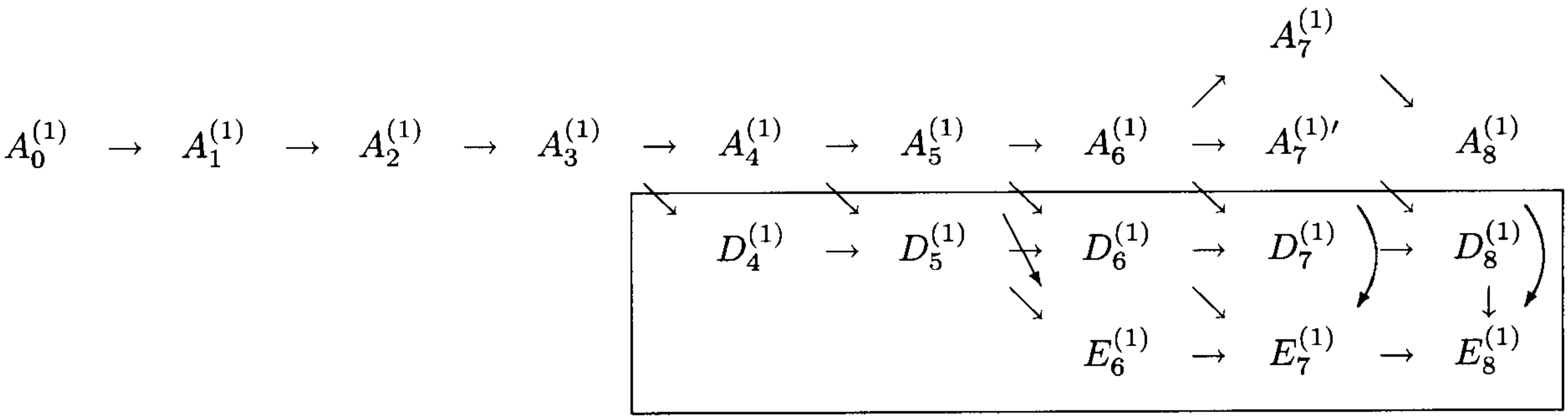
Sur les feuilletages associés aux équations du second ordre a points critiques fixes de P. Painlevé. Espaces de conditions initiales. *Jpn. J. Math.* **5** 1-79 (1979)

Unifying Property

Space of initial conditions is resolved at 9 points in $\mathbb{CP}^2$ (or 8 points in $\mathbb{P}^1$x$\mathbb{P}^1$)

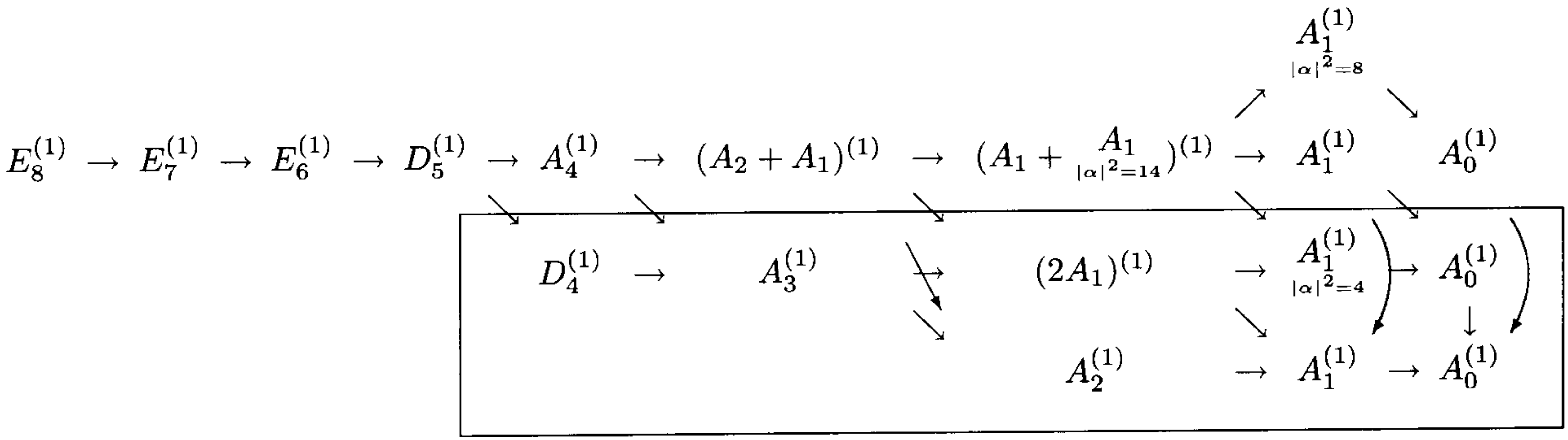


Equations on Rational Surfaces



Sakai 2001

Symmetries



Sakai 2001

Discrete Painlevé Equations

$$\mathrm{dP_I} : w_n \left(w_{n+1} + w_n + w_{n-1}\right) = z_n + d\, w_n$$

$$z_n = a + b\, n + c\,(-1)^n$$

$$\mathrm{dP_{II}} : w_{n+1} + w_{n-1} = \frac{z_n\, w_n + d}{1 - w_n^2}$$

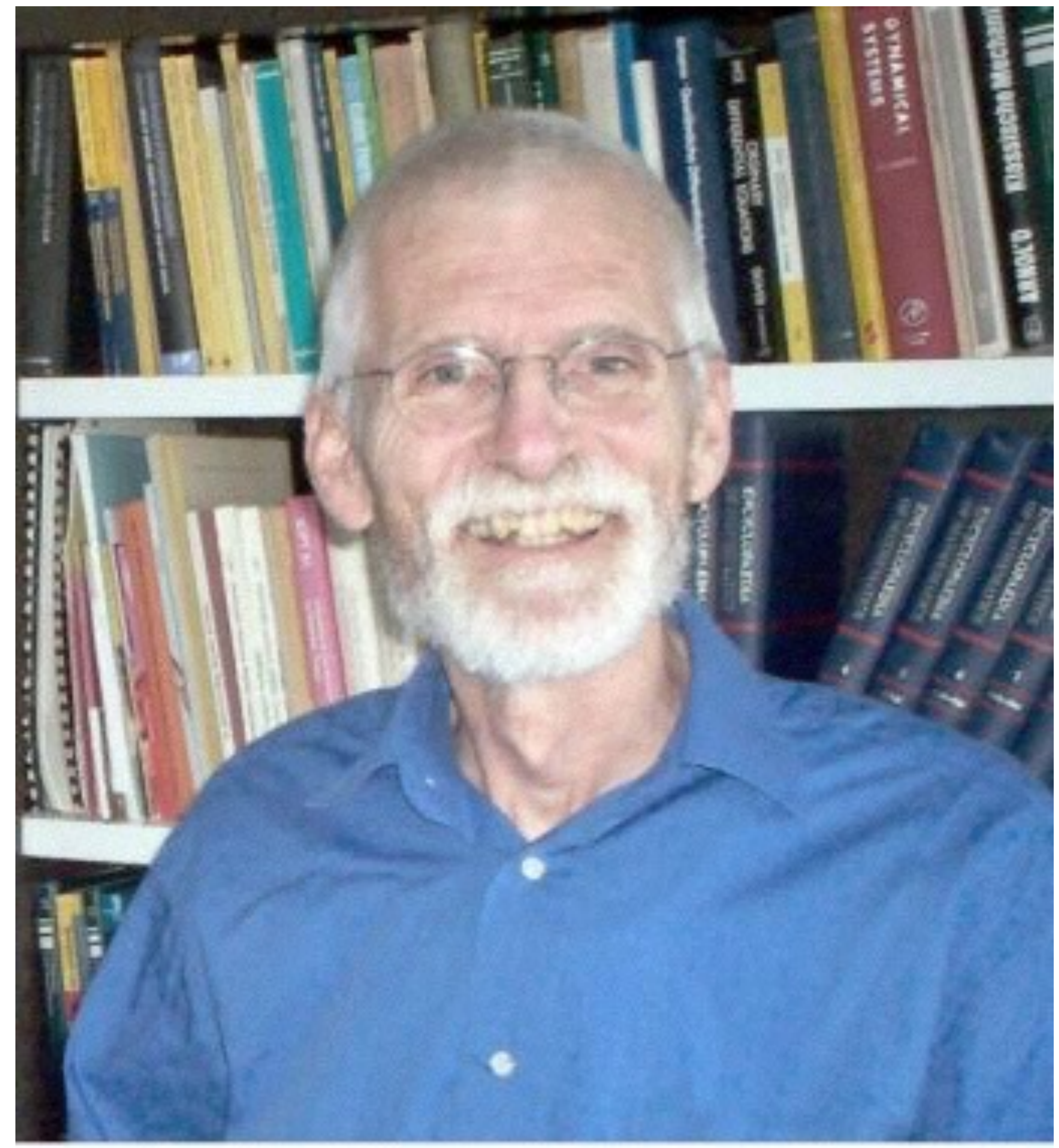
$$\mathrm{qP_{III}} : w_{n+1}\, w_{n-1} = cd\, \frac{(w_n - a\, q^n)(w_n - b\, q^n)}{(w_n - c)(w_n - d)}$$

$$\mathrm{dP_{IV}} : (w_{n+1} + w_n)\,(w_n + w_{n-1}) = \frac{(w_n^2 - a^2)\,(w_n^2 - b^2)}{(w_n - (a\, n + b))^2 - c^2}$$

$\vdots$ *& many more*

Geometry as a tool for Analysis

- Construct, compactify and regularize the initial value space
- Deduce behaviour of solutions in this space.
- Find global information about behaviours



Hans Duistermaat

1942-2010

Duistermaat & J, 2011

General Solutions

- In system form P_I is

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ 6\,w_1^2 - t \end{pmatrix}$$

- P_I has *t*-dependent Hamiltonian

$$H = \frac{w_2^2}{2} - 2\,w_1^3 + t\,w_1$$

Perturbed Form

- Or, in Boutroux's coordinates:

$$w_1 = t^{1/2}\,u_1(z),\;\; w_2 = t^{3/4}\,u_2(z) \quad z = \frac{4}{5}t^{5/4}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6\,u_1^2 - 1 \end{pmatrix} - \frac{1}{(5z)}\begin{pmatrix} 2\,u_1 \\ 3\,u_2 \end{pmatrix}$$

- a perturbation of a Hamiltonian system

$$E = \frac{u_2^2}{2} - 2\,u_1^3 + u_1 \;\Rightarrow\; \frac{dE}{dt} = \frac{1}{5t}\big(6E + 4u_1\big)$$

A Geometric Approach

- The values of *E* provide level curves of

$$\mathrm{P_I}: \ f_{\mathrm{I}}(x,y) = y^2 - 4x^3 + g_2 x, \ g_2 = 2$$

$$\mathrm{P_{II}}: \ f_{\mathrm{II}}(x,y) = y^2 - 2x^2 y - y,$$

$$\mathrm{P_{IV}}: \ f_{\mathrm{IV}}(x,y) = x^2 y + x y^2 + 2xy$$

- The level curves $f_{\mathrm{I}}(x,y) = g_3$ are well known in the theory of algebraic curves as the *Weierstrass cubic pencil*.

Projective Space

- What if *x, y* become unbounded?
- Use projective geometry: $x = \frac{u}{w}, y = \frac{v}{w}$

$$[x, y, 1] = [u, v, w] \in \mathbb{CP}^2$$

- The level curves of $\mathrm{P_I}$ are now

$$F_{\mathrm{I}} = wv^2 - 4u^3 + g_2 uw^2 + g_3 w^3$$

all intersecting at the *base point* [0, 1, 0].

- *Resolve* the flow through base points.

Resolution

- "Blow up" the singularity or base point:

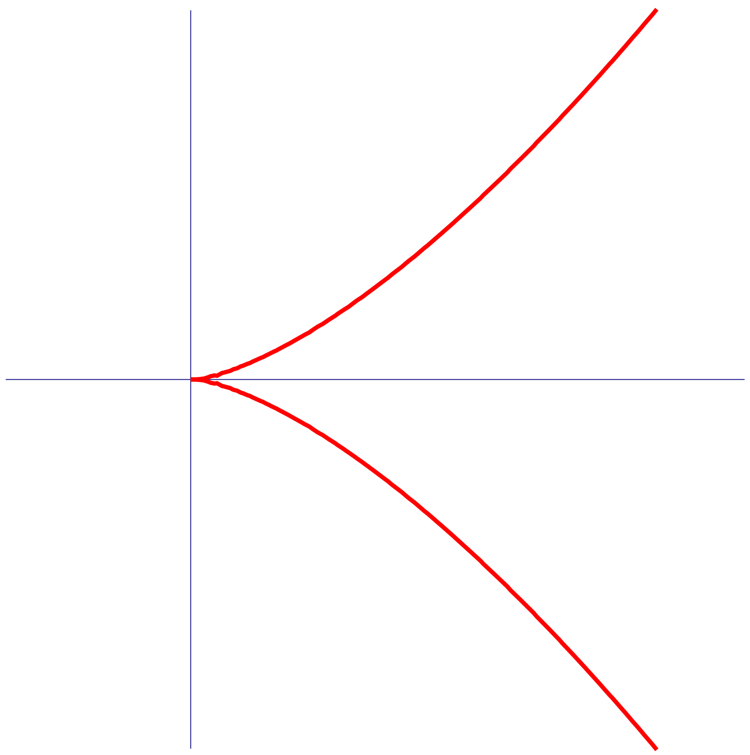
$$
\begin{aligned}
f(x,y) &= y^2 - x^3 \\
(x,y) &= (x_1, x_1\, y_1) \\
\Rightarrow & x_1^2\, y_1^2 - x_1^3 = 0 \\
\Leftrightarrow & x_1^2\, (y_1^2 - x_1) = 0
\end{aligned}
$$

- Note that

$$x_1 = x, y_1 = y/x$$

Example

$y^2 = x^3$



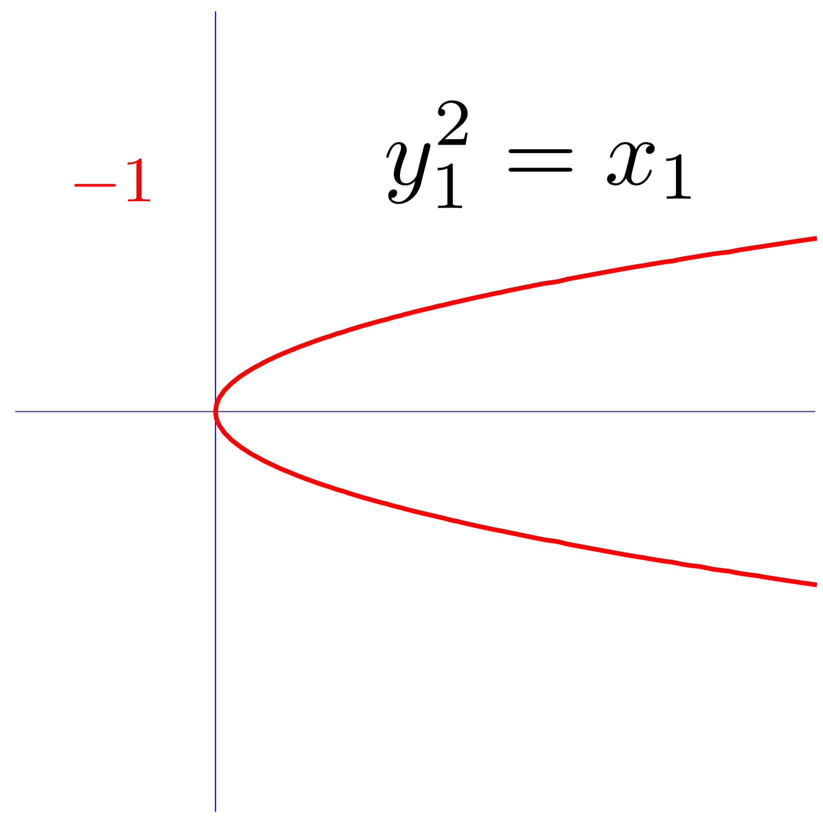
$$f(x, y) = y^2 - x^3$$

$$(x, y) = (x_1, x_1\, y_1)$$

$$f(x_1, x_1\, y_1) = x_1^2 (y_1^2 - x_1)$$

-1

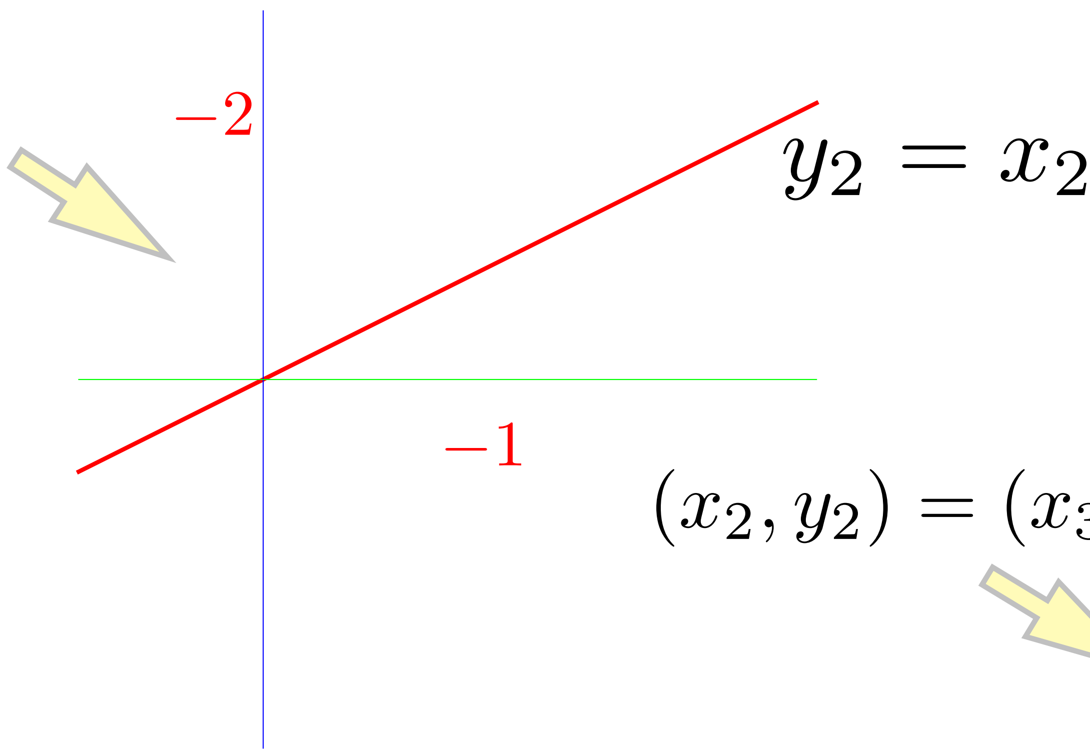
$y_1^2 = x_1$



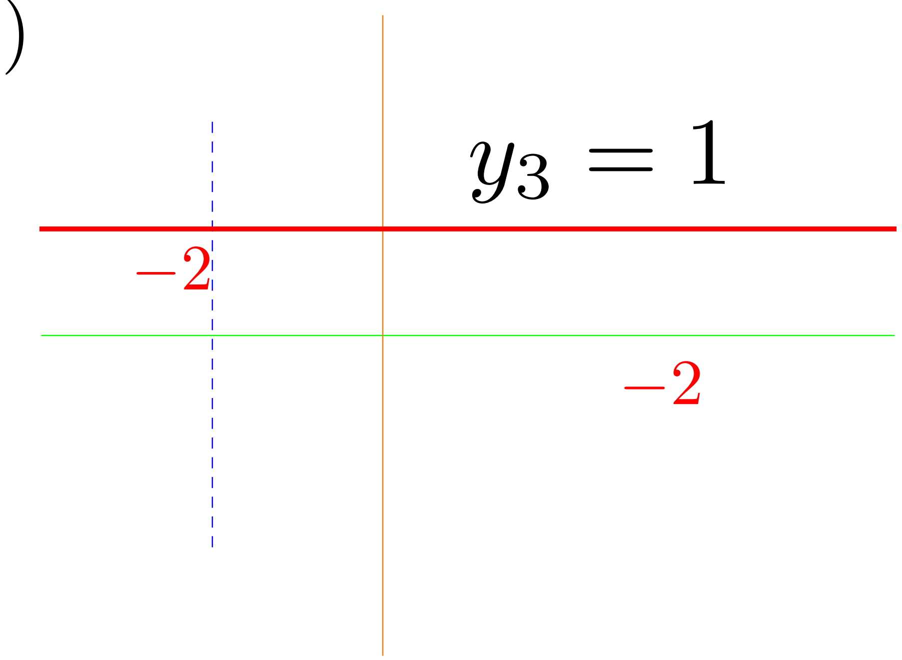
$$f_1(x_1, y_1) = y_1^2 - x_1$$

$$f_1(x_2\,y_2, y_2) = y_2(y_2 - x_2)$$

$$(x_1, y_1) = (x_2\,y_2, y_2)$$



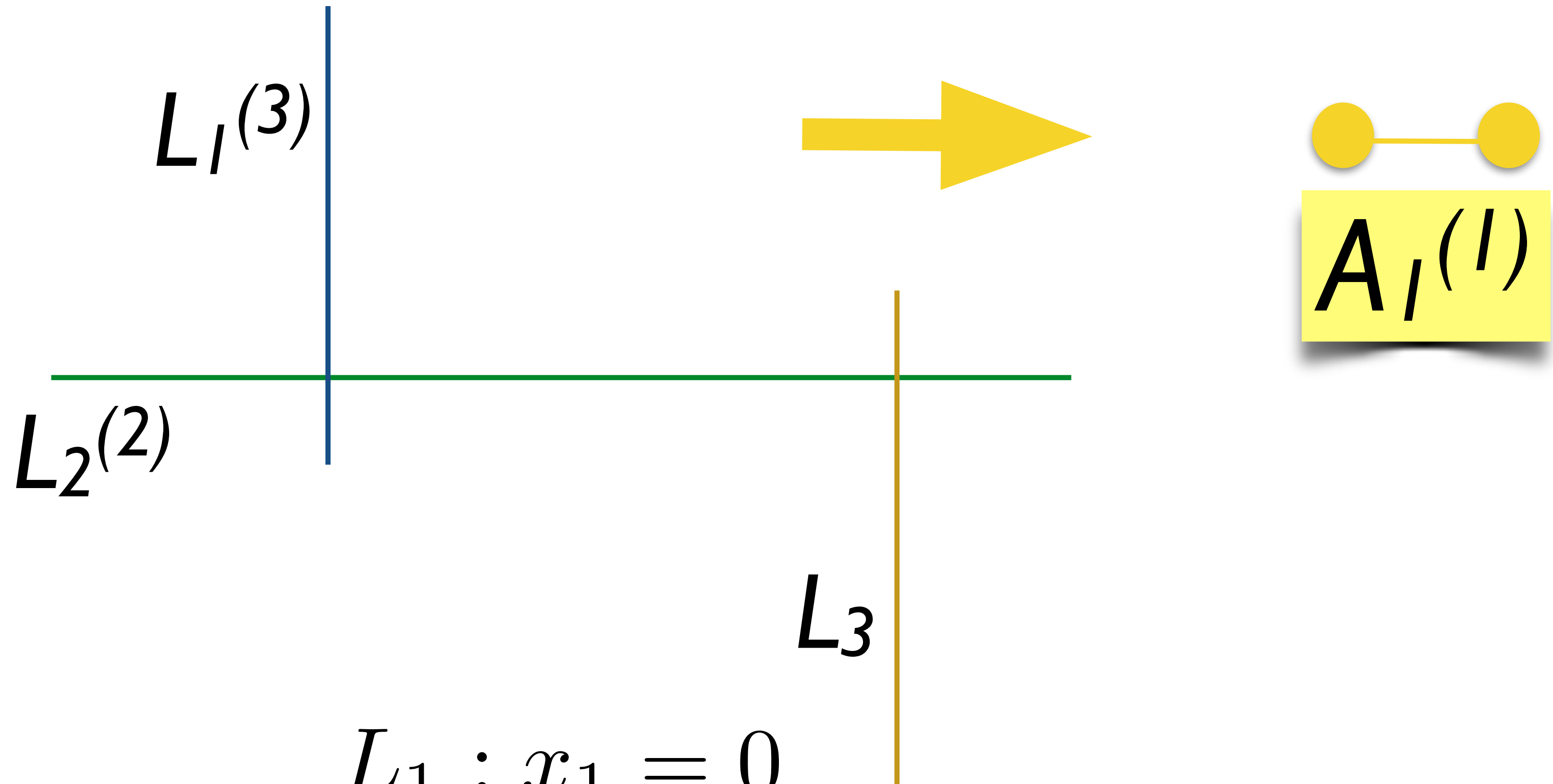
$$(x_2, y_2) = (x_3, x_3\,y_3)$$



$$f_2(x_2, y_2) = y_2 - x_2$$

$$f_2(x_3, x_3\,y_3) = x_3(y_3 - 1)$$

Initial-Value Space



$$L_1 : x_1 = 0$$

$$L_2 : y_2 = 0$$

$$L_3 : x_3 = 0$$

The space is compactified and regularised.

P_I, P_{II}, P_{IV}

P_I:

$$w_1 = t^{1/2}\, u_1(z),\;\; w_2 = t^{3/4}\, u_2(z) \quad z = \frac{4}{5} t^{5/4}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6\, u_1^2 - 1 \end{pmatrix} - \frac{1}{(5z)} \begin{pmatrix} 2\, u_1 \\ 3\, u_2 \end{pmatrix}$$

P_{II}:

$$w_1 = t^{1/2} u_1(z), w_2 = t\, u_2(z), z = \frac{2}{3}\, t^{3/2}$$

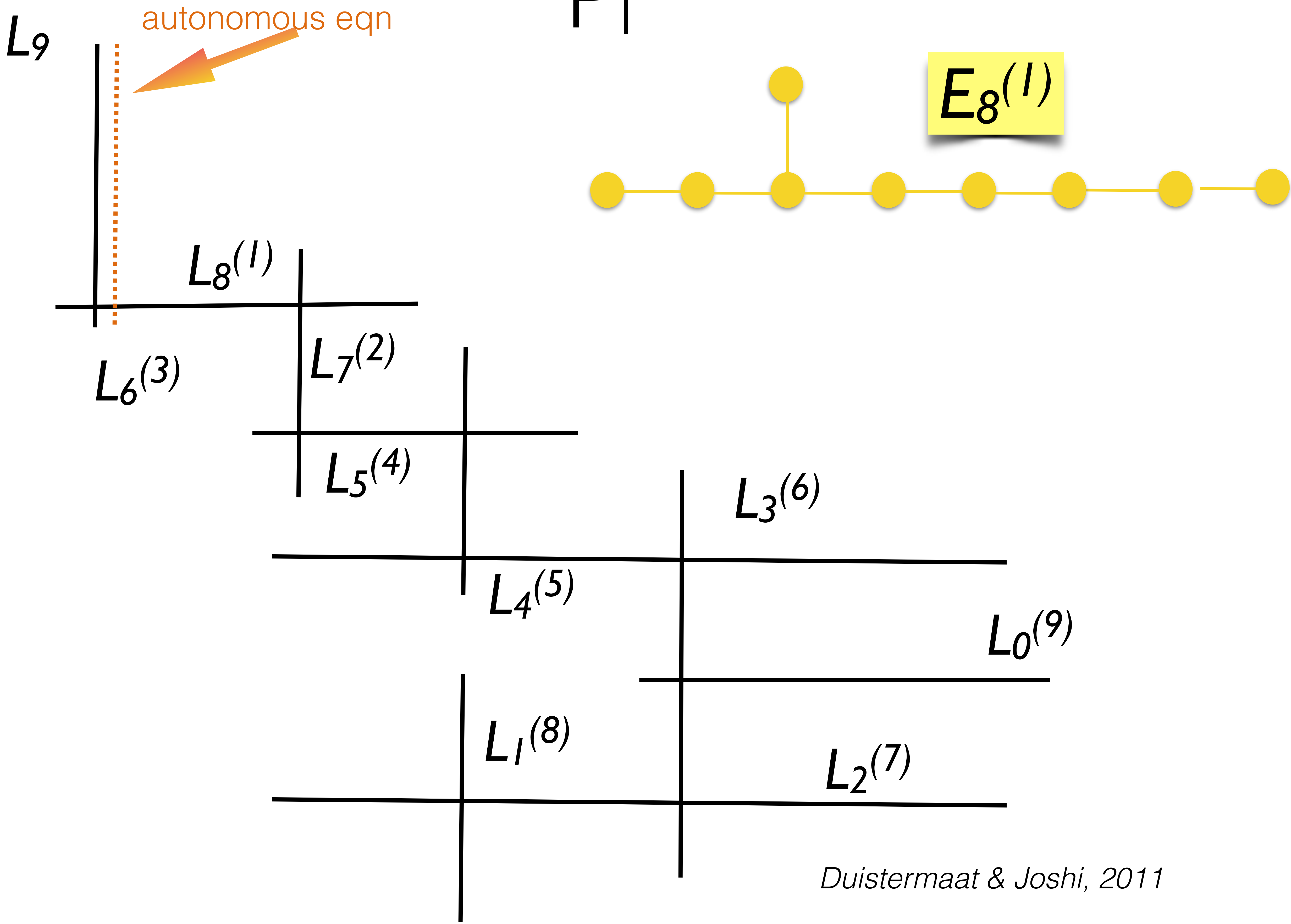
$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 - u_1^2 - \frac{1}{2} \\ 2 u_1 u_2 \end{pmatrix} - \frac{1}{3z} \begin{pmatrix} u_1 \\ -(2\alpha + 1) + 2\, u_2 \end{pmatrix}$$

P_{IV}:

$$w_1 = t\, u_1, w_2 = t\, u_2, z = \frac{t^2}{2}$$

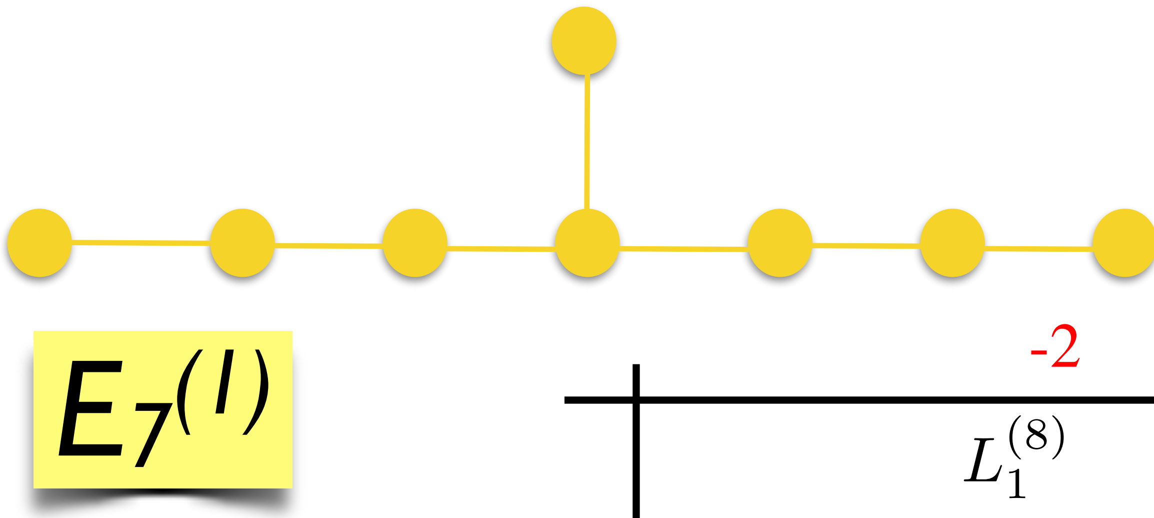
$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} -u_1(u_1 + 2\, u_2 + 2) \\ u_2(2 u_1 + u_2 + 2) \end{pmatrix} - \frac{1}{2z} \begin{pmatrix} 2\, \alpha_1 + u_1 \\ 2\, \alpha_2 + u_2 \end{pmatrix}$$

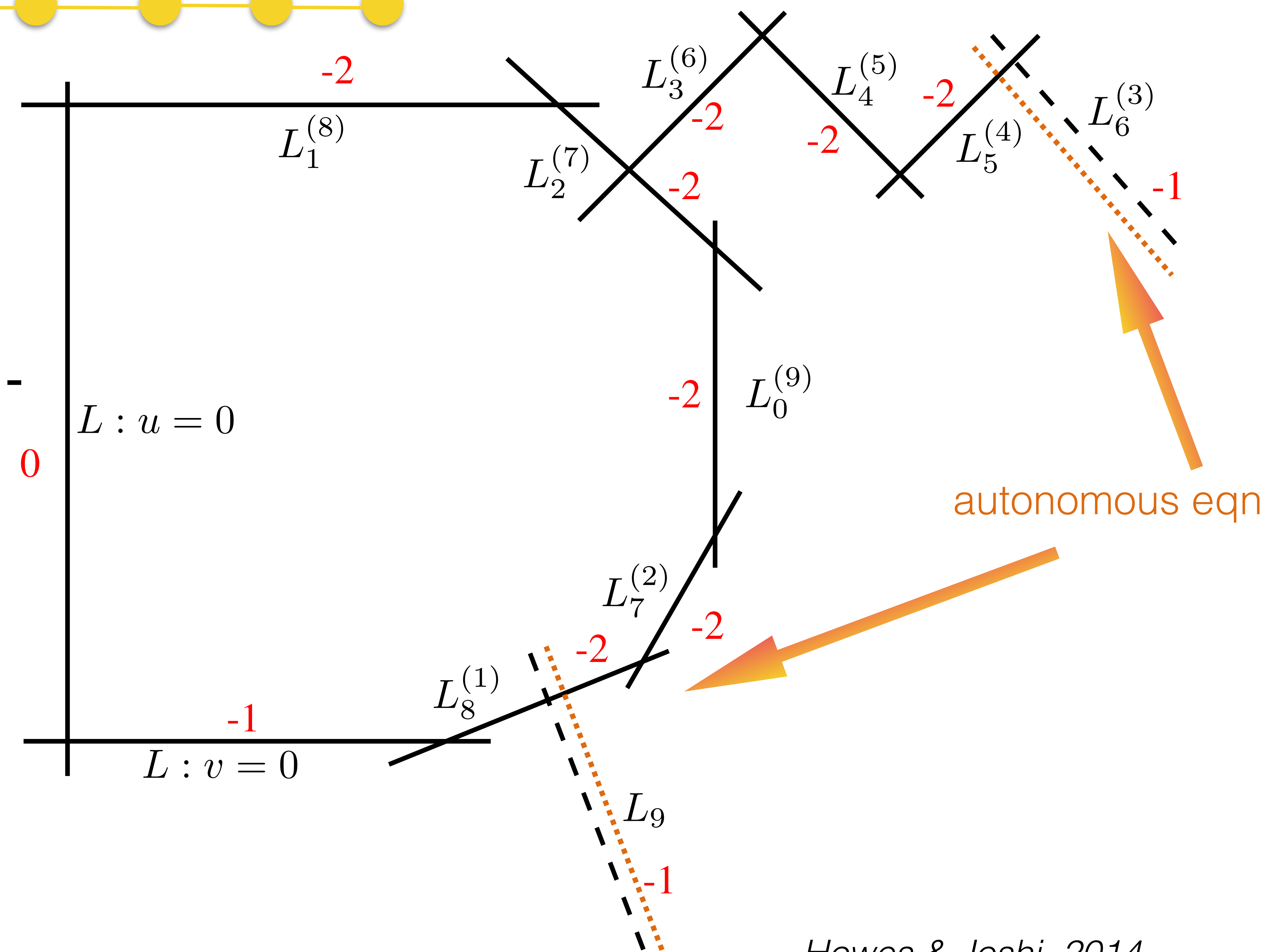
P_I



Duistermaat & Joshi, 2011

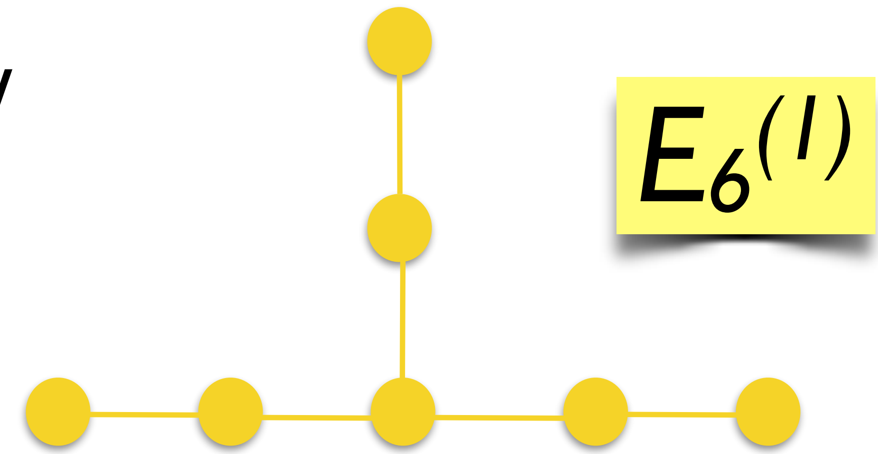
P_{II}

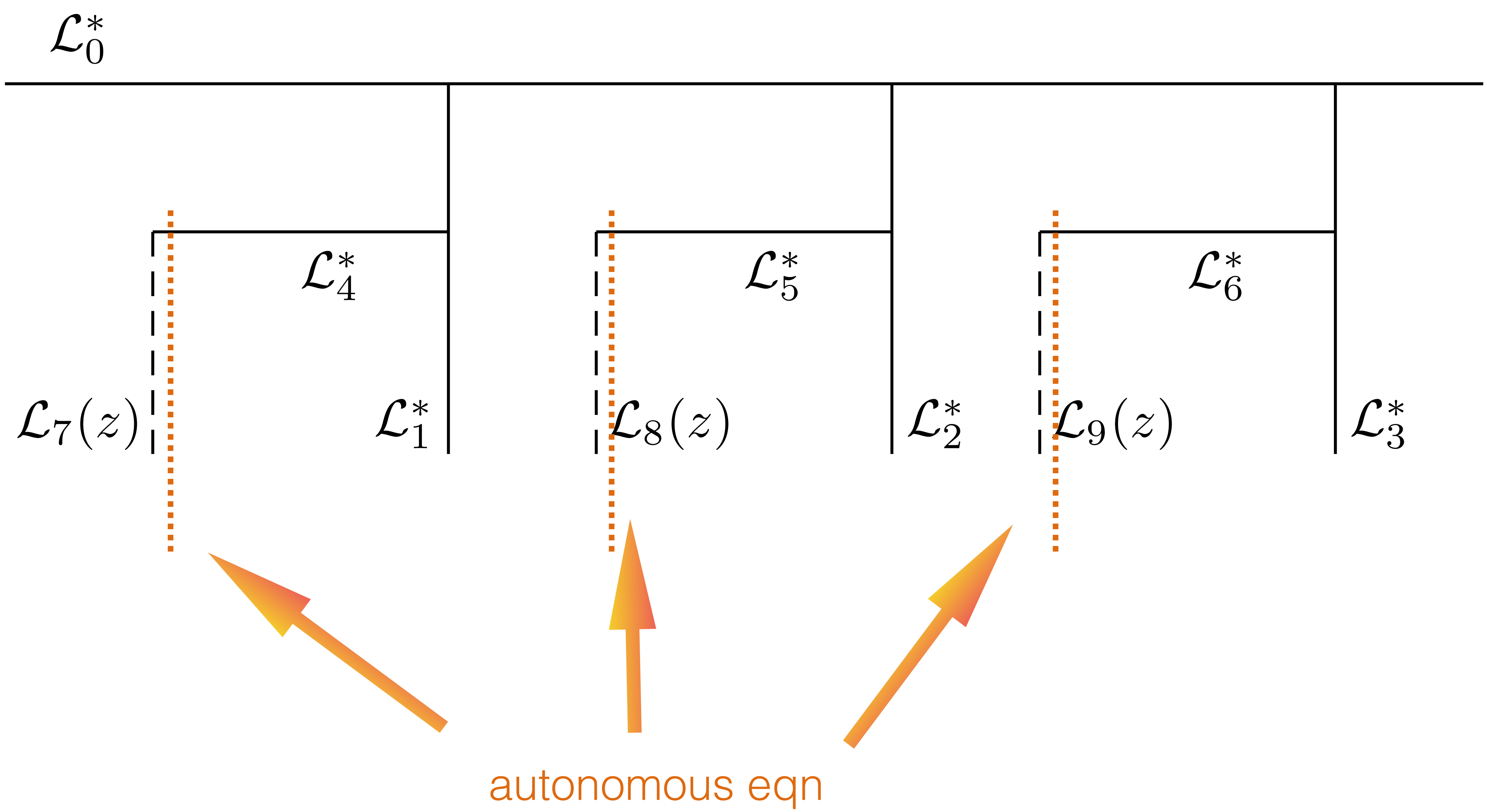




Howes & Joshi, 2014

P_{IV}





Joshi & Radnovic, 2015

Explicit Estimates

Proof. Recall that $L_8^{(1)} \backslash L_7^{(2)}$ is determined by the equation $u_{922} = 0$ and is parametrized by $u_{921} \in \mathbb{C}$. Moreover, L_9 minus one point not on $L_8^{(1)}$ corresponds to $u_{921} = 0$ and is parametrized by u_{922}. For the study of the solutions near the part $L_8^{(1)} \backslash L_7^{(2)}$ of I, we use the coordinates (u_{921}, u_{922}). Asymptotically for $u_{922} \to 0$ and bounded u_{921}, z^{-1}, we have

$$\dot{u}_{921} \sim -2^{-1} {u_{922}}^{-1}, \tag{4.1}$$

$$w_{92} \sim 2^6 u_{922}, \tag{4.2}$$

$$\dot{w}_{92}/w_{92} = 6(5z)^{-1} + \mathrm{O}({u_{922}}^2) = 6(5z)^{-1} + \mathrm{O}({w_{92}}^2), \tag{4.3}$$

$$q\, w_{92} \sim 1 - 2^8 (5z)^{-1} {u_{921}}^{-1}. \tag{4.4}$$

It follows from (4.3) that, as long as the solution is close to a given large compact subset of $L_8^{(1)} \backslash L_7^{(2)}$, $w_{92}(z) = (z/\zeta)^{6/5} w_{92}(\zeta)(1 + \mathrm{o}(1))$, where $z/\zeta \sim 1$ if and only if $|z - \zeta| \ll |\zeta|$. In view of (4.2), in this situation, u_{922} is approximately equal to a small constant, when (4.1) yields that $u_{921}(z) \sim u_{921}(\zeta) - 2^{-1} {u_{922}}^{-1}(z - \zeta)$, and it follows that $u_{921}(z)$, the affine coordinate on $L_8^{(1)} \backslash L_7^{(2)}$, fills an approximate disc centered at $u_{921}(\zeta)$ with radius $\sim R$, if z runs over an approximate disc centered at ζ with radius $\sim 2|u_{922}|R$. Therefore, if $|u_{922}(\zeta)| \ll 1/|\zeta|$, the solution at complex times z in a disk D centered at ζ with radius $\sim 2|u_{922}|R$ has the

Duistermaat & J (2011);

Global results for P_I, P_{II}, P_{IV}

- The union of exceptional lines is a repeller for the flow.
- There exists a complex limit set, which is non-empty, connected and compact.
- Every solution of P_I, every solution of P_{II} whose limit set is not {0}, and every non-rational solution of P_{IV} intersects the last exceptional line(s) infinitely many times => infinite number of movable poles and movable zeroes.

Duistermaat & J (2011); Howes & J (2014); J & Radnovic (2014)

What about discrete Painlevé Equations?

1. Find discrete analogue of Boutroux coordinates.
2. Resolve the space of initial values.
3. Obtain estimates to analyse results.

Consider the Contiguity Relations

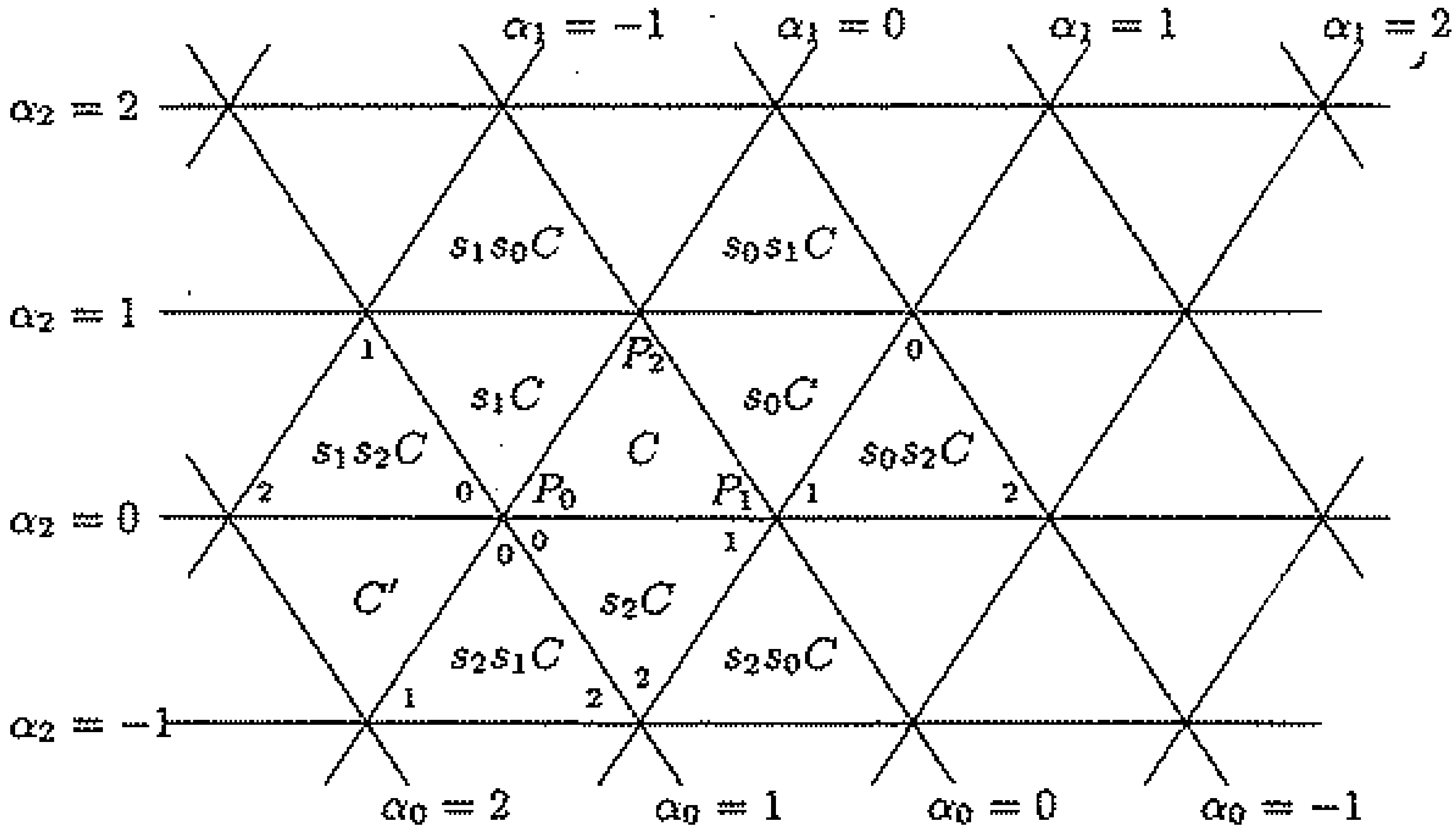


FIGURE 2.3. The Parameter Space of P_{IV}

Noumi, 2000

dP_I is a contiguity relation of P_{IV}

dP_I

$$x_{n+1} + x_n + x_{n-1} = \frac{\alpha n + \beta + c(-1)^n}{x_n} + \gamma$$

This arises as a contiguity relation of P_{IV} . Using

$$u_n = x_{2n},\ v_n = x_{2n+1}$$

we obtain

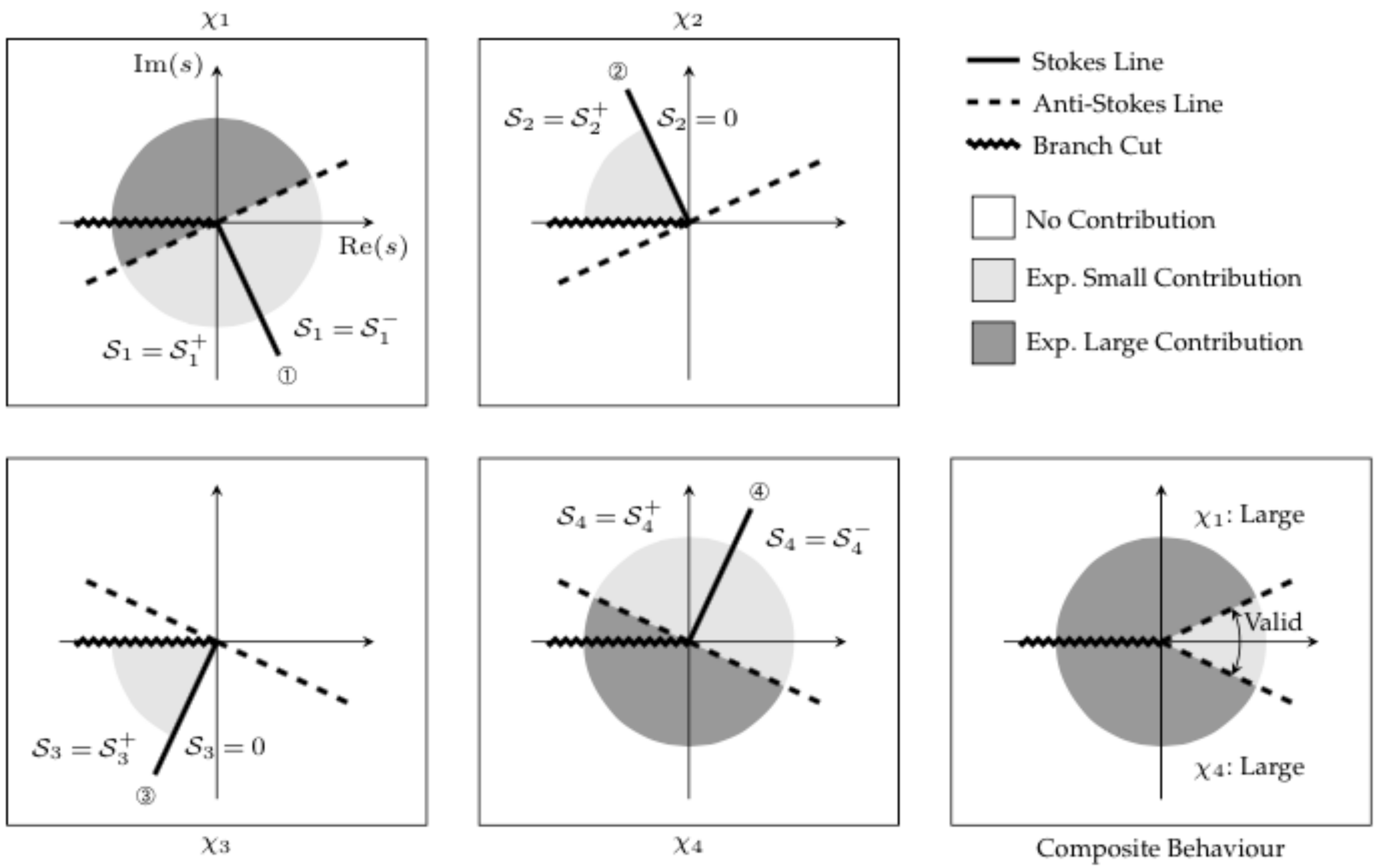
$$u_{n+1} + v_{n+1} + u_n = \frac{2\alpha n + \alpha + \beta - c}{v_{n+1}} + \gamma,$$

$$v_{n+1} + u_n + v_n = \frac{2\alpha n + \beta + c}{u_n} + \gamma$$

Asymptotic Series Solutions

- Consider the case *c=0.*
- Take $s = \epsilon n$ and $u_n = \dfrac{U(s,\epsilon)}{\epsilon^{1/2}}, v_n = \dfrac{V(s,\epsilon)}{\epsilon^{1/2}}$
- Then
$$U(s,\epsilon) \sim \sum_{m=0}^{\infty} \epsilon^{m/2}\, U_m(s), V(s,\epsilon) \sim \sum_{m=0}^{\infty} \epsilon^{m/2}\, V_m(s)$$
are divergent asymptotic series solutions, containing exponentially small terms, hidden beyond all orders.

Stokes sectors



(a) General Stokes Structure

J. & Lustri, 2015

Boutroux-like coordinates

Scaling

$$u_n = \sqrt{n} y_n, \ v_n = \sqrt{n} z_n$$

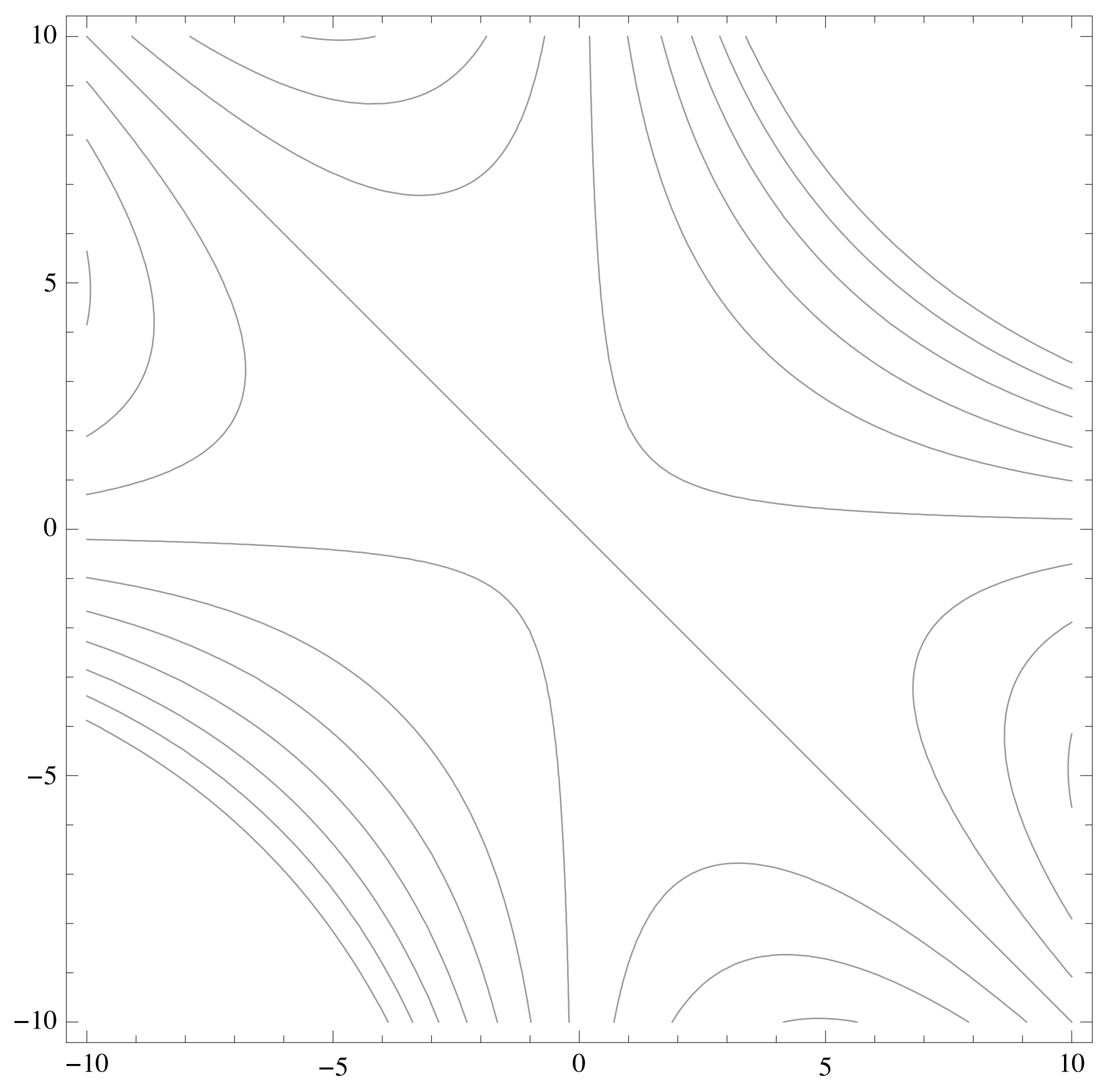
yields

$$y_{n+1} + y_n + z_{n+1} = \frac{2\alpha}{z_{n+1}} + \frac{\gamma}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right),$$

$$z_{n+1} + y_n + z_n = \frac{2\alpha}{y_n} + \frac{\gamma}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right)$$

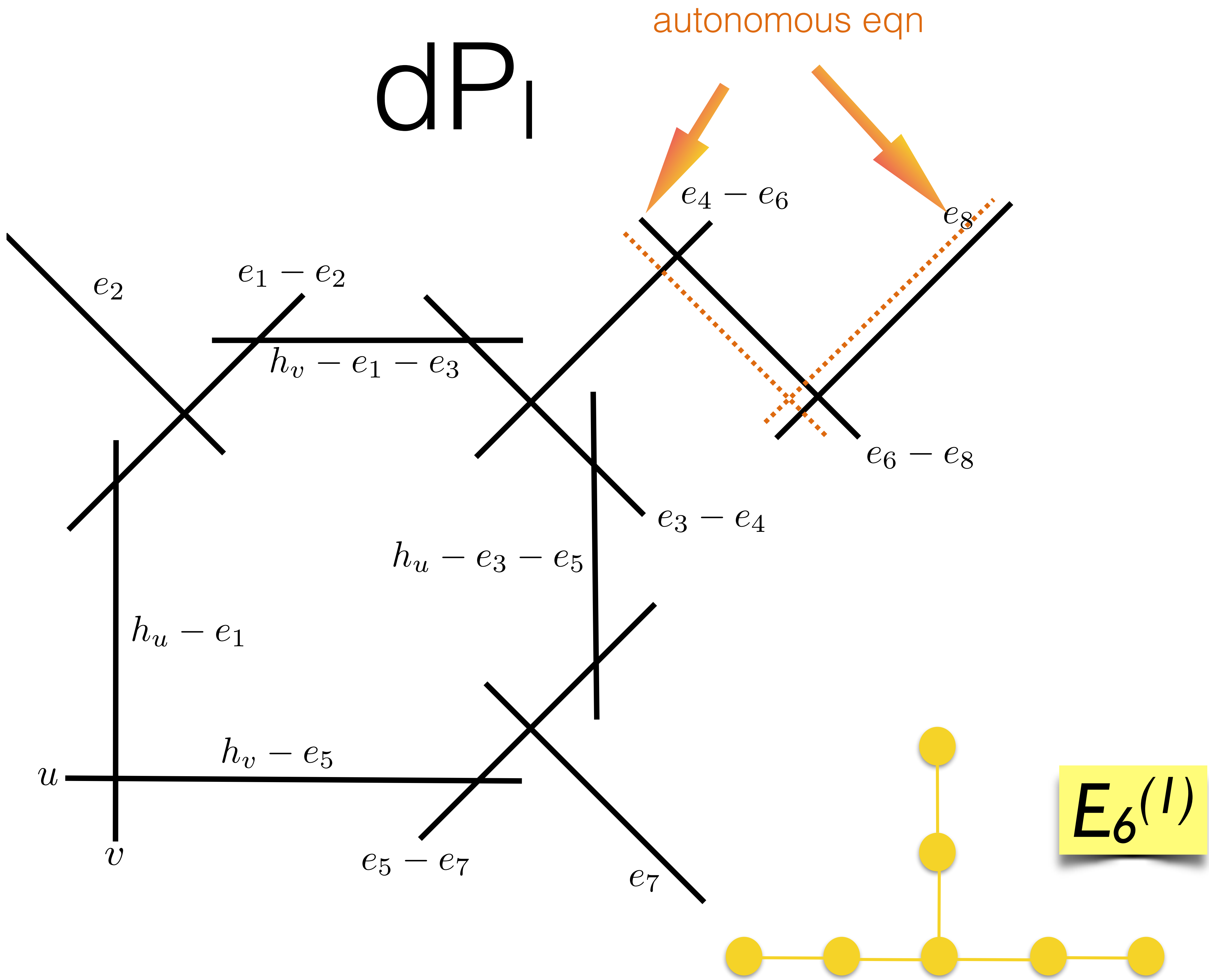
Leading-order

$$K(x,y) = x^2 y + x y^2 - 2\alpha\, x - 2\alpha\, y$$



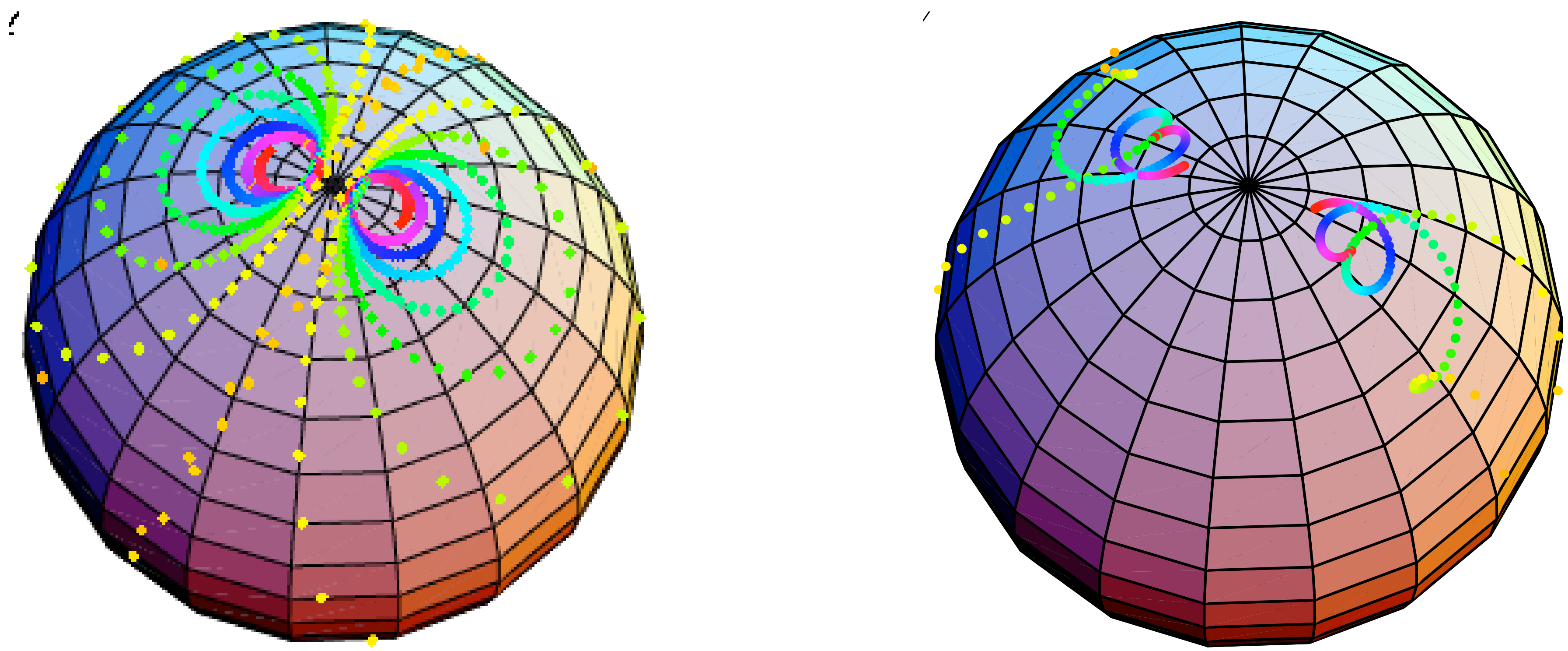
Elliptic curves

dP$_{\text{I}}$



autonomous eqn
$e_4 - e_6$
e_8
e_2
$e_1 - e_2$
$h_v - e_1 - e_3$
$e_6 - e_8$
$e_3 - e_4$
$h_u - e_3 - e_5$
$h_u - e_1$
$h_v - e_5$
u
v
$e_5 - e_7$
e_7
$E_6^{(1)}$

Solutions



Solution orbits of scalar dP1 on the Riemann sphere (where the north pole is infinity).

q-Discrete P_I

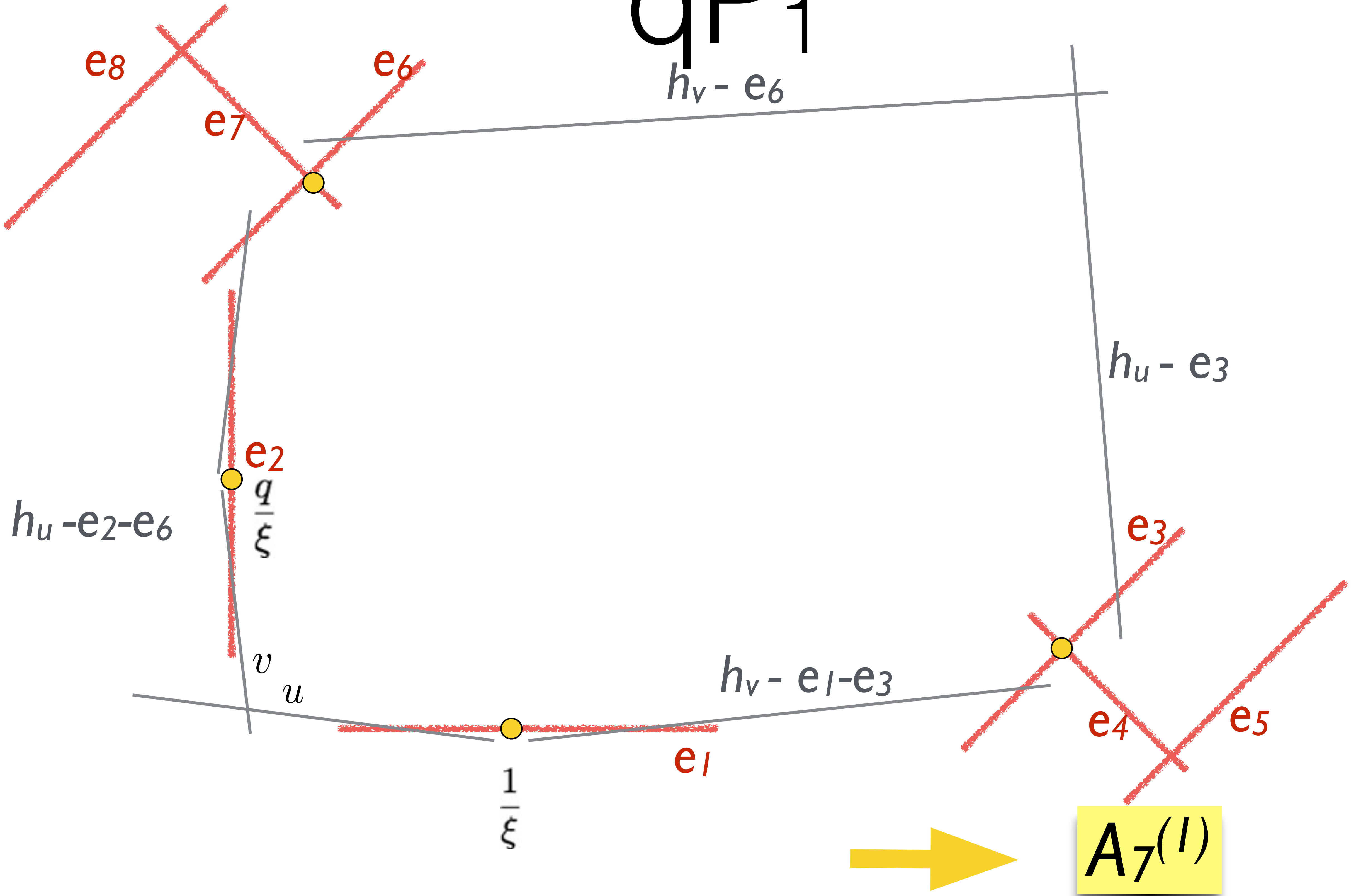
$$\overline{w}\,\underline{w} = \frac{1}{w} - \frac{1}{\xi\, w^2}$$

Almost stationary solutions:

$$\overline{w} \sim w,\ \ \underline{w} \sim w \ \text{ as } |\xi| \to \infty$$

$$\Rightarrow\ \ w(w^3 - 1) = \mathcal{O}(1/\xi)$$

qP_1



e_8
e_6
e_7
h_v - e_6
h_u - e_3
e_2
$\frac{q}{\xi}$
h_u -e_2-e_6
e_3
v
u
h_v - e_1-e_3
e_4
e_5
e_1
$\frac{1}{\xi}$
$A_7^{(1)}$

Near fixed points I

$$w(\xi) = \sum_{n=0}^{\infty} \frac{a_n}{\xi^n},$$

$$a_0^3 = 1,$$

$$a_1\,(q + 1 + q^{-1}) = -\,1,$$

$$\sum_{m=0}^{n} \sum_{j=0}^{n-m} \sum_{l=0}^{m} a_j\, a_{n-m-j}\, a_l\, a_{m-l} q^{(n-m-2\,j)} = a_n,\ \ n \geq 2.$$

Near fixed points II

$$w(\xi) = \sum_{n=1}^{\infty} \frac{b_n}{\xi^n}$$

$$b_1 = 1$$

$$b_2 = 0$$

$$b_3 = 0$$

$$b_n = \sum_{r=2}^{n-2} \sum_{k=1}^{r-1} \sum_{m=1}^{n-r-1} b_k \, b_{r-k} \, b_m \, b_{n-r-m} q^{(r-2\,k)}, \;\; n \geq 4$$

Base Points

$$\begin{cases} \bar{u} &= \dfrac{\xi u - 1}{\xi u^2 v}, \\ \bar{v} &= u, \end{cases} \Rightarrow (u, v) = (1/\xi, 0)$$

$$\begin{cases} \underline{u} &= v, \\ \underline{v} &= \dfrac{\xi v - q}{\xi u v^2}. \end{cases} \Rightarrow (u, v) = (0, q/\xi)$$

$$\begin{cases} \overline{u} &= \frac{U(\xi - U)}{\xi v} \\ \overline{v} &= \frac{1}{U} \end{cases} \Rightarrow (U, v) = (0, 0) \quad U = 1/u$$

similarly for $V = 1/v$

Results for qP_I

- The second type of series is divergent, valid in a large region.
- The corresponding true solution, called “quicksilver solution”, is analytic in this region. Corresponds to case of merging base points. Is it an analogue of the *tritronquée* solution?
- Exceptional lines are repellers for the flow.
- *J (2014); J. & Lobb (2015)*

Summary

- Dynamics of solutions of non-linear equations, whether they are differential or discrete, can be described globally and completely through geometry.
- Geometry appears to be the only analytic approach available in $\mathbb{C}$ for discrete equations.
- Finite properties?

Some of the geometric inquisitors who are part of the team at Sydney